A Geometric Vietoris-Begle Theorem, with an Application to Convex Subsets of Riesz Spaces

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Abstract

We show that a surjective map between compact connected ANR's is a homotopy equivalence if the fibers are contractible and either the domain is simply connected or the fibers are AR's. This is a geometric analogue of the Vietoris-Begle theorem. We use it to show that if L is a Hausdorff locally convex Riesz space, $x \in L$, the function $u_x : y \mapsto x \vee y$ is continuous, and $C \subset L$ is compact, convex, and metrizable, then $u_x|_C$ is a homotopy equivalence, so $u_x(C)$ is a compact AR.

1 Introduction

Recall that a metric space X is an *absolute neighborhood retract* (ANR) if, whenever $e : X \to Z$ is an embedding of X as a closed subset of another metric space Z, there is a neighborhood $U \subset Z$ of $e(X)$ and a retraction $r : U \to e(X)$, and X is an *absolute retract* if, whenever $e : X \to Z$ is an embedding of X as a closed subset of a metric space Z, there is a retraction $r : Z \to e(X)$. These spaces were extensively studied from the time of their introduction by Borsuk [4] until around 1970, but are now less well known. Elementary properties of these spaces are described in [15, Ch. 8]; from our point of view the most important elementary fact is that an ANR is an AR if

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and only if it is contractible ([15, Th. 8.2]). Other elementary properties will be developed as we proceed.

This note has two main points, the first of which is the following pair of results.

Theorem 1. *If X and Y are compact connected ANR's, $f : X \to Y$ is a continuous surjection, and, for each $y \in Y$, the fiber $f^{-1}(y)$ is an AR, then f is a homotopy equivalence.*

Our methods allow a weaker hypothesis on the fibers when X is simply connected.

Theorem 2. *If X and Y are compact connected ANR's, $f : X \to Y$ is a continuous surjection, X is simply connected, and, for each $y \in Y$, $f^{-1}(y)$ is contractible, then f is a homotopy equivalence.*

Prior results similar to Theorem 1 include various results in Section 3 of [2] and Proposition 2.1.8 of [19]. In all of those results X and Y are assumed to be finite dimensional. In particular, [2] uses Corollaries 12.14 and 12.15 of [5] to infer that Y is an ANR or an AR, and the hypotheses of those results include finite dimensionality. Thus the main generalization here is that finite dimensionality is not needed, so long as both X and Y are ANR's. That it suffices to only assume that the fibers are contractible when X is simply connected seems not to have been noticed before.

Theorems 1 and 2 are closely related to the Vietoris-Begle theorem. A space Z is *acyclic* with respect to a homology theory H_* with associated reduced homology $\tilde{H}_*$ (cohomology theory H^* with associated reduced cohomology $\tilde{H}^*$) if $\tilde{H}_n(Z) = 0$ ($\tilde{H}^n(Z) = 0$) for all $n = 0, 1, 2, \ldots$. Since homology and cohomology are invariant under homotopy, a contractible space is acyclic for any homology or cohomology theory. Of the various versions of the Vietoris-Begle theorem in the literature, we state two, which use Alexander-Spanier cohomology and homology respectively. The first might be regarded as the "standard" version. It asserts that if X and Y are paracompact Hausdorff spaces, $f : X \to Y$ is a closed continuous surjection, and, for some $n \geq 0$, $\tilde{H}^k(f^{-1}(y)) = 0$ for all $y \in Y$ and $k < n$, then $\tilde{H}^k(f) : \tilde{H}^k(X) \to \tilde{H}^k(Y)$ is an isomorphism for $k < n$ and an injection for $k = n$. A particularly elegant proof is given in [13]. Our proof of Theorem 2 uses the second, which is a dual result that was established by [18] and reproved by [8]. It asserts that if X and Y are compact metrizable spaces, f is a continuous surjection, and, for some $n \geq 0$, $\tilde{H}_k(f^{-1}(y)) = 0$ for all $y \in Y$ and $k < n$, then $\tilde{H}_k(f) : \tilde{H}_k(X) \to \tilde{H}_k(Y)$ is an isomorphism for $k < n$ and a surjection for $k = n$. Evidently Theorems 1 and 2 and the Vietoris-Begle theorem say quite similar things, in geometric and algebraic language respectively.

Our second main point is a consequence of Theorems 1 and 2. Let L be a Hausdorff locally convex Riesz space. That is, L is a Hausdorff locally convex topological vector space over the reals endowed with a partial order $\geq$ such that: a) if $x \geq y$, then $x + z \geq y + z$ for all $z \in L$ and $\alpha x \geq \alpha y$ for all $\alpha \geq 0$; b) any two elements $x, y \in L$ have a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$. For $x \in L$ let $u_x, d_x : L \to L$ be the functions $u_x(y) = x \vee y$ and $d_x(y) = x \wedge y$.

Theorem 3. *Suppose that $C \subset L$ is compact, convex, and metrizable, $x \in L$, and $D = u_x(C)$ ($D = d_x(C)$). If u_x (d_x) is continuous, then D is an ANR and $u_x|_C : C \to D$ ($d_x|_C : C \to D$) is a homotopy equivalence, so D is an AR.*

Remarks: (a) We construct an example satisfying the hypotheses of Theorem 2 but not those of Theorem 1. Kinoshita [11] (see also [15, 8.1]) created an example that came to be known as the *tin can with a roll of toilet paper*. This is the space $T = (D \times \{0\}) \cup (C \times [0,1]) \cup (S \times [0,1]) \subset \mathbb{R}^3$ where:

$$D = \{\, x \in \mathbb{R}^2 : \|x\| \le 1 \,\}, \quad C = \{\, x \in \mathbb{R}^2 : \|x\| = 1 \,\}, \quad S = \{\, \tfrac{\theta}{1+\theta}(\cos\theta, \sin\theta) : 0 \le \theta < \infty \,\}.$$

Of course T is compact. There is an obvious contraction of T that deformation retracts vertically onto $D \times \{0\}$, then compresses that set. Kinoshita gave a continuous function from T to itself that does not have a fixed point, so, in view of the Eilenberg-Montgomery [9] fixed point theorem, T cannot be an AR. Let $S^2 = \{\, z \in \mathbb{R}^3 : \|z\| = 1 \,\}$. It is easy to see[1] that there is a homeomorphism $h : S^2 \times (0,\infty) \to \mathbb{R}^3 \setminus T$ such that $T = \bigcap_{\varepsilon > 0} \overline{h(S^2 \times (0,\varepsilon))}$. For $y \in \mathbb{R}^3 \setminus T$ let $h^{-1}(y) = (p(y), \alpha(y))$. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the function

$$f(y) = \begin{cases} 0, & y \in T, \\ \alpha(y)p(y), & y \notin T. \end{cases}$$

(b) The proof of Theorem 3 uses the additional generality (in comparison with prior results) of Theorems 1 and 2 when L is infinite dimensional, but even when L is finite dimensional it seems that the proof requires some version of the results above. Methods that suffice in particular cases (a line segment in $\mathbb{R}^n$ for large n is a good example) are diverse and do not easily generalize.

(c) For $x \in L$ let $|x| = (x \vee 0) - (x \wedge 0)$. A set $A \subset L$ is *solid* if, for all $y \in A$, A contains all $x \in L$ such that $|x| \le |y|$, and L is *locally solid* if its topology has a base at the origin consisting of solid sets. A result of Roberts and Namioka (e.g., [1, p. 55]) asserts that a Riesz space is locally solid if and only if the function $(x,y) \mapsto x \vee y$ is continuous, and this is the case if and only if $(x,y) \mapsto x \wedge y$ is continuous. There is an example ([1, p. 56]) for which u_0 continuous even though the space is not locally solid.

(d) Among various ways that C may be metrizable even if L is not, we mention that Varadarajan [17] has shown that if L is the space of measures on a compact metric space with the weak

[1] Here is an explicit construction. Let L be a line in $\mathbb{R}^2$ that passes through the origin, and let $C_L = (L \times \mathbb{R}) \cap S^2$. For $\varepsilon > 0$ let $D_{L.\varepsilon} = P_{L,\varepsilon} \cup Q_{L,\varepsilon} \cup R_{L,\varepsilon}$ where $P_{L,\varepsilon} = \{\, x \in L : \|x\| \le 1 + \varepsilon \,\} \times \{-\varepsilon\}$, $Q_{L,\varepsilon} = \{\, x \in L : \|x\| = 1 + \varepsilon \,\} \times [-\varepsilon, 1+\varepsilon]$, and $R_{L,\varepsilon}$ is the union of $\{\, x \in L : 1 \le \|x\| \le 1+\varepsilon \,\} \times \{1+\varepsilon\}$, the singleton $\{(0,0,1+\varepsilon)\}$, and for each pair of consecutive points $x_0, x_1 \in L \cap S$ the set $\{\, ((1-t)x_0 + tx_1, \max\{1+\varepsilon - \min\{t, 1-t\}\|x_1 - x_0\|/\varepsilon, \varepsilon\}) : t \in [0,1] \,\}$. It is easy to construct maps $h_{L,\varepsilon} : C_L \times \{\varepsilon\} \to D_{L,\varepsilon}$ that combine to give a satisfactory h.

topology, then the lattice cone $L_+ = \{ x \in L : x \geq 0 \}$ is metrizable, but L is metrizable only under quite restrictive conditions. This case is common in economic applications.

(e) In economics the existence of equilibrium is frequently proved by applying the Kakutani fixed point theorem and its infinite dimensional generalizations to upper hemicontinuous convex valued correspondences. The author was led to wonder whether Theorem 3 might be true because it gives a method of passing from a convex valued correspondence to a contractible valued correspondence, to which the Eilenberg-Montgomery fixed point theorem might be applied.

2 The Proofs of Theorems 1 and 2

In this section X and Y are compact connected ANR's and $f : X \to Y$ is a continuous surjection. The proofs of Theorems 1 and 2 verify, respectively, the hypotheses of the following two results, which give sufficient conditions, in terms of the induced maps of homotopy and homology groups, for a function to be a homotopy equivalence. These are Theorem 1 (p. 1133) and Theorem 3 (p. 1135) of [20] respectively. (Actually the hypotheses of that Theorem 1 are somewhat weaker.)

Proposition 1. *If $f_n : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n, then f is a homotopy equivalence.*

Fix a point $x_0 \in X$, and let $y_0 = f(x_0)$. Let $\tilde{X}$ and $\tilde{Y}$ be the universal covering spaces of X and Y, with respect to the base points x_0 and y_0, and let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the lift of f with respect to these base points.

Proposition 2. *If $f_1 : \pi_1(X) \to \pi_1(Y)$ is an isomorphism and $\tilde{H}_n(\tilde{f}) : \tilde{H}_n(\tilde{X}) \to \tilde{H}_n(\tilde{Y})$ is an isomorphism for each $n = 2, 3, \ldots$, then f is a homotopy equivalence.*

We first present the proof of Theorem 1, then explain how to modify the argument to prove Theorem 2. The following fact will be used several times.

Lemma 1. *If $g : A \to B$ is continuous, A is compact, B is Hausdorff, $b \in B$, and $U \subset A$ is an open neighborhood of $g^{-1}(b)$, then there is an open $V \subset B$ containing b such that $g^{-1}(V) \subset U$.*

Proof. Otherwise for each open $V \subset B$ containing b there would be an $a_V \in A \setminus U$ such that $g(a_V) \in V$. Since $A \setminus U$ is compact, a subnet of $\{a_V\}$ would converge to one of its elements, say a. Since B is Hausdorff, continuity gives $g(a) = b$, but then $a \in g^{-1}(b) \cap (A \setminus U) = \emptyset$. □

Because they are metric spaces, X and Y can be embedded in Banach spaces B_X and B_Y (e.g., [15, Th. 6.3]). Fix retractions $r_X : U_X \to X$ and $s_Y : V_Y \to Y$ of neighborhoods $U_X \subset B_X$ and $V_Y \subset B_Y$ of X and Y. Let $W \subset Y \times Y$ be a neighborhood of the diagonal $\{ (y, y) : y \in Y \}$

such that $(1-t)y_0 + ty_1 \in V_Y$ for all $(y_0, y_1) \in W$ and $t \in [0,1]$. A *compressive pair* is a pair (V', V) of open subsets of Y such that $V \times V \subset W$, $V' \subset V$, and there is a continuous $\xi : f^{-1}(V') \times [0,1] \to f^{-1}(V)$ such that $\xi(\cdot, 0)$ is the identity function of $f^{-1}(V')$ and $\xi(\cdot, 1)$ is a constant function.

Lemma 2. *If $V \subset Y$ is open, $V \times V \subset W$, $y \in V$, and $f^{-1}(y)$ is an AR, then there is an open $V' \subset V$ containing y such that (V', V) is a compressive pair.*

Proof. Fix a contraction $c : f^{-1}(y) \times [0,1] \to f^{-1}(y)$ and a retraction $r : U \to f^{-1}(y)$ where $U \subset X$ is a neighborhood of $f^{-1}(y)$. Since $f^{-1}(y)$ is compact,

$$U' = \{\, x \in U : (1-t)x + tr(x) \in r_X^{-1}(f^{-1}(V)) \text{ for all } t \in [0,1] \,\}$$

is an open neighborhood of $f^{-1}(y)$. Lemma 1 implies that there is an open $V' \subset V$ containing y such that $f^{-1}(V') \subset U'$. Let $\xi : f^{-1}(V') \times [0,1] \to f^{-1}(V)$ be the function

$$\xi(x,t) = \begin{cases} r_X((1-2t)x + 2tr(x)), & 0 \le t \le \frac{1}{2}, \\ c(r(x), 2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

□

A *compressive cover* is a collection $\mathcal{V}$ of compressive pairs such that $\{\, V' : (V', V) \in \mathcal{V} \,\}$ is a cover of Y. We say that an compressive cover $\tilde{\mathcal{V}}$ is a *star refinement* of $\mathcal{V}$ if, for each $(\tilde{V}', \tilde{V}) \in \tilde{\mathcal{V}}$, there is a $(V', V) \in \mathcal{V}$ such that

$$\bigcup_{(\tilde{V}_0', \tilde{V}_0) \in \tilde{\mathcal{V}}, \tilde{V}_0 \cap \tilde{V} \neq \emptyset} \tilde{V}_0 \subset V'.$$

Suppose each $f^{-1}(y)$ is an AR. Obviously Lemma 2 implies that compressive covers exist. Moreover, for any given compressive cover $\mathcal{V}$ there is an compressive cover $\tilde{\mathcal{V}}$ that star refines it: the Lebesgue number lemma gives an $\varepsilon > 0$ such that for each $y \in Y$ there is some $(V', V) \in \mathcal{V}$ such that V' contains the ε-ball centered at y, and Lemma 2 gives a compressive cover $\tilde{\mathcal{V}}$ such that for each $(\tilde{V}', \tilde{V}) \in \tilde{\mathcal{V}}$ the diameter of $\tilde{V}$ is less than $\varepsilon/2$.

For each positive integer k let D^k and S^{k-1} be the closed unit ball in $\mathbb{R}^k$ and its boundary. Note that if (V', V) is a compressive pair with ξ as above, then any continuous $\eta_\partial : S^{k-1} \to f^{-1}(V')$ has a continuous extension $\eta : D^k \to f^{-1}(V)$ given by $\eta(tp) = \xi(\eta_\partial(p), t)$ for $p \in S^{k-1}$ and $t \in [0,1]$.

Proposition 3. *Let K be a finite simplicial complex, let L be a subcomplex, and let $g : K \to Y$ and $\eta : L \to X$ be maps such that $f\eta = g|_L$. Then there is a continuous extension $\gamma : K \to X$ of η such that $(g(p), f(\gamma(p))) \in W$ for all $p \in K$.*

Proof. (See [3, p. 436] for a more detailed description of the construction.) Let n be the dimension of K. Choose compressive covers $\mathcal{V}_k$ for $k = 0, \ldots, n$ such that $\mathcal{V}_{k-1}$ is a star refinement of $\mathcal{V}_k$ for each $k = 1, \ldots, n$. After sufficient repeated barycentric subdivision, for each k-simplex σ of K there is some $(V', V) \in \mathcal{V}_k$ such that $g(\sigma) \subset V$. Let $K^{(k)}$ be the k-skeleton of K. Proceeding inductively, for $k = 0, \ldots, n$ we will construct extensions $\gamma_k : K^{(k)} \cup L \to X$ of η such that for each k-simplex σ of K there is some $(V', V) \in \mathcal{V}_k$ such that $\gamma(\sigma) \subset f^{-1}(V)$. First construct an extension $\gamma_0 : K^{(0)} \cup L \to X$ by letting the image $\gamma_0(v)$ of a vertex of K that is not in L be any element of $f^{-1}(g(v))$. (Of course $\gamma_0(v) \in f^{-1}(V)$ for any $(V', V) \in \mathcal{V}_0$ such that $g(v) \in V$.) Now suppose that γ_{k-1} has already been constructed, σ is a k-simplex of K that is not in L, and τ is a facet of σ. Let $(\tilde{V}', \tilde{V})$ be an element of $\mathcal{V}_{k-1}$ such that $g(\tau) \subset \tilde{V}$. Since every facet of σ intersects τ and $\mathcal{V}_{k-1}$ is a star refinement of $\mathcal{V}_k$, there is a $(V', V) \in \mathcal{V}_k$ such that $g(\partial\sigma) \subset V'$, and there is an extension $\gamma_k|_\sigma : \sigma \to X$ of $\gamma_{k-1}|_{\partial\sigma}$ such that $\gamma_k(\sigma) \subset f^{-1}(V)$. Combining such extensions for all k-simplices in K that are not in L constructs γ_k. Finally let $\gamma = \gamma_n$. ☐

Proof of Theorem 1. For any $n \geq 1$ and continuous $g : S^n \to Y$ the last result gives a continuous $\gamma : S^n \to X$ such that $(g(p), f(\gamma(p))) \in W$ for all $p \in S^n$, so that $h(p,t) = s_Y((1-t)g(t) + tf(\gamma(p)))$ is a homotopy between g and $f\gamma$. Thus f_n is surjective. If $\eta : S^n \to X$ is continuous and $g : D^{n+1} \to Y$ is an extension of $f\eta$, then the last result gives a continuous extension $\gamma : D^{n+1} \to X$ of η. Thus f_n is injective. Since each f_n is an isomorphism, Proposition 1 implies that f is a homotopy equivalence. ☐

We now turn to the proof of Theorem 2.

Lemma 3. *If $y \in Y$ and $f^{-1}(y)$ is path connected, then, for any open $V \subset Y$ containing y, there is an open $V' \subset V$ containing y such that any function $\eta_\partial : \{-1, 1\} \to f^{-1}(V')$ has a continuous extension $\eta : [-1, 1] \to f^{-1}(V)$.*

Proof. Since $f^{-1}(y)$ is compact there is a $\delta > 0$ such that the open δ-ball B_δ in B_X around $f^{-1}(y)$ is contained in $r_X^{-1}(f^{-1}(V))$. Lemma 1 gives an open $V' \subset V$ containing y such that $f^{-1}(V') \subset B_\delta$. Suppose η_∂ is a function from $\{-1, 1\}$ to $f^{-1}(V')$. Choose $x_{-1}, x_1 \in f^{-1}(y)$ such that the distance from $\eta_\partial(-1)$ to x_{-1} and the distance from $\eta_\partial(1)$ to x_1 are both less than δ. A satisfactory extension η can be constructed by combining reparameterizations of the three paths $t \mapsto r_X((1-t)\eta_\partial(-1) + tx_{-1})$, a path $\pi : [0,1] \to f^{-1}(y)$ with $\pi(-1) = x_{-1}$ and $\pi(1) = x_1$, and the path $t \mapsto r_X((1-t)x_1 + t\eta_\partial(1))$. ☐

Proof of Theorem 2. The last result implies that there is an open cover $\{ V' : (V', V) \in \mathcal{V}\}$ of Y where $\mathcal{V}$ is a collection of pairs (V', V) such that V and V' are open subsets of Y, $V \times V \subset W$,

$V' \subset V$, and any function $\eta_\partial : \{-1,1\} \to f^{-1}(V')$ has a continuous extension $\eta : [-1,1] \to f^{-1}(V)$. Consider a continuous $g : S^1 \to Y$. After sufficient subdivision S^1 is a simplicial complex, each of whose 1-simplices σ satisfies $g(\partial\sigma) \subset V'$ for some $(V',V) \in \mathcal{V}$. For each vertex v of this complex choose $\gamma_0(v) \in f^{-1}(g(v))$. As in the proof of Proposition 3, there is an extension $\gamma_1 : S^1 \to X$ of γ_0 such that $(g(p), f(\gamma(p))) \in W$ for all $p \in S^1$, so that $h(p,t) = s_Y((1-t)g(t) + tf(\gamma(p)))$ is a homotopy between g and $f\gamma$. Thus f_1 is surjective.

Since X is simply connected, it follows that Y is simply connected, so $\tilde{X}$, $\tilde{Y}$, and $\tilde{f}$ are (up to irrelevant formalities) just X, Y, and f. As we mentioned previously, since each fibre $f^{-1}(y)$ is contractible, it is acyclic, and $\tilde{X} = X$ is compact, so the dual Vietoris-Begle theorem of Volovikov-Ahn and Dydak implies that $\tilde{H}_n(\tilde{f})$ is an isomorphism for all $n \geq 2$, after which Proposition 2 implies that f is a homotopy equivalence. (The dual Vietoris-Begle theorem is specific to Alexander-Spanier homology, and Whitehead uses singular homology. However, it is well known that Alexander-Spanier homology agrees with Čech homology on compact Hausdorff spaces, and Čech and singular homology agree on ANR's [7, 12, 14].) □

3 The Proof of Theorem 3

Recall the setting of Theorem 3: L is a Hausdorff locally convex Riesz space, $C \subset L$ is compact, convex, and metrizable, $x \in L$, u_x is continuous, and $D = u_x(C)$. (We only prove the assertion for u_x because exactly the same argument proves the parallel assertion for d_x.) A translate of an ANR contained in L is an ANR, and a composition of a homotopy equivalence with a translation is a homotopy equivalence, so we may assume that $x = 0$. We write u in place of u_0.

We first show that D is a compact connected ANR. Of course D is compact and connected because it is the continuous image of a compact connected space.

The Urysohn metrization theorem [10, p. 125] asserts that a regular T_1 space is metrizable if it has a countable base. By assumption L is Hausdorff, hence T_1. As a topological vector space, L is regular [16, p. 16]. These properties are inherited by subspaces, so D is regular and T_1. Since C is compact and metrizable, it is easy to construct a countable base for it. Consider $y \in D$ and a neighborhood V of y. Since $u^{-1}(y)$ is compact, it is covered by a finite union of base sets for C that are contained in $u^{-1}(V)$. Lemma 1 gives an open $V' \subset V$ containing y such that $u^{-1}(V')$ is contained in this finite union. Since u is surjective, $u(u^{-1}(V')) = V'$, so the interior of the image of the finite union contains y, and of course it is contained in V. We have shown that the set of interiors of images of finite unions of base sets for C is a base for D, and of course this base is countable, so D is metrizable.

The following sufficient condition for a space to be an ANR is well known (e.g., [15, Prop. 8.3]).

Lemma 4. *If Z is a Hausdorff locally convex topological vector space, $K \subset Z$ is convex, $U \subset K$ is (relatively) open, $A \subset U$ is metrizable, and $r : U \to A$ is a retraction, then A is an ANR*[2]*.*

Standard results imply that, since D is metrizable, it can be embedded as a closed subset of a convex subset E of a Banach space (e.g., [15, Th. 6.3]) and Lemma 4 implies that D is an ANR if it is a retract of E. For each $w \in E \setminus D$ let B_w be the open ball in E centered at w whose radius is one half of the distance from w to D. Since metric spaces are paracompact, the open cover $\{B_w \cap E\}$ of $E \setminus D$ has a locally finite refinement $\mathcal{U}$. Let $\{\varphi_U\}_{U \in \mathcal{U}}$ be a partition of unity subordinate to $\mathcal{U}$. For each $U \in \mathcal{U}$ choose an $x_U \in C$ such that the distance from U to $u(x_U)$ is less than twice the distance from U to D, define $\rho : E \setminus D \to D$ by setting

$$\rho(z) = u\Big(\sum_U \varphi_U(z) x_U\Big),$$

and let $r : E \to D$ be the function that is the identity on D and ρ on $E \setminus D$. Evidently r is a retraction if we can show that it is continuous. Since D is closed in E, r is continuous at each point in $E \setminus D$. Consider a point $y \in D$ and a neighborhood $V \subset D$. We need to find a neighborhood $V'' \subset E$ of y such that $r(V'') \subset V$.

Let W be a convex neighborhood of $u^{-1}(y)$ that is contained in $u^{-1}(V)$. (To prove that such a W exists consider that, because L is locally convex, for each $x \in u^{-1}(y)$ there is a convex neighborhood A_x of the origin such that $(x + 2A_x) \cap C \subset u^{-1}(V)$. If $x_1 + A_{x_1}, \ldots, x_k + A_{x_k}$ is a finite subcover of $\{\, x + A_x : x \in u^{-1}(y) \,\}$ and $A = \bigcap_i A_{x_i}$, then $(u^{-1}(y) + A) \cap C \subset u^{-1}(V)$.) Lemma 1 gives a neighborhood $V' \subset V$ of y such that $u^{-1}(V') \subset W$. Let $\delta > 0$ be small enough that V' contains the ball of radius δ (in D) centered at y, and let V'' be the ball of radius $\delta/5$ (in E) centered at y. Of course $r(D \cap V'') = D \cap V'' \subset V' \subset V$.

Consider a point $z \in V'' \setminus D$, a $U \in \mathcal{U}$ that contains z, and a $w \in E \setminus D$ such that $U \subset B_w$. The radius of B_w is less than the distance from y to z, so B_w is contained in the ball of radius $2\delta/5$ centered at z. The distance from U to $u(x_U)$ is also less than twice the distance from y to z, so the distance from z to $u(x_U)$ is less than $4\delta/5$. Thus the distance from y to $u(x_U)$ is less than δ, so $u(x_U) \in V'$ and $x_U \in W$. This is true for all $U \in \mathcal{U}$ such that $z \in U$, so $\sum_U \varphi_U(z) x_U \in W$ and $\rho(z) \in u(W) \subset V$. This completes the proof that D is an ANR.

We now show that the fiber $u^{-1}(y)$ above each $y \in D$ is convex. Let $L_+ = \{\, x \in L : x \geq 0 \,\}$ be the lattice cone of L. The Riesz decomposition property asserts that if $x_0, x_1, w \in L_+$ and

[2] For the sake of self containment we include the proof. Suppose that X is a metric space, $e : A \to X$ maps A homeomorphically onto $e(A)$, which is closed. A generalization of the Tietze extension theorem due to Dugundji [6] implies that $e^{-1} : e(A) \to A$ has a continuous extension $j : X \to K$. (Dugundji's proof is a variant of our proof below that D is an ANR, and after reading that the reader may have little difficulty constructing the argument.) Then $V = j^{-1}(U)$ is a neighborhood of $e(A)$, and $e \circ r \circ j|_V : V \to e(A)$ is a retraction.

$w \leq x_0 + x_1$, then there are $w_0, w_1 \in L_+$ such that $w_0 \leq x_0$, $w_1 \leq x_1$, and $w_0 + w_1 = w$. (To prove this let $w_0 = w \wedge x_0$ and $w_1 = w - w_0$. Clearly $w_0, w_1 \geq 0$ and $w_0 + w_1 = w$. Finally $w_1 = w - w \wedge x_0 = 0 \vee (w - x_0) \leq 0 \vee x_1 = x_1$.) Suppose that $x_0, x_1 \in C$, $u(x_0) = y = u(x_1)$, and $0 < \alpha < 1$. Of course $y \geq 0$ and $y = (1-\alpha)y + \alpha y \geq (1-\alpha)x_0 + \alpha x_1$, so y is an upper bound of $\{0, (1-\alpha)x_0 + \alpha x_1\}$, and we need to show that it is the least upper bound. Suppose that $y \geq y' \geq 0$ and $y' \geq (1-\alpha)x_0 + \alpha x_1$. Then $0 \leq y - y' \leq (1-\alpha)(y - x_0) + \alpha(y - x_1)$, and the Riesz decomposition property gives $w_0, w_1 \in L_+$ such that $w_0 \leq (1-\alpha)(y - x_0)$, $w_1 \leq \alpha(y - x_1)$, and $w_0 + w_1 = y - y'$. We have $x_0 \leq y - w_0/(1-\alpha) \leq y - w_0$, and $y - w_0 \geq 0$ because $w_0 \leq w_0 + w_1 = y - y' \leq y$. Thus $y - w_0$ is an upper bound of $\{0, x_0\}$, which contradicts $x_0 \vee 0 = y$ if $w_0 \neq 0$, so $w_0 = 0$. Symmetrically, $w_1 = 0$, so $y' = y$.

Since C is convex, it is simply connected. Since L is locally convex and C and the fibers $u^{-1}(y)$ are convex and metrizable, Lemma 4 implies that they are ANR's, and of course they are AR's because they are convex. Finally D is an ANR. Thus $u|_C : C \to D$ satisfies the hypotheses of both Theorems 1 and 2, so $u|_C : C \to D$ is a homotopy equivalence.

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