Fixed Point Theorems

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Abstract

This entry gives statements of the Tarski fixed point theorem and the main versions of the topological fixed point principle that have been applied in economic theory. Pointers are given to literature concerned with proofs of Brouwer's theorem, and with algorithms for computing approximate fixed points. The topological results are all consequences of a slightly weakened version of the Eilenberg-Montgomery (1946) fixed point theorem. The axiomatic characterization of the Leray-Schauder fixed point index (which is even more powerful) is also stated, and its application to issues concerning robustness of sets of equilibria is explained.

fixed point theorems

The Brouwer (1910) fixed point theorem and its descendants are key mathematical results underlying the foundations of economic theory.

Let $f : X \to X$ be a function from a space to itself. A **fixed point** of f is a point $x^* \in X$ that is mapped to itself by f: $f(x^*) = x^*$. A **fixed point theorem** is a result asserting that, under some hypotheses, the set of fixed points of f is nonempty. A simple example with many applications is:

Theorem 1 (Contraction Mapping Theorem). *If the metric space (X, d) is complete (recall that this means that every Cauchy sequence is convergent) and there is a number $c \in (0, 1)$ such that $d(f(x), f(x')) \leq cd(x, x')$ for all $x, x' \in X$, then f has a unique fixed point.*

Another example illustrating the importance of the general notion of completeness, but otherwise based on quite different principles, is:

Theorem 2 (Tarski's Fixed Point Theorem). *Let $(X, \leq)$ be a complete lattice: $\leq$ is a partial ordering of X and every subset of X has a greatest lower bound and a least upper bound. If $f : X \to X$ is monotone—that is, $f(x) \leq f(x')$ whenever $x \leq x'$—then there are fixed points $\underline{u}, \overline{u} \in X$ such that $\underline{u} \leq x$ whenever $x \leq f(x)$ and $x \leq \overline{u}$ whenever $f(x) \leq x$.*

This result is foundational for the theory of strategic complementarities (e.g., Milgrom and Shannon (1994), Echenique (2005)) and has been applied to growth theory by Hopenhayn and Prescott (1992).

The rest of our discussion is devoted to results related to Brouwer's fixed point theorem. A topological space has the **fixed point property** if every continuous map from the space to itself has a fixed point. Brouwer's theorem states that a nonempty compact convex subset of a Euclidean space has the fixed point property. This celebrated result underlies many of the advanced results of topology, and was a pivotal event in the development of algebraic topology, which has influenced many areas of mathematics. In the half century following Brouwer's paper the theory of fixed points was extended in various directions, yielding several generalizations of Brouwer's result that are themselves famous theorems. Early in the postwar period fixed point theorems were used by Arrow and Debreu (1954), McKenzie (1959), Nash (1950, 1951), and Debreu (1952) to prove the fundamental equilibrium existence results of theoretical economics: every economy with finitely many goods and agents has a competitive equilibrium; every finite normal form game has a Nash equilibrium. Fixed point theory continues to

play an important role in the extensive body of research that grew out of these fundamental discoveries.

Useful books devoted to fixed point theory include Border (1985), which emphasizes results used in economic theory, Brown (1971), which develops the theory of the fixed point index using the methods of algebraic topology, and Dugundji and Granas (2003), which comprehensively surveys the topic from the point of view of applications to analysis and topology. The latter book features extensive historical information concerning the development, and the developers, of the subject.

Proofs and Algorithms

Since Brouwer's theorem is a breakthrough result, one should expect proofs to reveal deep mathematical principles, and in fact Brouwer's work was a major stimulus to the development of the subject that is now known as algebraic topology. Eventually Sperner (1928) distilled a relatively simple combinatoric argument out of the topological ferment of that era. Although this argument is the most popular in graduate education in economics, in the author's opinion the exposition in Milnor (1965) of an argument due to Hirsch is worth whatever additional effort it entails, because the student also learns Sard's theorem, which is another fundamental result of the last century with important applications in economic theory. Although the substance of the argument in Milnor (1978) appears to be less useful, its brevity and elementary character are stunning. The proof of McLennan and Tourky (2005) is also relatively simple, and displays how Kakutani's theorem follows easily from the existence of Nash equilibrium for a special class of two person games, which is one of the simplest manifestations of the fixed point principle.

Computation of approximate fixed points has many applications in economics and other fields, and is an important topic of research. Iteration of a function is only guaranteed to work when the the function is a contraction, as in Theorem 1, but this method is often practical for functions that do not satisfy this condition. Other methods are derived from proofs of Brouwer's theorem. The method pioneered by Scarf (Scarf (1973), Doup (1988)) is a method of moving through the simplices of a simplicial subdivision of the simplex. It is justified by a refinement of the proof of Sperner's lemma. The proof derived from Sard's theorem points toward homotopy methods, which have a huge literature (García and Zangwill (1981), Algower and Georg (1990)). The proof in McLennan and Tourky (2005) also points toward algorithms in which the equilibria of certain two person games give rise to approximate fixed points.

Variants

We will give statements of the main forms in which the fixed point principle is applied in economic theory. Let X and Y be metric spaces. A **correspondence** $F : X \to Y$ assigns a nonempty $F(x) \subset Y$ to each $x \in X$. When $Y = X$, a point x^* is said to be a **fixed point** if $x^* \in F(x^*)$. If P is any property of sets, then F is P **valued** if each image $F(x)$ has property P. It is **upper semicontinuous** (u.s.c.) if it is compact valued and, for each $x \in X$ and each neighborhood V of $F(x)$, there is a neighborhood U of x such that $F(x') \subset V$ for all $x' \in U$. It is not hard to show that if Y is compact, then F is u.s.c. if and only if its graph

$$\mathrm{Gr}(F) = \{\, (x, y) \in X \times Y : y \in F(x) \,\}$$

is closed. We think of a function as a singleton-valued correspondence, in which case upper semicontinuity coincides with the usual notion of continuity.

Economic models frequently give rise to sets of optimal individual choices that are convex, but may have more than one element. For this reason the most prominent fixed point theorem in economic applications is:

Theorem 3 (Kakutani (1941)). *If X is a nonempty compact convex subset of a Euclidean space and $F : X \to X$ is an u.s.c. convex valued correspondence, then F has a fixed point.*

The following variant is tailored for applications in general equilibrium theory, where one is searching for a price vector that equates supply and demand in all markets.

Theorem 4 (Debreu-Gale-Kuhn-Nikaido Lemma). *Let*

$$\Delta := \{\, p \in \mathbb{R}^n_+ : \sum_{j=1}^{n} p_i = 1 \,\}$$

be the $n - 1$ dimensional simplex. If $Z : \Delta \to \mathbb{R}^n$ is an u.s.c.c.v. correspondence satisfying $p \cdot z = 0$ for all $p \in \Delta$ and all $z \in Z(p)$, then there is a $p^* \in \Delta$ and $z^* \in Z(p^*)$ such that $z^* \leq 0$.

The following result of Shapley (1973b,a) (see also Herings (1997) and references cited therein) generalizes the famous K-K-M theorem of Knaster et al. (1929). It has important applications to the theory of the core and other aspects of cooperative game theory and general equilibrium theory.

Theorem 5 (K-K-M-S-Theorem). *Let $\mathcal{N} = 2^{\{1,\dots,n\}} \setminus \emptyset$, and for $S \in \mathcal{N}$ let $\Delta^S := \{\, x \in \Delta : x_i = 0 \text{ for all } i \notin S \,\}$. If $\{C^S\}_{S \in \mathcal{N}}$ is a collection of closed sets such that $\Delta^T \subset \bigcup_{S \subset T} C^S$ for all $T \in \mathcal{N}$, then there is $\mathcal{B} \subset \mathcal{N}$ and numbers $\lambda_S \geq 0$ for $S \in \mathcal{B}$ such that $\sum_{i \in S \in \mathcal{B}} \lambda_S = 1$ for all $i = 1, \dots, n$ (such a $\mathcal{B}$ is called a* **balanced collection***) and $\bigcap_{S \in \mathcal{B}} C^S \neq \emptyset$.*

The original K-K-M theorem is the special case in which $C^S = \emptyset$ whenever S has more than one element. That is, $C_1 \cap \cdots \cap C_n \neq \emptyset$ whenever $C_1, \dots, C_n \subset \Delta$ are closed sets satisfying $\Delta^T \subset \bigcup_{i \in T} C_i$ for all $T \in \mathcal{N}$.

Generalizations

During the first half of the last century there emerged a sequence of increasingly general versions of Brouwer's theorem. Let X and X' be metric spaces, and let $\varphi : X \to X'$ be a homeomorphism. A point $x^* \in X$ is a fixed point of a continuous function $f : X \to X$ if and only if $\varphi(x^*)$ is a fixed point of $\varphi \circ f \circ \varphi^{-1}$, so the fixed point property is invariant under homeomorphism. Compactness and continuity are invariant properties, but the assumptions of convexity and finite dimensionality in Brouwer's theorem seem too strong, as does the assumption of convex valuedness in Kakutani's theorem. One is led to search for weaker, topological assumptions that imply the fixed point property.

Let Y be another metric space. A continuous function

$$h : X \times [0, 1] \to Y$$

is called a **homotopy**. For each $0 \leq t \leq 1$ let $h_t = h(\cdot, t) : X \to Y$. We think of "continuously deforming" h_0 into h_1, with the variable t representing time, and we say that h_0 and h_1 are **homotopic**. The space X is **contractible** if the identity function on X is homotopic to a constant function. If X is convex, then for any $x_0 \in X$ the function

$$h(x, t) = x_0 + (1 - t)(x - x_0)$$

is such a homotopy, so convex sets are contractible. It was conjectured that nonempty compact contractible sets have the fixed point property, but eventually counterexamples were discovered by Kinoshita (1953) and others.

A **retraction** of X onto a subset A is a continuous function $r : X \to A$ whose set of fixed points is A, so that $r(a) = a$ for all $a \in A$. In this circumstance we say that A is a **retract** of X. One point of interest is that if X has the fixed point property, then so does A: if $g : A \to A$ is continuous,

then $g \circ r : X \to A \subset X$ has a fixed point x^*, and $x^* = g(r(x^*)) = g(x^*)$ because x^* must be in A.

The subspace A is a **neighborhood retract** if there is an open $U \supset A$ and a retraction $r : U \to A$. A continuous function $e : X \to Y$ is an **embedding** if it is injective and $e^{-1} : e(X) \to X$ is continuous, i.e., e is a homeomorphism onto its image. A metric space X is an **absolute neighborhood retract** (ANR) if $e(X)$ is a neighborhood retract whenever $e : X \to Y$ is an embedding of X in a metric space Y. The class of ANR's is large, encompassing many important types of spaces such as manifolds, simplicial complexes, and convex sets, and there is an extensive theory (e.g., Borsuk (1967)) that cannot be described here. One may think of an ANR as a space that has bounded complexity, in a certain sense, in a neighborhood of each of its points[1].

Eilenberg and Montgomery (1946) gave a fully satisfactory generalization of Brouwer's theorem: F has a fixed point whenever X is a nonempty compact acyclic ANR and $F : X \to X$ is an u.s.c. acyclic valued correspondence. Acyclicity is a concept from algebraic topology that cannot be defined here; the important point for us is that contractible sets are acyclic, and that the loss of generality in passing from acyclicity to contractibility is of slight concern in economic theory.

Contractible valued correspondences that are not convex valued appear in McLennan (1989a) and Reny (2005). There are many applications in economics of the special case of the Eilenberg-Montgomery theorem in which X is convex (but possibly infinite dimensional) and F is convex valued, for which relatively simple and direct proofs were given by Fan (1952) and Glicksberg (1952). In turn this result is more general than both Kakutani's theorem and the well known Schauder (1930) fixed point theorem.

The Leray-Schauder Fixed Point Index

Consider the fixed points of the function from $[0, 1]$ to itself shown in Figure 1. The points A and C are qualitatively similar, and qualitatively different from B. In the one dimensional setting one can easily see that if the function is differentiable, and its graph is not tangent to the diagonal at any of its fixed points, then the number of fixed points of the first type must

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[1] An example of a space that is *not* an ANR is the union X of the unit circle centered at the origin in $\mathbb{R}^2$ and the set $\{ (1 - \theta^{-1})(\cos\theta, \sin\theta) : 1 \leq \theta < \infty \}$. If X was an ANR, then there would exist a retraction of a neighborhood $U \subset \mathbb{R}^2$ onto X, and the retraction would take small connected neighborhoods of $(1, 0)$ in U to small connected neighborhoods of $(1, 0)$ in X, but small neighborhoods of $(1, 0)$ in X are disconnected.
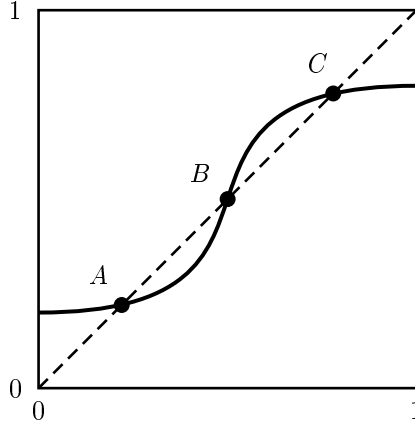
Figure 1

be one greater than the number of fixed points of the second type. In particular, the number of fixed points must be odd. These properties extend to smooth functions $f : C \to C$, where C is an n-dimensional convex set, that intersect the diagonal in the "expected" manner: the Jacobian of $\mathrm{Id}_C - f$ is nonsingular. Debreu (1970) used Sard's theorem (e.g., Milnor (1965)) to show that for an exchange economy with fixed preferences, the excess demand function generated by a "generic" endowment vector has well behaved equilibria, and Dierker (1972) showed that the qualitative conclusions described above hold in this circumstance. Mas-Colell (1985) summarizes the extensive literature descended from these seminal contributions.

The **Leray-Schauder fixed point index** generalizes these aspects of the theory to correspondences, to sets of fixed points that are not singletons, and to general ANR's. Suppose X is a nonempty compact ANR, $U \subset X$ is open and $\overline{U}$ is its closure. A correspondence $F : \overline{U} \to X$ is **index admissible** if it is u.s.c. and does not have any fixed points in its boundary $\overline{U} \setminus U$. Let $\mathcal{I}_X$ be the set of index admissible contractible valued correspondences $F : \overline{U} \to X$ where $U \subset X$ is open. A homotopy $h : \overline{U} \times [0,1] \to X$ is **index admissible** if each h_t is index admissible.

The next result gives an axiomatic characterization of a number $\Lambda_X(F)$. When there are finitely many fixed points the Additivity axiom allows us to think of $\Lambda_X(F)$ as the sum of their indices. When $X \subset \mathbb{R}^n$, $f : \overline{U} \to X$ is a smooth function, and x is a fixed point in the interior of X with $\mathrm{Id}_{\mathbb{R}^n} - Df(x)$ nonsingular, the index of x is $+1$ or -1 according to whether the determinant of $\mathrm{Id}_{\mathbb{R}^n} - Df(x)$ is positive or negative.

Theorem 6. *There is a unique function* $\Lambda_X : \mathcal{I}_X \to \mathbb{Z}$ *satisfying:*

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(I1) (Normalization) If $c : X \to X$ is a constant function, then $\Lambda_X(c) = 1$.

(I2) (Additivity) If $F : \overline{U} \to X$ is in $\mathcal{I}_X$, $U_1, \ldots, U_r$ are disjoint open subsets of U, and F has no fixed points in $\overline{U} \setminus (U_1 \cup \ldots \cup U_r)$, then

$$\Lambda_X(F) = \sum_{i=1}^{r} \Lambda_X(F|_{\overline{U}_i}).$$

(I3) (Homotopy) If $h : \overline{U} \times [0, 1] \to X$ is an index admissible homotopy, then $\Lambda_X(h_0) = \Lambda_X(h_1)$.

(I4) (Continuity) For each $F : \overline{U} \to X$ in $\mathcal{I}_X$ there is a neighborhood $W \subset \overline{U} \times X$ of $Gr(F)$ such that $\Lambda_X(F') = \Lambda_X(F)$ for all $F' : \overline{U} \to X$ with $F' \in \mathcal{I}_X$ and $Gr(F') \subset W$.

The index is closed related to the Brouwer degree of a function between manifolds of the same dimension. These ideas evolved from the time of Brouwer's work until O'Neill (1953) achieved the axiomatic expression of the concept (for functions) given above.

Theorem 1 has many important consequences. To begin with note that if $F \in \mathcal{I}_X$ has no fixed points, then Additivity implies that

$$\Lambda_X(F) = \Lambda_X(F|_\emptyset) = \Lambda_X(F|_\emptyset) + \Lambda_X(F|_\emptyset) = 0.$$

Therefore F must have a fixed point whenever $\Lambda_X(F) \neq 0$. If $f : X \to X$ is a continuous function, then $\Lambda_X(f)$ is called the **Lefschetz number** of f. The famous Lefschetz (1923) fixed point theorem states that f has a fixed point if its Lefschetz number is nonzero, and provides connections to algebraic topology that give tools for computing the Lefschetz number.

We now use the following approximation result to recover the weak version of the Eilenberg-Montgomery theorem stated above, thereby showing that Theorem 6 embodies the fixed point principle. This result generalizes Kakutani's method of passing from Brouwer's theorem to his result, and it plays an important role in one method of proving Theorem 6.

Theorem 7 (Mas-Colell (1974), McLennan (1989b)). *If X is a compact ANR, $U, V \subset X$ are open with $\overline{V} \subset U$, $F : \overline{U} \to X$ is an u.s.c. contractible valued correspondence, and $W \subset \overline{U} \times X$ is a neighborhood of $Gr(F)$, then there is a continuous function $f : \overline{V} \to X$ with $Gr(f) \subset W$.*

Suppose that $F : X \to X$ is an u.s.c. contractible valued correspondence. Applying the last result with $U = V = X$ and W as in (I4), we find that

there is a continuous function $f : X \to X$ with $\Lambda_X(f) = \Lambda_X(F)$. If X is contractible, so that there is a homotopy $h : X \times [0,1] \to X$ with $h_0 = \mathrm{Id}_X$ and h_1 a constant function, then $j(x,t) = f(h(x,t))$ is a homotopy with $j_0 = f$ and j_1 a constant function, so Homotopy and Normalization imply that $\Lambda_X(f) = 1$. We conclude that $\Lambda_X(F) = 1$, and that F necessarily has a fixed point.

Recall that a subset C of a metric space Y is **connected** if there do not exist open sets $V_1, V_2 \subset Y$ with $V_1 \cap V_2 = \emptyset$ and $V_1 \cap C \neq \emptyset \neq V_2 \cap C$. A subset of Y is a **connected component** if it is the union of all connected sets containing some point y. Each connected component is connected, and the connected components partition Y.

Suppose that X is a compact contractible ANR, that $F : X \to X$ is in $\mathcal{I}_X$, and that the set of fixed points of F has finitely many connected components $C_1, \ldots, C_r$. Additivity implies that each component C_i has a well defined index λ_i that depends on the restriction of F to an arbitrarily small neighborhood of C_i. Suppose that it is possible to show that $\lambda_i = 1$ for each i. Since additivity implies that $\sum_i \lambda_i = \Lambda_X(F) = 1$, it follows that $r = 1$. This style of proof of uniqueness is applicable to many economic settings, but usually more elementary methods are available. At present no alternative to its application in Eraslan and McLennan (2005) is known. It is more common to use the index to prove nonuniqueness: it suffices to display a connected component whose index is different from one.

The fixed point index has two other important properties.

Theorem 8. *(Multiplication) If X and Y and compact ANR's, $U \subset X$ and $V \subset Y$ are open, $F : \overline{U} \to X$ and $G : \overline{V} \to Y$ are index admissible contractible valued correspondences, and $F \times G : \overline{U} \times \overline{V} \to X \times Y$ is the correspondence that takes (x, y) to $F(x) \times G(y)$, then*

$$\Lambda_{X \times Y}(F \times G) = \Lambda_X(F) \cdot \Lambda_Y(G).$$

Theorem 9. *(Commutativity) If X and Y are compact ANR's and $f : X \to Y$ and $g : Y \to X$ are continuous functions, then*

$$\Lambda_X(g \circ f) = \Lambda_Y(f \circ g).$$

There is a more general version of Commutativity for functions defined on subsets of X and Y, but its statement involves technical complications. In view of the uniqueness asserted in Theorem 6, Multiplication and Commutativity are, in principle, consequences of (I1)-(I4), but it is not known how to prove them in this way. In practice these properties are treated

1

B

A

0

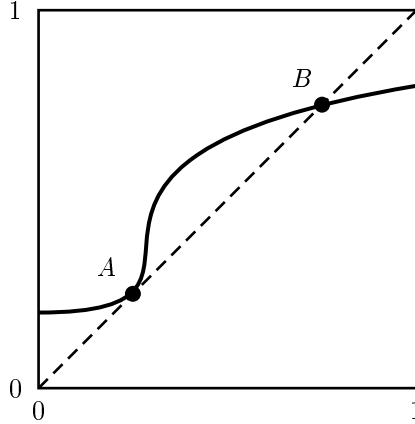0   1

Figure 2

as axioms and shepherded up the ladder of generality, one rung at a time, along with everything else. In fact Commutativity (which was introduced by Browder (1948) for this purpose) plays a critical role at one stage of this process.

Essential Sets of Fixed Points

The two fixed points in Figure 2 are qualitatively different. Arbitrarily small perturbations of the function have no fixed point near A, but this is not the case for B. In the terminology introduced by Fort (1950) A is **inessential** while B is **essential**. Let X be a compact contractible ANR, let $F : X \to X$ be an u.s.c. contractible valued correspondence, and let C be the set of fixed points of F. Kinoshita (1952) extended Fort's ideas to correspondences, and to sets of fixed points, defining an **essential set of fixed points** of F to be a compact $C' \subset C$ such that for any neighborhood U of C' there is a neighborhood W of $\mathrm{Gr}(F)$ such that any continuous function $f : X \to X$ with $\mathrm{Gr}(f) \subset W$ has a fixed point in U.

For any neighborhood U of C we can find a neighborhood W of $\mathrm{Gr}(F)$ that cannot have any fixed points outside of U, so C is essential. That is, without some additional condition, essentiality does not distinguish some fixed points from others. Following Kohlberg and Mertens (1986), one is led to consider minimal essential sets, which exist by virtue of the following argument. Let $B_1, B_2, \ldots$ be a listing of the open balls of rational radii centered at points in some countable dense subset of X. Define a sequence $K_0, K_1, K_2, \ldots$ inductively by setting $K_0 = C$ and, for $j \geq 1$, setting $K_j = K_{j-1} \setminus B_j$ if this set is essential and otherwise setting $K_j = K_{j-1}$. We claim that $K_\infty = \bigcap_j K_j$ is a minimal essential set. Any neighborhood

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U of K_∞ contains some K_j (the accumulation points of a sequence $\{x_j\}$ with $x_j \in K_j \setminus U$ must be outside U but also in each K_j, by compactness, hence in K_∞) and each K_j is essential, so K_∞ is essential. If there was a smaller essential set there would be some j such that $K_\infty \setminus B_j \neq K_\infty$ was essential, but then $K_{j-1} \setminus B_j$ would also be essential, in which case $K_\infty \cap B_j \subset K_j \cap B_j = \emptyset$.

Kinoshita (1952) showed that minimal essential sets are connected when X is convex and F is convex valued. Otherwise one could find a minimal essential set $C_1 \cup C_2$, where C_1 and C_2 are nonempty, compact, and disjoint. Then C_1 and C_2 are inessential, so there is a perturbation of F that has no fixed points near C_1 and another such perturbation of F has no fixed points near C_2. The main idea of Kinoshita's argument is that these can be combined, using convex combination with locally varying weights, to give a perturbation of F that has no fixed point near $C_1 \cup C_2$, thereby contradicting the assumption that $C_1 \cup C_2$ is essential.

Kinoshita's theorem is pertinent to the literature on refinements of Nash equilibrium that began with the introduction in Selten (1975) of perfect equilibrium. An important technique is to give a privileged status to those Nash equilibria that can be approximated by fixed points of certain perturbations of the given correspondence. In particular, it has important connections to the notion of strategic stability of Kohlberg and Mertens (1986).

The fixed point index also has implications for essential sets. For the sake of simplicity assume that C consists of finitely many connected components $C_1, \ldots, C_r$. (This condition holds in the application to Nash equilibrium.) Any C_i with nonzero index is essential, by Continuity. Since the sum of the indices is one, some C_i must have nonzero index, so a connected essential set exists. Harder arguments, which apply the Hopf theorem (Milnor (1965)) to "transport" fixed points of perturbations to a desired location, and to eliminate pairs of fixed points of opposite index, show that any proper subset of a C_i is inessential, and that C_i is inessential if its index is zero. Thus the minimal essential sets are precisely those C_i with nonzero index.

References

E. Algower and K. Georg. *Numerical Continuation Methods*. Springer Verlag, New York, 1990.

K.J. Arrow and G. Debreu. Existence of an equilibrium for a competitive economy. *Econometrica*, 22:265–290, 1954.

K. Border. *Fixed Point Theorems with Applications to Economics and Game Theory*. Cambridge University Press, Cambridge, 1985.

K. Borsuk. *Theory of Retracts*. Polish Scientific Publishers, Warsaw, 1967.

L.E.J. Brouwer. Uber abbildung von mannigfaltikeiten. *Mathematische Annalen*, 71:97–115, 1910.

F.E. Browder. *The Topological Fixed Point Theory and its Applications to Functional Analysis*. PhD thesis, Princeton University, 1948.

R.F. Brown. *The Lefschetz Fixed Point Theorem*. Scott Foresman and Co., Glenview, IL, 1971.

G. Debreu. A social equilibrium existence theorem. *Proc. Nat. Acad. Sci.*, 38:886–893, 1952.

G. Debreu. Economies with a finite set of equilibria. *Econometrica*, 38:387–392, 1970.

E. Dierker. Two remarks on the number of equilibria of an economy. *Econometrica*, 40:951–953, 1972.

T. Doup. *Simplicial Algorithms on the Simplotope*. Springer-Verlag, Berlin, 1988.

J. Dugundji and A. Granas. *Fixed Point Theory*. Springer-Verlag, New York, 2003.

F. Echenique. A short and constructive proof of Tarski's fixed point theorem. *International Journal of Game Theory*, 33:215–218, 2005.

S. Eilenberg and D. Montgomery. Fixed-point theorems for multivalued transformations. *American Journal of Mathematics*, 68:214–222, 1946.

H. Eraslan and A. McLennan. Uniqueness of stationary equilibrium payoffs in coalitional bargaining. Mimeo, University of Pennsylvania, 2005.

K. Fan. Fixed point and minimax theorems in locally convex linear spaces. *Proceedings of the National Academy of Sciences*, 38:121–126, 1952.

M.K. Fort. Essential and nonessential fixed points. *American J. of Mathematics*, 72:315–322, 1950.

C. B. García and W. I. Zangwill. *Pathways to Solutions, Fixed Points, and Equilibria*. Prentice-Hall, EnglewoodCliffs, NJ, 1981.

I.L. Glicksberg. A further generalization of the Kakutani fixed point theorem with applications to Nash equilibrium. *Proceedings of the American Mathematical Society*, 3:170–174, 1952.

P. J.-J. Herings. An extremely simple proof of the K-K-M-S theorem. *Economic Theory*, 10:361–367, 1997.

H. Hopenhayn and E. Prescott. Stochastic monotonicity and stationary distributions for dynamic economies. *Econometrica*, 60:1387–1406, 1992.

S. Kakutani. A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal*, 8:416–427, 1941.

S. Kinoshita. On essential components of the set of fixed points. *Osaka Mathematical Journal*, 4:19–22, 1952.

S. Kinoshita. On some contractible continua without the fixed point property. *Fundamentae Mathematicae*, 40:96–98, 1953.

B. Knaster, C. Kuratowski, and C. Mazurkiewicz. Ein beweis des fixpunktsatzes fur n-dimensionale simplexe. *Fundamenta Mathematicae*, 14:132–137, 1929.

E. Kohlberg and J.-F. Mertens. On the strategic stability of equilibria. *Econometrica*, 54:1003–1038, 1986.

S. Lefschetz. Continuous transformations of manifolds. *Proceedings of the National Academy of Sciences*, 9:90–93, 1923.

A. Mas-Colell. A note on a theorem of F. Browder. *Mathematical Programming*, 6:229–233, 1974.

A. Mas-Colell. *The Theory of General Economic Equilibrium: A Differentiable Approach*. Cambridge University Press, Cambridge, 1985.

L. W. McKenzie. On the existence of general equilibrium for a competitive market. *Econometrica*, 27:54–71, 1959.

A. McLennan. Consistent conditional systems in noncooperative game theory. *International Journal of Game Theory*, 18:141–174, 1989a.

A. McLennan. Fixed points of contractible valued correspondences. *International Journal of Game Theory*, 18:175–184, 1989b.

A. McLennan and R. Tourky. From imitation games to Kakutani. Mimeo, University of Minnesota, 2005.

P. Milgrom and C. Shannon. Monotone comparative statics. *Econometrica*, 62:157–180, 1994.

J. Milnor. *Topology from the Differentiable Viewpoint*. University Press of Virginia, Charlottesville, 1965.

J. Milnor. Analytic proofs of the "hairy ball theorem" and the Brouwer fixed-point theorem. *The American Mathematical Monthly*, 85(7):521–524, 1978.

J. Nash. *Non-cooperative Games*. PhD thesis, Mathematics Department, Princeton University, 1950.

J. Nash. Non-cooperative games. *Annals of Mathematics*, 54:286–295, 1951.

B. O'Neill. Essential sets and fixed points. *American J. of Mathematics*, 75: 497–509, 1953.

P. Reny. On the existence of monotone pure strategy equilibria in Bayesian games. Mimeo, University of Chicago, 2005.

H. E. Scarf. *The Computation of Economic Equilibria*. Yale University Press, New Haven, 1973.

J. Schauder. Der fixpunktsatz in funktionalräumen. *Studia Mathematica*, 2: 171–180, 1930.

R. Selten. Re-examination of the perfectness concept for equilibrium points of extensive games. *International J. of Game Theory*, 4:25–55, 1975.

L. S. Shapley. On balanced games without side payments—a correction. The Rand Paper Series Report No. p-4910/1, 1973a.

L.S. Shapley. On balanced games without side payments. In T.C. Hu and S.M. Robinson, editors, *Mathematical Programming Study*, pages 175–189. Academic Press, New York, 1973b.

E. Sperner. Neuer beweis für die invarianz der dimensionszahl und des gebietes. *Abh. Math. Sem. Univ. Hamburg*, 6:265–272, 1928.