The Index +1 Principle

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Abstract

In the simplest (generic) case the fixed point index assigns an index of +1 or -1 to each fixed point of a function or correspondence, and these indices sum to +1. If an isolated equilibrium is stable with respect to *any* process of adjustment to equilibrium that is "natural," in the sense that the agents adjust their strategies in directions that increase utility, or prices adjust in the general direction of excess demand, then the index of the equilibrium is +1. The *index +1 principle* asserts that consequently only index +1 equilibria are empirically relevant; we argue that this should be regarded as a fundamental principle of economic analysis. Since the fixed point index has a general axiomatic characterization, the index +1 principle is universally applicable to economic models in which equilibria are topological fixed points. It does not depend on insignificant details of model specification. The index +1 principle itself, and the hypothesis that processes of adjustment to equilibrium are natural, are strongly supported by experimental evidence. The index +1 principle can be seen as the multidimensional extension of Samuelson's correspondence principle.

Keywords: Fixed point index, refinements of equilibrium, vector field index, dynamical system, asymptotic stability, adjustment to equilibrium, Lyapunov function, converse Lyapunov theorem, experimental economics, Paul Samuelson, correspondence principle.

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1 Introduction

Economists generally reject equilibria of games and markets that are manifestly unstable. For example, an unstable equilibrium of a market for a single good, or the mixed equilibrium of a battle of the sexes, is thought to be impossible as an outcome that is *self-reproducing*, in the sense that it is expected to occur, and then does occur, repeatedly. However, the meaning of instability quickly becomes vague as the dimension of the space of endogenous variables increases. In physics and chemistry the notion of instability is clearly understood because the relevant theory gives a specific dynamic process of adjustment to equilibrium. In contrast, a market or a game cannot have a process of gradual adjustment to equilibrium that is well understood by the agents in the model because instead of conforming to the process, they would try to exploit it. Although economists may agree that unstable equilibria will not be observed, what this means in general is far from clear. For example, after proposing that dynamic stability (along with consistency (e.g., supply equals demand) and rational behavior of individuals) should be regarded as one of three basic properties of equilibrium in general, Dixon (1990) goes on to remark that "stability is often played down, since it is almost impossible to provide a coherent account of stability in economics."

This paper develops a simple, general, qualitative consequence of stability that rules out many equilibria because they are unstable for *any* plausible process of adjustment to equilibrium. As a refinement of equilibrium, it shares mathematical features with the various refinements of Nash equilibrium that have been widely discussed, and is (roughly speaking) as restrictive as some of the more powerful game theoretic refinements such as strategic stability. However, because it is rooted in dynamic stability, rather than intuition and reasoning concerning the nature of rationality, its motivation is more compelling and less speculative. In addition to being in accord with intuition, it is strongly supported by casual empiricism and existing experimental research.

The equilibria of an exchange economy, and the equilibria of a game, can be understood as the fixed points of an upper hemicontinuous convex valued correspondence. The *fixed point index* of a regular fixed point of a smooth function f is +1 or -1 according to the sign of the determinant of the matrix of partial derivatives of $\mathrm{Id} - f$. (A fixed point is *regular*, by definition, if this determinant does not vanish.) If all equilibria are regular (this case is generic for many models) the sum of the indices is +1.

The main thrust of this paper is that for the central models of economic theory, namely game theory and general equilibrium, and almost certainly many other models as well, the isolated equilibria with index -1 will not be self-reproducing outcomes because they cannot be dynamically stable with respect to *any* economically natural process of adjustment to equilibrium. We describe this as the *index +1 principle*. It should be thought of as a conjunction of two propositions:

(a) An isolated equilibrium will not be observed as a self-reproducing outcome if it is not stable with respect to *some* dynamic process of adjustment to equilibrium that is "natural."

(b) If an isolated equilibrium is stable with respect to a natural process of adjustment to equilibrium, then its index is $+1$.

The meaning of "natural" depends on the model. For game theory, a natural model will be one in which each agent adjusts her mixed strategy in a direction that is weakly utility improving, given the current mixed strategies of the other agents, and a mixed strategy profile is a rest point of the process if and only if it is a Nash equilibrium. A price adjustment process for an exchange economy will be natural if prices always adjust in a direction whose inner product with current excess demand is nonnegative[1] and a price vector is a rest point of the process if and only if it is a Walrasian equilibrium. For game theory and general equilibrium theory the only model of adjustment to equilibrium satisfying the rational expectations hypothesis is that equilibrium is reached immediately, so other models of adjustment, and reasoning based on their properties, necessarily have some behavioral aspect. In this sense (a) is an *hypothesis* rather than a logical consequence of other economic principles. As such, it can and should be assessed against experience and available evidence.

In contrast, (b) is a mathematical assertion that has been proved by Demichelis and Ritzberger (2003) for game theory and is proved herein for general equilibrium. The method of proof is expected to be generally applicable to models in which equilibria are topological fixed points.

We will argue that the index $+1$ principle should be regarded as a fundamental principle of economic analysis, but insofar as the phrase 'fundamental principle' has no definite, objective meaning, this assertion can only be regarded as a shorthand for a collection of propositions that lend substance to this view. In the remainder of this section we describe them briefly, and sketch how they will be developed in the remainder of the paper.

However, before taking up this task it is important to stress a major caveat: by a "principle" we do not mean a "law of economics" that holds without exception. The

[1] For the results of Section 3 it actually suffices to assume that the direction of adjustment is never a negative scalar multiple of the gradient of expected utility, in the case of games, or excess demand, in the case of markets.

validity of the index +1 principle relative to any particular model depends on finding a class of processes of adjustment to equilibrium that is compelling, in the sense that processes of adjustment outside this class are implausible, and which allows the logic of (b) to go through. The index +1 principle provides guidance concerning what one should typically expect, and patterns of reasoning that can be used to assess its validity in relation to particular models. Such a principle can be useful, illuminating, and powerful, even if there are certain examples for which the underlying chain of reasoning fails.

A fundamental principle should be widely, if not universally, applicable. In Section 2 we present the axiom system for the fixed point index, and the theorem asserting the unique existence of an index satisfying these axioms. This theorem encompasses the Eilenberg-Montgomery fixed point theorem (Eilenberg and Montgomery (1946)) which extends the Kakutani (1941) fixed point theorem to correspondences that are contractible valued, and the Lefschetz fixed point theorem, which extends fixed point theory to domains that are not contractible, hence nonconvex. Thus the index +1 principle can be applied to the most fundamental models of economic theory, namely general equilibrium theory and game theory, as well as all other models in which equilibrium can be expressed as a topological fixed point. This includes infinite dimensional models, and models for which the function or correspondence satisfies only minimal continuity conditions, and is not in any sense smooth. As we will explain in more detail, the sequential equilibrium concept of Kreps and Wilson (1982) can be put into this framework.

Roughly, the index assigns an integer to each set of fixed points that is *clopen* (both closed and open) in the relative topology of the set of fixed points. The Euler characteristic of a sufficiently well behaved compact set is the index of that set as the set of fixed points of that set's identity function. The Euler characteristic of a singleton is +1. The general expression of (b) above is that if a sufficiently well behaved clopen set of fixed points is stable with respect to natural dynamics, then its index is equal to its Euler characteristic. (In this sense the phrase "index +1 principle" is slightly misleading, but "index equals Euler characteristic principle" was rejected as insufficiently pithy.)

As the various mathematical arguments below illustrate, the index axioms constitute a powerful analytic framework which frequently yields simple, direct proofs. For example, although the determinant test alluded to above could be used to compute the indices of most of the equilibria of the examples considered in the body of the paper, in almost all cases it is easier to use reasoning based on the axioms. Independent of the merits of the index +1 principle, this axiom system should be better known in economics.

In Section 3 we describe two results that show that (b) above holds for the two central models of economic theory, namely game theory and general equilibrium theory. The first, due to Demichelis and Ritzberger (2003), asserts that if a connected component of the set of Nash equilibria of a game is stable with respect to a dynamic process of adjustment to equilibrium that is natural, in the sense described above, then the component's index is equal to its Euler characteristic. A variant of their argument is used to establish a similar result for connected components of the set of equilibrium prices of an exchange economy. The style of reasoning seen in the proofs should be applicable to a wide variety of models and notions of natural adjustment processes. Section 4 also describes a game studied by Balkenborg and Vermeulen (2016) that starkly illustrates conflicts between dynamic stability and various notions of strategic stability.

A fundamental principle should not be unduly sensitive to minor details of a model's formulation and presentation. A game-theoretic model of a social interaction necessarily excludes a host of pure strategies that are available in reality. Often the excluded strategies are manifestly irrational, and it is, perhaps, almost an automatic reflex to pass from this fact to the assumption that the predictions of the model will not be affected by whether the strategy is available. However, Ben-Porath and Dekel (1992) point out that certain concepts in the literature on refinements of Nash equilibrium are sensitive to the inclusion of possibilities for self-destructive behavior. In Section 4 we show that the index of a set of Nash equilibria does not depend on the inclusion or exclusion of pure strategies that are not best responses at any equilibrium in the set, so the index is insensitive to the inclusion or exclusion of strictly dominated pure strategies. We also present another result that establishes that the index +1 principle is insensitive to the details of the model's presentation: if a simple game (the social scientist's model) is well approximated by a more complex game (reality) then the index of any set of Nash equilibria of the simple game is the same as the index of the set of Nash equilibria of the complex game that lie above this set.

A fundamental principle should be reliable. Section 5 assesses the extent to which the index +1 is supported by empirical evidence. There does not seem to be an obvious violation of the principle in anecdotal or historical experience. Although it has not been the primary subject of any experimental study, there have been a fairly large number of studies of games and markets with multiple equilibria, and these studies provide substantial support. In addition, there is a fairly large literature studying adjustment to equilibrium in games and markets. We describe the models that have been proposed, and the experimental results that have been obtained, finding that on the whole they provide strong support for (a) above.

In Section 6 we consider the index +1 principle in relation to Samuelson's correspondence principle. We decompose a prototypical example of the correspondence principle, as Samuelson presented it concretely, into three steps, the first two of which are the 1-dimensional index +1 principle. In this sense the index +1 principle can be understood as having the correspondence principle as its 1-dimensional special case. The third step is the derivation of qualitative results for comparative statics, which quickly becomes problematic as the dimension increases. Nowadays economic praxis attaches relatively minor significance to consequences of the index +1 principle for comparative statics, but we enjoy a much richer array of theoretical models and empirical strategies, for which the index +1 principle has diverse consequences.

Section 7 summarizes and concludes.

2 The Fixed Point Index

In this section we present the axiomatic characterization of the fixed point index, emphasizing that it is applicable to domains of arbitrary (even infinite) dimension and to functions and correspondences that need not satisfy any analogue of smoothness or regularity, and which can have infinite sets of fixed points.

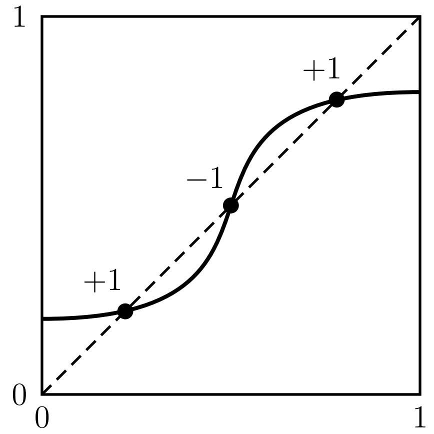

FIGURE 1. INDICES OF FIXED POINTS OF A UNIVARIATE FUNCTION

The key ideas are already illustrated by Figure 1, which shows the graph of a differentiable function $f : [0,1] \to [0,1]$ that has three fixed points. At two of these the graph of f goes from above the diagonal to below as we go from left two right. Each of these has index +1. At the third fixed point the graph of f goes from below the diagonal to above, and this fixed point has index -1. Since each of the fixed points of f is regular, in the sense that the derivative of $\mathrm{Id}_{[0,1]} - f$ does not vanish there, we can think of the index of a fixed point as the sign of the derivative of $\mathrm{Id}_{[0,1]} - f$ at that point. (In general Id_X denotes the identity function of the set X.) Note that the sum of the indices is

$+1$ for all C^1 functions whose fixed points are all regular. As we will see below, these phenomena are completely general.

The most general setting for the fixed point index depends on certain advanced topological concepts. If X and Y are sets, a *set valued function* $F : X \to Y$ assigns a set $F(x) \subset Y$ to each $x \in X$. (A function $f : X \to Y$ will be identified with the set valued function $x \mapsto \{f(x)\}$.) The *graph* of F is

$$\mathrm{Gr}(F) = \{\, (x, y) \in X \times Y : y \in F(x) \,\}.$$

If $C \subset X$ and $F : C \to X$ is a set valued function, the set of *fixed points* of F is

$$\mathcal{F}(F) = \{\, x \in C : x \in F(x) \,\}.$$

A set valued function $F : X \to Y$ is *compact* (convex, etc.) *valued* if each $F(x)$ is compact (convex, etc.). A *correspondence* is a nonempty valued set valued function. If X and Y are topological spaces, a correspondence $F : X \to Y$ is *upper hemicontinuous* if it is compact valued and, for each $x \in C$ and each neighborhood V of $F(x)$, there is a neighborhood $U \subset C$ of x such that $F(x') \subset V$ for all $x' \in U$.

A topological space X is *contractible* if the identity function is homotopic to a constant function. That is, there is a continuous function $c : X \times [0, 1] \to X$ such that $c(\cdot, 0) = \mathrm{Id}_X$ and $c(\cdot, 1)$ is a constant function. Any star shaped subset of a topological vector space is contractible, and in particular convex sets are contractible. The circle is an example of a space that is not contractible. (Although this is intuitive, it is not easy to prove.)

If Z is a topological space and $X \subset Z$, a *retraction of Z onto X* is a continuous function $r : Z \to X$ such that $r(x) = x$ for all $x \in X$. We say that X is a *retract* of Z. A topological space X is an *absolute retract* (AR) if, whenever Z is a metric space and (a homeomorphic image of) X is a closed subspace, X is a retract of Z. A topological space X is an *absolute neighborhood retract* (ANR) if, whenever Z is a metric space and (a homeomorphic image of) X is a closed subspace, X is a retract of some neighborhood $U \subset Z$ of X.

Basic characterization results give concrete intuition concerning these classes of spaces. A retract of a convex subset of a locally convex topological vector space is an AR, and any AR has a homeomorphic image of this sort. (E.g., Section 7.5 of McLennan (2016).) Similarly, a retract of an open subset of a convex subset of a locally convex topological vector space is an ANR, and any ANR has such a homeomorphic image. (E.g., Section 7.4 of McLennan (2016).) An important intuition is that each point of an ANR has a neighborhood that retains some of the simplicity of the neighborhood in the topological vector space. Using this result, one can show that manifolds, finite simplicial complexes, and convex subsets of normed linear spaces are ANR's, so from the point of view of economic modelling this is a quite general class of spaces. An ANR is an AR if and only if it is contractible. (Theorem 7.5.4 of McLennan (2016).)

A topological space X has *the fixed point property* if every continuous function from X to itself has a fixed point. The Eilenberg-Montgomery fixed point theorem (Eilenberg and Montgomery (1946)) implies that if X is a nonempty compact AR and $F : X \to X$ is an upper hemicontinuous contractible valued correspondence, then F has a fixed point. In addition to generalizing the Kakutani fixed point theorem by replacing the geometric hypothesis of convexity with the conceptually more satisfactory topological hypothesis of contractibility, this implies that every compact AR has the fixed point property. Whether an arbitrary nonempty compact contractible metric space necessarily has the fixed point property was unknown for several years, but Kinoshita (1953) gave an example of a compact contractible subset of $\mathbb{R}^3$ that is not an AR and does not have the fixed point property. In view of these results, the Eilenberg-Montgomery theorem seems maximally general, and ANR's seem to be the most general noncontractible setting in which topological fixed point theory is well behaved.

Fix an ANR X. If $C \subset X$ is compact and $F : C \to X$ is an upper hemicontinuous and contractible valued correspondence, F is *index admissible* if $\mathcal{F}(F)$ is contained in the topological interior $\operatorname{int} C = C \setminus \overline{(X \setminus C)}$ of C. (Often X is a naturally thought of as a subset of a larger space Y—for example, we usually think of the space of mixed strategy profiles of a game as embedded in a Euclidean space—in which case F is allowed to have fixed points in the boundary of X relative to its inclusion in Y.) Let $\mathcal{I}_X(C)$ be the set of index admissible correspondences $F : C \to X$, and let $\mathcal{I}_X = \bigcup \mathcal{I}_X(C)$ where the union is over all compact $C \subset X$.

Definition 1. *A* fixed point index *for X is a function $\Lambda_X : \mathcal{I}_X \to \mathbb{Z}$ satisfying:*

(I1) (Normalization) If $c : C \to X$ is a constant function whose value is in $\operatorname{int} C$, then

$$\Lambda_X(c) = 1.$$

(I2) (Additivity) If $F : C \to X$ is an element of $\mathcal{I}_X$, $C_1, \ldots, C_r$ are pairwise disjoint compact subsets of C, and $\mathcal{F}(F) \subset \operatorname{int} C_1 \cup \ldots \cup \operatorname{int} C_r$, then

$$\Lambda_X(F) = \sum_i \Lambda_X(F|_{C_i}).$$

(I3) (Continuity) For each $F : C \to X$ in $\mathcal{I}_X$ there is a neighborhood $U \subset C \times X$ of $\operatorname{Gr}(F)$ such that $\Lambda_X(\hat{F}) = \Lambda_X(F)$ for every $\hat{F} \in \mathcal{I}_X(C)$ with $\operatorname{Gr}(\hat{F}) \subset U$.

An important method is to approximate a correspondence with a function. The following result (see Ch. 9 of McLennan (2016) for a proof) implies that this is always possible.

Proposition 1. *Suppose that X and Y are ANR's, $C \subset X$ is compact, and $F : C \to Y$ is an upper semicontinuous contractible valued correspondence. Then for any neighborhood U of $\mathrm{Gr}(F)$ there is a continuous function $f : C \to Y$ such that $\mathrm{Gr}(f) \subset U$.*

The finite dimensional version of this result for convex valued correspondences was the method Kakutani (1941) used to prove his extension of Brouwer's theorem. It was generalized by Cellina (1969, 1970), extended to contractible valued correspondences by Mas-Colell (1974), and to ANR's by McLennan (1991).

To begin the process of understanding axioms (I1)–(I3) we observe that the existence of an index for X implies quite general fixed point theorems. If $F \in \mathcal{I}_X$ and $\mathcal{F}(F) = \emptyset$, then Additivity gives both $\Lambda_X(F) = \Lambda_X(F|_\emptyset)$ and $\Lambda_X(F) = \Lambda_X(F|_\emptyset) + \Lambda_X(F|_\emptyset)$, so $\Lambda_X(F) = 0$. Therefore $\mathcal{F}(F) \neq \emptyset$ whenever $\Lambda_X(F) \neq 0$. Let X be a compact AR, and let $c : X \times [0, 1] \to X$ be a contraction. If $f : X \to X$ is a continuous function, then $f \circ c$ (the function $(x, t) \mapsto c(t, f(x))$ works equally well) is a homotopy between f and a constant function, so Normalization and Continuity imply that $\Lambda_X(f) = 1$. Now let $F : X \to X$ be an upper hemicontinuous correspondence. The Eilenberg-Montgomery fixed point theorem (Eilenberg and Montgomery (1946)) asserts that if F is acyclic[2] valued, then $\mathcal{F}(F) \neq \emptyset$. Contractible spaces are acyclic, so the following weaker assertion is a corollary: if F is contractible valued, then $\mathcal{F}(F) \neq \emptyset$. (Spaces that are acyclic but not contractible are rare "in nature," at least in economic theory, so one expects this version to be adequate for economic applications.) The index provides a different route to this conclusion: the fact that $\Lambda_X(f) = 1$ whenever $f : X \to X$ is a continuous function and Proposition 1 combine with Continuity to imply that $\Lambda_X(F) = 1$ whenever $F : X \to X$ is an upper hemicontinuous contractible valued correspondence.

If X is a compact ANR, the *Euler characteristic* of X is $\chi(X) = \Lambda_X(\mathrm{Id}_X)$ where $\mathrm{Id}_X : X \to X$ is the identity function. The connection with dynamics investigated in Section 3 suggests that the following is the most general formulation of the index +1 principle: *if a connected component of the set of equilibria of an economic model is an ANR, a necessary condition for it to be stable with respect to some natural adjustment dynamics and thus potentially empirically relevant, is that its index agrees with its Euler characteristic.* The Euler characteristic of a singleton is +1 (this follows from Normalization) so the special case of the index +1 principle motivating its name is: in order

[2] Acyclicity is a concept defined in terms of homology: a space X is *acyclic* if $H_0(X, \mathbb{Z}) = \mathbb{Z}$ and $H_i(X, \mathbb{Z}) = 0$ for all $i > 0$.

to be potentially empirically relevant, the index (relative to the excess demand or best response correspondence) of an isolated equilibrium must be $+1$.

The Continuity axiom is closely related to invariance under homotopy. Let $C \subset X$ be compact. A *homotopy* is an upper hemicontinuous contractible valued correspondence

$$H : C \times [0, 1] \to X.$$

We usually think of H as a continuous (in the appropriate topology) function taking each t to a correspondence $H_t = H(\cdot, t)$ "at time t." An *index admissible homotopy* (IAH) is a homotopy H such that each H_t is index admissible. If H is an IAH, then Continuity implies that $\Lambda(H_t)$ is a (locally constant, hence) constant function of t, so

$$\Lambda(H_0) = \Lambda(H_1).$$

Many applications of Continuity invoke this fact, which we describe as *Homotopy*.

In view of Continuity, $\Lambda_X(F) \neq 0$ implies that the set of fixed points is *essential* in the sense of Kinoshita (1952): for any neighborhood V of $\mathcal{F}(F)$ there is a neighborhood $U \subset C \times X$ of $\mathrm{Gr}(F)$ such that if $F' : C \to X$ is upper hemicontinuous and contractible valued, then $\mathcal{F}(F') \cap V \neq \emptyset$. (To prove this note that if U is the neighborhood given by Continuity, then $U \setminus \{ (x, x) : x \in X \setminus V \}$ is also a neighborhood of $\mathrm{Gr}(F)$.) Theorem 15.3 of McLennan (2016) provides a converse for convex valued correspondences: if $X \subset \mathbb{R}^m$ is compact and convex, $C \subset X$ is compact, $F : C \to X$ is upper hemicontinuous, convex valued, and index admissible, $\mathcal{F}(F)$ is connected, and $\Lambda_X(F) = 0$, then any neighborhood $U \subset C \times X$ of $\mathrm{Gr}(F)$ contains the graph of a continuous function $f : C \to X$ with $\mathcal{F}(f) = \emptyset$.

In effect Λ_X assigns an index to each subset of the set of fixed points that is *clopen* (both closed and open) in the subspace topology of this set. That is, if $F : C \to X$ is an element of $\mathcal{I}_X$ and $K \subset \mathcal{F}(F)$ is clopen in the relative topology of $\mathcal{F}(F)$, then there is a compact neighborhood $C' \subset C$ such that $C' \cap \mathcal{F}(F) = K$, and Additivity implies that $\Lambda_X(F|_{C'}) = \Lambda_X(F|_{C' \cap C''}) = \Lambda_X(F|_{C''})$ for any two such neighborhoods C' and C''. Let $\Lambda_F(K)$ denote this common value.

We now explain how the main methods of computing the index follow from the axioms. First consider linear functions $\ell : \mathbb{R}^m \to \mathbb{R}^m$ restricted to the unit disk $D = \{ x \in \mathbb{R}^m : \|x\| \leq 1 \}$. Elementary matrix analysis shows that the set of ℓ whose unique fixed point is zero has two path connected components, and continuity implies that the sign of the determinant of $\mathrm{Id}_{\mathbb{R}^m} - \ell$ is constant on each component. Normalization and Continuity imply that if ℓ is in the component containing the constant zero function, so that this determinant is positive, then $\Lambda(\ell|_D) = +1$. More generally, if $f : D \to D$ is a

C^1 function that has a unique fixed point x^* in the interior of D, and $|\mathrm{Id}_{\mathbb{R}^m} - Df(x^*)|$ is positive, then $\Lambda(f) = +1$. As Figure 1 suggests, such an f is homotopic to a function g with three fixed points, at two of which $|\mathrm{Id}_{\mathbb{R}^m} - Dg(x^*)|$ is positive and at one of which it is negative. Since (by Continuity) the sum of the indices must be $+1$, if x^* is the fixed point at which $|\mathrm{Id}_{\mathbb{R}^m} - Dg(x^*)|$ is negative, then $\Lambda_g(x^*) = -1$.

In a Euclidean setting a continuous function can be approximated by a C^1 function whose fixed points are all regular, in the sense that the relevant determinant does not vanish. (E.g., Hirsch (1976), Section 2.2.) Thus, in a Euclidean setting, the index of a clopen set of fixed points can in principle always be computed by constructing a suitable approximation, but of course this can be quite tedious in practice. Three ideas often simplify the computation. First, if K is a clopen set of fixed points of F and one can construct any sort of approximation of F that has no fixed points near K, then $\Lambda_F(K) = 0$. Second, if the set of fixed points of F has finitely many connected components, say $K_1, \ldots, K_r$, $\Lambda_X(F)$ is known (for instance because X is contractible) and $\Lambda_F(K_1), \ldots, \Lambda_F(K_{r-1})$ are already known, then

$$\Lambda_F(K_r) = \Lambda_X(F) - \Lambda_F(K_1) - \cdots - \Lambda_F(K_{r-1}).$$

Finally the Commutativity axiom introduced below sometimes allows one to compute the index of a function by relating it to a function whose index is known.

We now introduce two other properties of the index.

(I4) (Commutativity[3]) If X and Y are compact ANR's, and $f : X \to Y$ and $g : Y \to X$ are continuous functions, then

$$\Lambda_X(g \circ f) = \Lambda_Y(f \circ g).$$

(I5) (Multiplication) If X and Y are ANR's, $F : C \to X$ is an element of $\mathcal{I}_X$, $G : D \to Y$ is an element of $\mathcal{I}_Y$, and $F \times G$ is the correspondence taking (x, y) to $F(x) \times G(y)$, then

$$\Lambda_{X \times Y}(F \times G) = \Lambda_X(F) \cdot \Lambda_Y(G).$$

Theorem 1. *There is a unique assignment of an index Λ_X to each ANR X that satisfies (I1)–(I4), and this assignment also satisfies (I5).*

[3]This is actually a special case that is adequate for our applications. The general version of Commutativity is: if X and $\hat{X}$ are ANR's, C, D, $\hat{C}$, and $\hat{D}$ are compact sets with $D \subset C \subset X$ and $\hat{D} \subset \hat{C} \subset \hat{X}$, $g : C \to \hat{X}$ and $\hat{g} : \hat{C} \to X$ are continuous with $g(D) \subset \operatorname{int} \hat{C}$ and $\hat{g}(\hat{D}) \subset \operatorname{int} C$, $\hat{g} \circ g|_D$ and $g \circ \hat{g}|_{\hat{D}}$ are index admissible, and $g(\mathcal{F}(\hat{g} \circ g|_D)) = \mathcal{F}(g \circ \hat{g}|_{\hat{D}})$, then $\Lambda_X(\hat{g} \circ g|_D) = \Lambda_{\hat{X}}(g \circ \hat{g}|_{\hat{D}})$.

The proof of Theorem 1 is much too lengthy to present here in complete detail. It has several stages. First, the unique existence of an index is established for smooth index admissible functions $f : C \to \mathbb{R}^m$ (where C is a compact subset of $\mathbb{R}^m$) whose fixed points are all regular, using the methods of differentiable topology. This index satisfies Commutativity by virtue of a nontrivial result of linear algebra, and Multiplication is a consequence of elementary properties of the determinant. The second stage uses the fact that continuous functions can be approximated by smooth functions to extend the index to continuous index admissible functions on Euclidean spaces. A compact subset of an ANR can be "dominated," in a certain sense, by a compact subset of a Euclidean spaces that is the closure of its interior (Theorem 7.6.4 of McLennan (2016)). Using this fact, one can show that if X is an ANR, $C \subset X$ is compact, and $f : C \to X$ is an index admissible function, then f can be approximated by $f \circ g' \circ g$ where C' is a compact subset of some $\mathbb{R}^m$, $g : C \to C'$ and $g' : C' \to C$ are continuous. Continuity and the general version of Commutativity imply that

$$\Lambda_X(f) = \Lambda_X(f \circ g' \circ g) = \Lambda_{\mathbb{R}^m}(g \circ f \circ g').$$

This implies that there is at most one index for ANR's that satisfies (I1)–(I4), and one can verify that the index defined by this formula does indeed satisfy (I1)–(I5). Finally the index is extended from functions to contractible valued correspondences using a suitable generalization of Proposition 1.

Historically the fixed point index evolved out of the general development of algebraic topology in the first half of the 20$^{\text{th}}$ century, beginning with the work of Poincaré and Brouwer. Lefschetz and Hopf brought the fixed point theorem to the context of manifolds in the 1920's, and Leray and Schauder provided the initial formulation of the index during the next two decades. In his Ph.D. thesis Browder (1948) used Commutatitivy to extend the index to ANR's as sketched above (we will see that Commutativity has other uses as well) and the index was given an axiomatic formulation by O'Neill (1953). Book length treatments include Brown (1971), Dugundji and Granas (2003), and Górniewicz (2006), which study the subject from the point of view of pure mathematics, and McLennan (2016), which is tailored to meet the needs of economists and economic applications.

The fixed point index for regular economies was introduced in general equilibrium theory by Dierker (1972, 1974). It plays a role in the analysis of the Lemke-Howson algorithm in Shapley (1974). Hofbauer (1990) applies the vector field index (defined below) to dynamic issues in evolutionary game theory, and Ritzberger (1994) applies it to game theory systematically.

Mas-Colell et al. (1995) mention that it is, in principle, possible to prove that the set of equilibria has a single connected component by showing that there are finitely many components, each of which has index +1. At the time they wrote there was no example in which uniqueness could not be proven by simpler methods, but Eraslan and McLennan (2011) use this method to prove a uniqueness result, and at present no more direct proof of their result is known.

We now discuss the fixed point index in relation to game theory. A *semi-algebraic set* is a subset of a Euclidean space that is a defined by a logical formula that is built up from polynomial equations and inequalities using 'and,' 'or,' 'not,' and parenthesis, according to the usual rules. More concretely, a semi-algebraic set is a finite union of sets that are defined by conjunctions of polynomial equations and inequalities. Fundamental results of semi-algebraic geometry (e.g., Bochnak et al. (1987)) imply that a semi-algebraic set has finitely many connected components, each of which is a path connected ANR. The application of semi-algebraic geometry to game theory was pioneered by Blume and Zame (1993), and that paper is recommended for a quick introduction to the subject's foundational results.

For a finite strategic form game the set of Nash equilibria is a semi-algebraic set, so it has finitely many connected components, each of which is closed and thus clopen in the relative topology of the set of Nash equilibria. Of course the set of Nash equilibria is also the set of fixed points of the best response correspondence, so each of its connected components has an index, and the sum of the indices is +1. A component whose index is nonzero is "robust" with respect to payoff perturbations and trembles: Continuity implies that for any neighborhood U of the component there is a neighborhood V of the game's payoffs and a neighborhood W of 0 in the space of trembles (in the sense of Selten (1975)) such that for any payoff in V, and for any tremble in W, the perturbed game has a Nash equilibrium in U. Wu and Jiang (1962) say that the component is *essential* if it is robust with respect to payoff perturbations (this condition is weaker than the component being essential in the sense of Kinoshita (1952)) and robustness with respect to trembles implies that the component contains a strategically stable set in the sense of Kohlberg and Mertens (1986). Thus the requirement that a component has a nonzero index is already a quite strong refinement of the Nash equilibrium concept, which may be further refined (without sacrificing existence) by replacing 'nonzero' with 'positive' or 'odd.' Of course the index is agnostic concerning which subsets of a component should be regarded as "solutions" as per the discussion in Kohlberg and Mertens (1986). Also, as we will see later, dynamic stability can favor a component of index zero.

The use of Kakutani's theorem to prove the existence of Nash equilibrium of a normal form game, and to prove the existence of a competitive equilibrium in a general equilibrium setting, are of course very well known. It is less well known that the set of sequential equilibria (Kreps and Wilson (1982)) of an extensive form game is also (the projection of) the set of fixed points of an upper hemicontinuous contractible valued correspondence, so that index theory can be applied. We now briefly sketch the main ideas as they are developed in McLennan (1989a).

First we quickly review the sequential equilibrium concept of Kreps and Wilson (1982). Let a finite extensive form game with perfect recall be given. (We assume that the reader is familiar with the setting and terminology of Kreps and Wilson (1982). Our discussion is slightly more general since we allow for the possibility that there are information sets at which Nature chooses that are not initial.) A *behavior strategy* is an assignment, to each information set, of a probability distribution over the set of actions that are available there, which must agree with the given distribution if the information set is controlled by Nature; it is *interior* if each probability distribution assigns positive probability to all actions. A *belief* is an assignment, to each information set, of a probability distribution over (the set of nodes in) the information set. An *assessment* is a pair consisting of a behavior strategy and a belief. If play is governed by an interior behavior strategy, then every node occurs with positive probability, so there is an induced belief that assigns the induced conditional distribution to each information set. An *interior consistent assessment* is an assessment in which the behavior strategy is interior and the belief is the one induced by it. The set of *consistent assessments* is the closure of the set of interior consistent assessments.

Given a behavior strategy and a node, one can compute an expected payoff for each player conditional on reaching the node. Given an assessment, an information set, and an action at that information set, averaging (with weights given by the belief at the information set) of the expected payoffs of the nodes resulting from choosing the action at each of the nodes in the information set, computes an expected payoff (conditional on reaching the information set) for the agent who controls the information set. An assessment is *myopically rational* if, at each information set controlled by a player, the behavior strategy assigns all probability to actions that maximize this expected utility. A *sequential equilibrium* is a consistent, myopically rational assessment. Kreps and Wilson show that, because the game satisfies perfect recall, a sequential equilibrium is fully rational in the sense that there is no information set at which an agent can improve her conditional expected utility by changing her behavior at that and/or other information sets further down the game tree.

Given a consistent assessment, a *sequential best response* is a consistent assessment that, at each information set, assigns all probability to actions that maximize conditional expected utility for the given consistent assessment. The *sequential best response correspondence* is the correspondence that assigns the set of such best responses to each consistent assessment. This is a correspondence mapping the set of consistent assessments into itself that is upper hemicontinuous (because the relevant conditional expected utilities are continuous functions) and which has the set of sequential equilibria as its set of fixed points. However, whether this best response correspondence is contractible valued (which would imply that the set of set of consistent assessments is contractible) is unknown.

To obtain a well behaved correspondence we pass to a setting that keeps track of more conditional probabilities. Let Z be a finite set. For any nonempty subset E let $\Delta(E)$ be the set of probability measures on E. A *conditional system* (Rényi (1970), Myerson (1986)) is an assignment of a probability measure $p(\,\cdot\,|E) \in \Delta(E)$ to each nonempty $E \subset Z$ satisfying

$$p(C|E) = p(C|D) \cdot p(D|E)$$

for all $C \subset D \subset E \subset Z$ with D and E nonempty. A conditional system is *interior* if all such probabilities are positive. The space of conditional systems is the closure of the set of interior conditional systems, and is homeomorphic to a $(|Z| - 1)$-dimensional ball (McLennan (1989b)).

Now let Z be the set of terminal nodes of the given finite extensive form game. A conditional system gives conditional probabilities over all nonempty subsets of Z, so it encodes a belief and, for each node in each information set, a distribution over the set of actions at the information set conditional on the node being reached. A conditional system is *consistent* if, for each information set, the distributions over actions at the various nodes in the information set are all the same, and this distribution is the given one if the information set is controlled by Nature. That is, a conditional system is consistent if it has an unambiguous projection to a behavior strategy, so a consistent conditional system projects onto an assessment, and in fact the set of consistent assessments is precisely the image of this projection. The set of consistent conditional systems is the closure of the set of interior consistent conditional systems, and it is also homeomorphic to a closed Euclidean ball (McLennan (1989a)).

Given a consistent conditional system p, the expected utility conditional on reaching an information set and choosing an action there is the one computed using the data of the associated consistent assessment, as we described above. A *conditional best response* to p is a consistent conditional system that assigns all probability at each information set to the actions that maximize these conditional expected utilities, and the *conditional best response correspondence* assigns the set of such conditional best responses to each p. As above, since the relevant expected utilities are continuous functions of p, the conditional best response correspondence is upper hemicontinuous. McLennan (1989a) shows that for any specification of a nonempty set of actions at each information set not controlled by Nature, the set of consistent conditional systems that do not assign any probability to other actions is contractible. Therefore the conditional best response correspondence is contractible valued, and satisfies the hypotheses of the Eilenberg-Montgomery fixed point theorem.

The set of sequential equilibria and the set of fixed points of the conditional best response correspondence are both semi-algebraic sets, so each has finitely many path connected components. Since a continuous function maps connected sets to connected sets, each component of the set of fixed points of the conditional best response correspondence is mapped into a component of the set of sequential equilibria by the projection described above. If one defines the index of a component of the set of sequential equilibria to be the sum of the indices of the components of the set of fixed points of the conditional best response correspondence that lie above it, then the sum of the indices of the components of the set of sequential equilibria is $+1$, and there necessarily exist components with odd index and components with positive index. By Continuity, any component of the set of fixed points of the conditional best response correspondence with nonzero index is robust with respect to perturbations of the conditional best response correspondence, in the sense that sufficiently nearby perturbations will have fixed points near the set, and a component of the set of sequential equilibria will be robust with respect to perturbations of the extensive form payoffs (for example) if such a component lies above it. In this connection we should mention an important result of Kreps and Wilson (1982): for generic extensive form payoffs there are finitely many paths (distributions on Z) induced by sequential equilibria. When there are finitely many paths, each component of the set of sequential equilibria and each component of the set of fixed points maps to a single path, and a path is robust with respect to perturbations of the payoffs if one of the components of the set of fixed points of the conditional best response correspondence that maps to it has nonzero index.

3 Natural Dynamics

In certain biological applications of game theory the pure strategies are thought of as biological traits (often genes, but other interpretations are possible) while the payoffs are reproduction rates such as the expected number of offspring surviving to adulthood. If we imagine that members of today's population are first matched with each other randomly, after which they play the game and achieve whatever reproductive success, this gives rise to a discrete time dynamic system: the proportion of the population playing a certain strategy tomorrow is the proportion of today's population playing that strategy times the average number of offspring, normalized by dividing by the size of the population. Stability of an equilibrium with respect to such dynamics motivates the concept of an evolutionarily stable strategy (Maynard Smith and Price (1973)). The study of evolutionary dynamics began in biology with Taylor and Jonker (1978) and eventually migrated to economics, where it generated a large literature that is surveyed by (among others) Weibull (1995), Vega-Redondo (1996), Samuelson (1997), Hofbauer and Sigmund (1998), Fudenberg and Levine (1998), Cressman (2003), and Sandholm (2010).

Such dynamics are sensibly motivated when applied to simple organisms, and also to large populations of myopic agents, such as drivers in a traffic network, whose aggregate behavior adjusts gradually to changing circumstances. (Microfoundations of strategic dynamics are discussed in Section 1.2 of Sandholm (2010).) When agents are sophisticated creatures who are aware of the adjustment process and can respond rapidly, gradual adjustment processes are conceptually problematic because of the possibility of exploiting the adjustment process instead of following it. This is equally the case for processes such as tatonnement that have been proposed as models of adjustment to equilibrium in markets. In fact the only adjustment process that is fully consistent with the principal of rational expectations (the agents understand the model and behave optimally relative to its predictions) is to go to equilibrium immediately. Thus all models of dynamics of strategic adjustment for sophisticated agents are to some extent behavioral, with some aspect of bounded rationality. In addition, since the agents whose behavior is being studied cannot have very precise knowledge of the dynamic adjustment process, there is every reason to expect the social scientist to have at best a quite vague understanding of it.

The considerations suggest that if reliable conclusions can be obtained from dynamic considerations, they must be based on coarse qualitative properties of the dynamics. Following Demichelis and Ritzberger (2003), we say that a Nash equilibrium of a strategic form game is *potentially stable* if it is locally stable with respect to *some* dynamic adjustment process that is *natural*, in the sense of not adjusting any component of a profile of mixed strategies in a direction that lowers the agent's expected utility. (Section 5.2 of Sandholm (2010) discusses related concepts.) Similarly, an equilibrium of an exchange economy is potentially stable if it is locally stable with respect to some dynamic adjustment process in price space that never adjusts prices in a direction that is a negative multiple of excess demand. The most direct form of the hypothesis being advanced here is that if an isolated equilibrium is not potentially stable, then it will not be observed as an outcome that is self-reproducing in the sense of being expected to occur, and then occurring, repeatedly.

This section presents two results that show that for the central models of economic theory, namely finite strategic form games and general equilibrium, if an isolated equilibrium is potentially stable, then its index is +1. More generally, if a connected component of the set of equilibria is potentially stable, and it is an ANR, then its index must coincide with its Euler characteristic.

This is to be understood as a quite weak consequence of the presumption that a self-reproducing outcome must be stable. There is no suggestion that it exhausts the consequences of dynamic stability, and in fact in Section 5 we will see games with equilibria that are evidently unstable, but which are not rejected by this criterion.

We now introduce necessary formalities concerning vector fields, the vector field index, and dynamics. Let H be an n-dimensional affine subspace of $\mathbb{R}^m$, and let L be the n-dimensional linear subspace of $\mathbb{R}^m$ that is parallel to H. (The Euclidean setting provides certain technical simplifications, but all concepts and results extend to vector fields on manifolds.)

If $S \subset H$, a *vector field*[4] on S is a continuous function $\nu : S \to L$. An *equilibrium* of ν is a point $p \in S$ such that $\nu(p) = 0$. Let $\mathcal{E}(\nu)$ be the set of equilibria of ν. A vector field $\nu : C \to L$ is *index admissible* if C is a compact subset of H and $\mathcal{E}(\nu)$ is contained in the interior of C. Let $\mathcal{V}_H(C)$ be the set of index admissible vector fields on C, and let $\mathcal{V}_H = \bigcup \mathcal{V}_H(C)$ where the union is over all compact $C \subset H$.

Definition 2. *A* vector field index *for H is a function* $\mathrm{ind}_H : \mathcal{V}_H \to \mathbb{Z}$ *satisfying:*

(I1) (Normalization) If p is an element of the interior of C and $\nu : C \to L$ is the vector field $\nu(q) = q - p$, then $\mathrm{ind}_H(\nu) = +1$.

(I2) (Additivity) If $\nu : C \to X$ is an element of $\mathcal{V}_H$, $C_1, \ldots, C_r$ are pairwise disjoint compact subsets of C, and $\mathcal{E}(\nu) \subset \mathrm{int}\, C_1 \cup \ldots \cup \mathrm{int}\, C_r$, then

$$\mathrm{ind}_H(\nu) = \sum_i \mathrm{ind}_H(\nu|_{C_i}).$$

(I3) (Continuity) For each $\nu : C \to L$ in $\mathcal{V}_H$ there is a neighborhood $U \subset C \times L$ of $\mathrm{Gr}(\nu)$ *such that* $\mathrm{ind}_H(\hat{\nu}) = \mathrm{ind}_H(\nu)$ *for every $\hat{\nu} \in \mathcal{V}_H(C)$ with* $\mathrm{Gr}(\hat{F}) \subset U$.

[4] For game theory it is also natural to consider the more general notion of a *differential inclusion* which assigns a nonempty set of tangent vectors to each point. Smirnov (2002) provides an introduction to the relevant theory.

For a vector field $\nu : C \to L$ let $f_\nu : C \to H$ be the function $f_\nu(p) = p + \nu(p)$, and for a continuous function $f : C \to H$ let $\nu_f : C \to L$ be the vector field $\nu_f(p) = f(p) - p$. Of course $\nu_{f_\nu} = \nu$ and $f_{\nu_f} = f$, and ν is an index admissible vector field if and only if f_ν is an index admissible function. If Λ_H is a fixed point index for H, then $\nu \mapsto (-1)^n \Lambda_H(f_\nu)$ is a vector field index for H. Conversely, if ind_H is a vector field index, then $f \mapsto (-1)^n \text{ind}(\nu_f)$ is a fixed point index (for functions) for H. Thus existence and uniqueness of the vector field index follows easily from existence and uniqueness of the fixed point index.

The sign $(-1)^n$ could be avoided by replacing the Normalization axiom above with the requirement that if $p \in \text{int}\, C$, then the index of $q \mapsto p - q$ is $+1$, but unfortunately (for us at least) the definition above is standard. The equations $\text{ind}_H(\nu) = \Lambda_H(f_{-\nu})$ and $\Lambda_H(f) = \text{ind}(-\nu_f)$ are perhaps the simplest expressions of the relationship between the fixed point index and the vector field index.

A *vector field homotopy* on a set $S \subset H$ is a continuous function $\eta : S \times [0,1] \to L$. Let $\eta_t = \eta(\cdot, t)$ denote the vector field "at time t." If η is a vector field homotopy on a compact C, it is *index admissible* if each η_t is index admissible. If this is the case, then Continuity implies that $\text{ind}_H(\eta_0) = \text{ind}_H(\eta_1)$.

Let $\Sigma \subset H$ be nonempty, closed, and convex. For $p \in H$ let $r(p)$ be the nearest point in Σ. Evidently $r : H \to \Sigma$ is a retraction. We claim that r is Lipschitz with Lipschitz constant 1. To see this note that if $p, p' \in H$ and $r(p) \neq r(p')$, then the line segment between $r(p)$ and $r(p')$ is contained in Σ, so it makes an obtuse angle with the line segment between p and $r(p)$ (otherwise some point between $r(p)$ and $r(p')$ would be closer to p than $r(p)$) and also with the line segment between p' and $r(p')$. Therefore $\|r(p) - r(p')\| \leq \|p - p'\|$.

If ν is a vector field on $S \subset \Sigma$, then we say that ν is *not outward pointing* if, for all $p \in S$, $\nu(p)$ points into any half space that contains Σ and has p in its boundary. Formally this condition is that $\langle \nu(p), \eta \rangle \geq 0$ for all $\eta \in L$ such that $\Sigma \subset \{\, y \in L : \langle y, \eta \rangle \geq \langle p, \eta \rangle \,\}$.

In game theory equilibria are frequently in the boundary (relative to the topology of H) of Σ. We extend the vector field index to this situation as follows. If ν is a not outward pointing vector field on a compact $C \subset \Sigma$, we say that ν is *index admissible* if it does not have any equilibria in the boundary $C \cap \overline{\Sigma \setminus C}$ of C relative to the topology of Σ. Let $\tilde{C}$ be a compact subset of H such that

$$(\tilde{C} \cap \overline{H \setminus \tilde{C}}) \cap \Sigma = C \cap \overline{\Sigma \setminus C}.$$

For example, we could take $\tilde{C} = \{\, p \in r^{-1}(C) : \|r(p) - p\| \leq \varepsilon \,\}$ for some $\varepsilon > 0$. Then $\tilde{\nu}|_{\tilde{C}}$ is index admissible in the previous sense, so $\text{ind}_H(\tilde{\nu}|_{\tilde{C}})$ is defined. By Additivity this number does not depend on the choice of $\tilde{C}$, so we can define $\text{ind}_\Sigma(\nu)$ to be the common value of $\mathrm{ind}_H(\tilde{\nu}|_{\tilde{C}})$ for all suitable $\tilde{C}$. It is easy to see that ind_Σ satisfies suitably modified versions of Normalization, Additivity, and Continuity. In particular, if A is the set of equilibria of ν and C' is a compact subset of C that contains A in its interior, then $\mathrm{ind}_\Sigma(\nu|_{C'}) = \mathrm{ind}_\Sigma(\nu)$, and we can denote this number by $\mathrm{ind}_\nu(A)$.

We now discuss dynamics. Let $\tilde{Z}$ be an open subset of H, and let $\tilde{\nu} : \tilde{Z} \to L$ be a locally Lipschitz[5] vector field. If $a < b$, a C^1 function $\gamma : [a, b] \to \tilde{Z}$ is a *finite trajectory* of $\tilde{\nu}$ if $\gamma'(t) = \tilde{\nu}(\gamma(t))$ for all t. A *trajectory* of $\tilde{\nu}$ is a C^1 function from a (closed, open, half open, bounded, or unbounded) interval of $\mathbb{R}$ whose restriction to every compact subinterval is a finite trajectory. Foundational results of the theory of ordinary differential equations imply that for any compact $K \subset \tilde{Z}$ there is an $\varepsilon > 0$ such that there is a unique $\tilde{\Phi} : K \times (-\varepsilon, \varepsilon) \to \tilde{Z}$ such that for each $p \in K$, $\tilde{\Phi}(p, 0) = p$ and $\tilde{\Phi}(p, \cdot)$ is a trajectory of $\tilde{\nu}$. In addition, $\tilde{\Phi}$ is continuous, and if $\tilde{\nu}$ is C^r for some $1 \le r \le \infty$, then $\tilde{\Phi}$ is C^r. From this it follows without great difficulty that there is a maximal $\tilde{W} \subset \tilde{Z} \times \mathbb{R}$, called the *flow domain* of $\tilde{\nu}$, such that:

(a) For each $p \in \tilde{Z}$, $\{\, t \in \mathbb{R} : (p, t) \in \tilde{W} \,\}$ is an interval containing 0.

(b) There is a continuous $\tilde{\Phi} : \tilde{W} \to \tilde{Z}$ such that for each $p \in \tilde{Z}$, $\tilde{\Phi}(p, 0) = p$ and $\tilde{\Phi}(p, \cdot)$ is a trajectory of $\tilde{\nu}$.

In fact $\tilde{W}$ is open, there is a unique function $\tilde{\Phi}$ satisfying (b) that is called the *flow* of $\tilde{\nu}$, $\tilde{\Phi}$ is continuous, and if $\tilde{\nu}$ is C^r, then so is $\tilde{\Phi}$.

Let $Z \subset \Sigma$ be open, and let $\nu : Z \to L$ be a locally Lipschitz vector field that is not outward pointing. Let $\tilde{Z} = r^{-1}(Z)$, and for $p \in \tilde{Z}$ let

$$\tilde{\nu}(p) = \nu(r(p)) + r(p) - p.$$

If $p \notin \Sigma$, then $r(p) - p$ is the normal vector of a bounding hyperplane of Σ at $r(p)$, so if $p \in r^{-1}(S)$, then $\langle r(p) - p, \nu(r(p)) \rangle \ge 0$ and thus $\tilde{\nu}(p) \ne 0$. In particular the only equilibria of $\tilde{\nu}$ are the equilibria of ν.

Since ν is locally Lipschitz and r is Lipschitz, $\tilde{\nu}$ is locally Lipschitz. Let $\tilde{W}$ and $\tilde{\Phi}$ be the flow domain and flow of $\tilde{\nu}$. Proposition 15.4 of McLennan (2016) implies that $\tilde{\Phi}(\tilde{W} \cap (Z \times \mathbb{R}_+)) \subset Z$. From this it follows that for any compact $K \subset Z$ there is an $\varepsilon > 0$ such that there is a unique $\Phi : K \times [0, \varepsilon) \to Z$ such that for each $p \in K$, $\Phi(p, 0) = p$ and $\Phi(p, \cdot)$ is a trajectory of ν. As above, it follows without difficulty that there is a maximal $W \subset Z \times \mathbb{R}_+$, called the *forward flow domain* of ν, such that for each $p \in Z$,

[5] Recall that a function $f : X \to Y$ between metric spaces is *Lipschitz* if there is a constant $\ell \ge 0$ such that $d(f(x), f(x')) \le \ell d(x, x')$ for all $x, x' \in X$, and it is *locally Lipschitz* if each $x \in X$ has a neighborhood U such that $f|_U$ is Lipschitz.

$\{ t \in \mathbb{R}_+ : (p,t) \in W \}$ is an interval containing 0, and there is a continuous $\Phi : W \to Z$, called the *forward flow* of ν, such that for each $p \in Z$, $\Phi(p,0) = p$ and $\Phi(p,\cdot)$ is a trajectory of ν. Again, if ν is C^r, then so is Φ. As with homotopies, it often works well to treat the temporal argument as a subscript, writing $\Phi_t(p)$ in place of $\Phi(p,t)$. If $S \subset Z$, $I \subset \mathbb{R}$ and $S \times I \subset W$, then $\Phi_I(S) = \{ \Phi_t(p) : p \in S \text{ and } t \in I \}$, and there are abbreviations such as $\Phi_t(S)$ in place of $\Phi_{\{t\}}(S)$ that should cause no confusion.

A set $A \subset Z$ is *forward invariant* if $A \times \mathbb{R}_+ \subset W$ and $\Phi_{\mathbb{R}_+}(A) \subset A$. The set A is *Lyapunov stable* if, for any neighborhood $U \subset Z$ of A, there is a neighborhood $U' \subset U$ such that $U' \times \mathbb{R}_+ \subset W$ and $\Phi_{\mathbb{R}_+}(U') \in U$. The set A is *uniformly attractive* if there is a neighborhood $U \subset Z$ of A such that $U \times \mathbb{R}_+ \subset W$ and, for any neighborhood $U' \subset Z$ of A, there is a $T \geq 0$ such that $\Phi_{[T,\infty)}(U) \subset U'$. Combining these conditions, the set A is *uniformly asymptotically stable* if it is compact, forward invariant, Lyapunov stable, and uniformly attractive. It turns out (McLennan (2016), Lemma 15.6) that if A is compact and asymptotically stable, then it is uniformly asymptotically stable.

The *domain of attraction of* A is

$$D(A) = \{ p \in Z : \limsup_{t \to \infty} d(\Phi_t(p), A) = 0 \}.$$

(Here d is the usual Euclidean metric on $\mathbb{R}^n$, extended to points and closed sets by taking the minimum distance from the point to points in the set.) A *Lyapunov function* for $D(A)$ and A is a continuous function $\mathcal{L} : D(A) \to [0,\infty)$ such that:

(a) $\mathcal{L}^{-1}(0) = A$.

(b) For each $p \in D(A)$ the ν*-derivative* $\nu\mathcal{L}(p) = \frac{d}{dt}\mathcal{L}(\Phi_t(p))|_{t=0}$ is defined, and there is a continuous $a : (0,\infty) \to (0,\infty)$ such that $\nu\mathcal{L}(p) \leq -a(d(p,A))$ for all $p \in D(A)$.

(c) For every neighborhood U of A there is $\varepsilon > 0$ such that $\mathcal{L}^{-1}([0,\varepsilon]) \subset U$.

It is intuitive and very well known that if A is compact and there is a Lyapunov function for A, then A is uniformly asymptotically stable. The converse—if A is uniformly asymptotically stable, then there is a Lyapunov function for A—is also true. This is a highly nontrivial result with a rather complicated history that is sketched by Nadzieja (1990). Briefly, a sequence of partial solutions, over several decades, eventually culminated in a complete (in the sense that the Lyapunov function can be required to be C^∞) solution of the problem for vector fields on open subsets of manifolds by Wilson (1969).

Theorem 15.13 of McLennan (2016) extends the result to not outward pointing vector fields on convex sets, and more generally to vector fields that are not outward pointing on analogous subsets of manifolds. The idea is that if A is asymptotically stable for ν, then (Theorem 15.12 of McLennan (2016)) it is also asymptotically stable for $\tilde{\nu}$. Wilson's theorem gives a Lyapunov function for A and $\tilde{\nu}$ whose restriction to the portion of the domain lying in Σ is Lyapunov function for A and ν.

Suppose that A is compact and asymptotically stable. The domain of attraction $D(A)$ does not (by definition) contain any equilibria of ν outside of A, so for a compact neighborhood $C \subset D(A)$ of A, $\nu|_C$ is index admissible. In addition, Additivity implies that $\text{ind}_H(\nu|_C)$ does not depend on the choice of C, and we denote the common value by $\text{ind}_\nu(A)$. There is the following relationship between asymptotic stability and the index.

Theorem 2. *If $A \subset Z$ is compact, asymptotically stable, and an ANR, then*

$$\text{ind}_{-\nu}(A) = \chi(A).$$

This is essentially due to Demichelis and Ritzberger (2003). Although the special case when A is a single point is a well known result in the theory of dynamical systems, it seems that the more general result was not developed in that literature (e.g., Krasnosel'ski and Zabreiko (1984), Th. 52.1) even though it could have physical applications. Since generic payoffs for an extensive form game give rise to associated normal forms with infinitely many Nash equilibria, the additional generality is pertinent to that setting.

Here we only sketch the main ideas of one of the proofs given by Demichelis and Ritzberger, referring the reader to Ch. 15 of McLennan (2016) for more details. The converse Lyapunov theorem implies that there is a Lyapunov function $\mathcal{L}$ for ν and A. For $\varepsilon > 0$ let $A_\varepsilon = \mathcal{L}^{-1}([0, \varepsilon])$. Since A is an ANR, it is a retract of some neighborhood of itself, which contains A_ε for sufficiently small ε, so we can fix ε for which there is a retraction $r : A_\varepsilon \to A$. Since A is compact, it has a compact neighborhood, and we can require that A_ε is contained in some such neighborhood, so we may assume that A_ε is compact.

For $p \in D(A)$ let $\tau(p) = \inf\{\, t \geq 0 : \Phi_t(p) \in A_\varepsilon \,\}$. Since Φ is continuous, for any $c \in \mathbb{R}$ the sets $\tau^{-1}((-\infty, c))$ and $\tau^{-1}((c, \infty))$ are open, so τ is continuous. Therefore the function $p \mapsto \Phi_{\tau(p)}(p)$ is a retraction of $D(A)$ onto A_ε, which implies that A_ε is a retract of an open subset of Σ and thus an ANR. If $i : A \to A_\varepsilon$ is the inclusion, then Commutativity gives $\chi(A) = \Lambda_A(r \circ i) = \Lambda_{A_\varepsilon}(r \circ i) = \Lambda_{A_\varepsilon}(r)$.

It is easy to see that for small $t > 0$, Φ_t and $f_{t\nu}|_{A_\varepsilon}$ are index admissible homotopic. Fix such a t. Then $s \mapsto s\nu$ gives a homotopy of vector fields between $t\nu$ and ν that is index admissible homotopic (in the obvious sense) so $\Lambda_{A_\varepsilon}(\Phi_t) = \Lambda_{A_\varepsilon}(f_{t\nu}|_{A_\varepsilon}) = \text{ind}(-t\nu|_{A_\varepsilon}) = \text{ind}(-\nu|_{A_\varepsilon}) = \text{ind}_{-\nu}(A)$. Since r is a retraction, continuity implies that the set of $p \in A_\varepsilon$ such that A_ε contains the line segment between p and $r(p)$ is a neighborhood of A_ε. Let T be large enough that $\Phi_T(A_\varepsilon)$ is contained in this neighborhood. We can follow the homotopy $s \mapsto \Phi_s$ from Φ_t to Φ_T, use convex combination to create a homotopy between Φ_T and $r \circ \Phi_T$, then follow the homotopy $s \mapsto r \circ \Phi_s$ from $r \circ \Phi_T$ to r, thereby creating an index admissible homotopy between Φ_t and r, so Homotopy gives $\Lambda_{A_\varepsilon}(\Phi_t) = \Lambda_{A_\varepsilon}(r)$. Combining the various results above gives $\text{ind}_{-\nu}(A) = \Lambda_{A_\varepsilon}(\Phi_t) = \Lambda_{A_\varepsilon}(r) = \chi(A)$.

We now state the main result from Demichelis and Ritzberger (2003). Let

$$G = (S_1, \ldots, S_n, u_1, \ldots, u_n)$$

be a strategic form game. That is, $S_1, \ldots, S_n$ are finite sets of *pure strategies* and $u_1, \ldots, u_n$ are real valued functions whose domain is the set $S = S_1 \times \cdots \times S_n$ of *pure strategy profiles*. Let $N = \{1, \ldots, n\}$. For any nonempty finite set X let $\Delta(X) = \{\, \mu : X \to [0,1] : \sum \mu(x) = 1 \,\}$ be the set of probability measures on X. The set of *mixed strategies* for agent $i \in N$ is $\Sigma_i = \Delta(S_i)$. Let $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ be the set of *mixed strategy profiles*. Abusing notation, let u_i also denote the multilinear extension of u_i to Σ:

$$u_i(\sigma) = \sum_{s \in S} \Big(\prod_{h \in N} \sigma_h(s_h) \Big) u_i(s).$$

A mixed strategy profile σ^* is a *Nash equilibrium* if $u_i(\sigma^*) \geq u_i(\tau_i, \sigma^*_{-i})$ for all i and $\tau_i \in \sigma_i$. (As usual (τ_i, σ^*_{-i}) denotes the mixed strategy profile obtained from σ^* by replacing σ^*_i with τ_i.) Agent i's set of *best responses* to $\sigma \in \Sigma$ is

$$B_i(\sigma) = \{\, \tau_i \in \Sigma_i : u_i(\tau_i, \sigma_{-i}) \geq u_i(\tau'_i, \sigma_{-i}) \text{ for all } \tau'_i \in \Sigma_i \,\}.$$

The *best response correspondence* is the correspondence $B : \Sigma \to \Sigma$ given by $B(\sigma) = B_1(\sigma) \times \cdots \times B_n(\sigma)$. Of course B is an upper hemicontinuous convex valued correspondence whose fixed points are the Nash equilibria of G.

For each i let $H_i = \{\, \tau_i \in \mathbb{R}^{S_i} : \sum \tau_i(s_i) = 1 \,\}$ and $L_i = \{\, \tau_i \in \mathbb{R}^{S_i} : \sum \tau_i(s_i) = 0 \,\}$, and let $H = H_1 \times \cdots \times H_n$ and $L = L_1 \times \cdots \times L_n$. A vector field $\nu : \Sigma \to H$ can be regarded as an n-tuple $(\nu_1, \ldots, \nu_n)$ of vector fields $\nu_i : \Sigma \to L_i$, and an element of L_i can be identified with the element of L with the same i-component and all other components zero. The vector field ν is a *payoff consistent selection dynamics* if ν is not outward pointing and $Du_i(\sigma)\nu_i(\sigma) \geq 0$ for all $\sigma \in \Sigma$ and $i \in N$, and it is a *Nash dynamics* if, in addition, $\nu(\sigma) = 0$ if and only if σ is a Nash equilibrium.

Theorem 3 (Demichelis and Ritzberger (2003))**.** *If ν is a Nash dynamics and A is a connected component of the set of Nash equilibria that is uniformly asymptotically stable for ν, then $\Lambda_B(A) = \text{ind}_{-\nu}(A) = \chi(A)$.*

Remark: Demichelis and Ritzberger (Remark 2, p. 60) show that every payoff consistent selection dynamics can be perturbed to a Nash dynamics by adding an arbitrarily small amount of the dynamics introduced by Brown and von Neumann (1950). Thus the hypotheses of this result can be weakened by requiring only that ν is a payoff consistent selection dynamics.

To prove Theorem 3 one first observes that Theorem 2 implies that $\text{ind}_{-\nu}(A) = \chi(A)$. The next result completes the argument.

Proposition 2. *If ν is a Nash dynamics and A is a connected component of the set of Nash equilibria, then* $\text{ind}_{-\nu}(A) = \Lambda_B(A)$.

Proof. Since the set of Nash equilibria is semi-algebraic, there is a compact $C \subset \Sigma$ with $A = \mathcal{F}(B|_C)$ that has no Nash equilibria in its topological boundary $\partial C = C \cap \overline{\Sigma \setminus C}$. We have $\text{ind}_{-\nu}(A) = \text{ind}_\Sigma(-\nu|_C)$.

For $\sigma \in \Sigma$ and $i \in N$ let $\gamma_i(\sigma) \in \mathbb{R}^{S_i}$ be given by

$$\gamma_i(\sigma; s_i) := \max\{u_i(s_i, \sigma_{-i}) - u_i(\sigma), 0\},$$

and let $b(\sigma) = (b_1(\sigma), \dots, b_n(\sigma)) \in \Sigma$ be given by

$$b_i(\sigma; s_i) := \frac{\sigma_i(s_i) + \gamma_i(\sigma; s_i)}{\sum_{t_i \in S_i} \sigma_i(t_i) + \gamma_i(\sigma; t_i)}.$$

(These functions were introduced by Nash (1950).) Let $\beta(\sigma) = b(\sigma) - \sigma$. Then the set of Nash equilibria is the set of fixed points of the function b, and it is the set of vector field equilibria of the vector field β. Evidently β is a Nash dynamics.

We claim that for any $t \in [0, 1]$, $(1-t)\nu + t\beta$ is a Nash dynamic. It is evidently not outward pointing with $Du_i(\sigma)((1-t)\nu_i(\sigma) + t\beta_i(\sigma)) \geq 0$ for all $\sigma \in \Sigma$ and $i \in N$. If σ is not a Nash equilibrium, and $t > 0$, then this inequality holds strictly for some i, so that $(1-t)\nu_i(\sigma) + t\beta_i(\sigma) \neq 0$. If σ is a Nash equilibrium, then $\nu(\sigma) = \beta(\sigma) = 0$. Theorem 15.3 of McLennan (2016) implies that $\text{ind}_\Sigma(-\nu|_C) = \text{ind}_\Sigma(-\beta|_C)$. The homotopy $J : C \times [0, 1] \to \Sigma$ given by

$$J(\sigma, t) = \{\, (1-t)b(\sigma) + t\tau : \tau \in B(\sigma) \,\}$$

is clearly upper hemicontinuous and convex valued. For any $\sigma \in \partial C$ there is some i such that $\sigma_i \notin B_i(\sigma)$, so that $u_i(\tau_i, \sigma_{-i}) > u_i(\sigma)$ for every $\tau_i \in B_i(\sigma)$. Since ν is a Nash dynamics, $u_i(b_i(\sigma), \sigma_{-i}) \geq u_i(\sigma)$, so $u_i(\tau_i, \sigma_{-i}) \geq u_i(\sigma)$ for all t and $\tau \in J_t(\sigma)$, with strict inequality if $t > 0$. Therefore J_t does not have any fixed point in ∂C if $t > 0$, and J_0 does not have any fixed point in ∂C because all the equilibria of ν are Nash equilibria. Thus J is index admissible, so Continuity implies that

$$\text{ind}_{-\nu}(A) = \text{ind}_{-\beta}(A) = \Lambda_\Sigma(b|_C) = \Lambda_\Sigma(J_0) = \Lambda_\Sigma(J_1) = \Lambda_\Sigma(B|_C) = \Lambda_B(A).$$

□

An interesting example due to Balkenborg and Vermeulen (2016) illustrates the Demichelis-Ritzberger theorem, methods by which the index may be computed, and the contrast between strategic and dynamic stability. Suppose that $S_1 = \cdots = S_n$, and that for each $i \in N$ and $s \in S$, $u_i(s) = 0$ if $s_1 = \cdots = s_n$ and otherwise $u_i(s) = 1$. In words, the outcome is a failure for everyone if and only if everyone chooses the same pure strategy. Balkenborg and Vermeulen say that such a G is a *minimal diversity game*.

The set of Nash equilibria is easily computed. First suppose that σ is a completely mixed equilibrium. Then each agent i is indifferent between any two $a, b \in S_1 = \cdots = S_n$, so $\prod_{k \neq i} \sigma_k(a) = \prod_{k \neq i} \sigma_k(b)$, which implies that $\sigma_i(a)/\sigma_i(b) = \prod_{k \in N} \sigma_k(a) / \prod_{k \in N} \sigma_k(b) = \sigma_j(a)/\sigma_j(b)$ for all i and j. Therefore $\sigma_1 = \cdots = \sigma_n$, and consequently indifference between all pure strategies implies that σ is the mixed strategy profile ρ in which each i assigns equal probability to all pure strategies, which is evidently a Nash equilibrium.

Now suppose that σ is a Nash equilibrium with $\sigma_i(a) = 0$ for some agent i and pure strategy a. Every $j \neq i$ can insure a payoff of 1 by playing a, so $\sigma \in \Gamma = \{\, \sigma \in \Sigma : u_1(\sigma) = \cdots = u_n(\sigma) = 1 \,\}$. Conversely, each element of Γ is a Nash equilibrium since the common payoff is maximized. Thus the set of Nash equilibria is $\{\rho\} \cup \Gamma$.

Consider the vector field ν on Σ given by $\nu_i(\sigma)(a) = \sigma_i(a)(u_i(a, \sigma_{-i}) - u_i(\sigma))$. Direct computation shows that ν is a payoff consistent selection dynamics. By an argument similar to the one given above (cf. Section 4 of Balkenborg and Vermeulen (2016)) the only equilibria of this vector field outside of Γ are the mixed strategy profiles σ such that $\sigma_1 = \cdots = \sigma_n$ and each σ_i assigns equal probability to all elements of some subset of the common set of pure strategies and no probability to pure strategies outside this set. In particular, Γ has a neighborhood with no other equilibria of ν. Near Γ the flow of ν increases the common utility, so Γ is uniformly asymptotically stable. Therefore the index of Γ is its Euler characteristic.

Balkenborg and Vermeulen show that Γ is homeomorphic to a sphere of dimension $(m-1)(n-1)-1$ where $m = |S_1| = \cdots = |S_n|$ is the common number of pure strategies. We now compute the Euler characteristics of spheres. Consider the orthogonal transformation of $\mathbb{R}^2$ with matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Applying this to each factor of the d-fold cartesian product of $\mathbb{R}^2$ gives an orthogonal transformation ℓ of $\mathbb{R}^{2d}$ whose only fixed point is the origin, and the restriction of this map to the unit sphere gives a function from the unit sphere to itself that is homotopic to the identity and which has no fixed points. Thus the Euler characteristic of an odd dimensional sphere is 0. Applying ℓ and $\mathrm{Id}_{\mathbb{R}}$ to the factors of the product $\mathbb{R}^{2d} \times \mathbb{R}$ gives an orthogonal transformation of $\mathbb{R}^{2d+1}$ whose restriction to the unit sphere is a function from the unit sphere to itself that is homotopic to the identity and which has two fixed points, which may be thought of as the North and South Poles. At each of these the derivative of the function is ℓ, and since

$$|\mathrm{Id}_{\mathbb{R}^{2d}} - \ell| = \begin{vmatrix} 1-\cos\theta & -\sin\theta \\ \sin\theta & 1-\cos\theta \end{vmatrix}^d = (2-2\cos\theta)^d > 0,$$

the index of each is $+1$. Thus the Euler characteristic of an even dimensional sphere is $+2$. Since the sum of the index of Γ and the index of $\{\rho\}$ is $+1$, the index of $\{\rho\}$ is $+1$ or -1 according to whether $(m-1)(n-1)-1$ is odd or even.

In particular, $\{\rho\}$ is essential in the sense of Kinoshita (1952). Balkenborg and Vermeulen establish that it is stable in the sense of Kojima et al. (1985), so that is satisfies all of the strategic stability concepts surveyed by Hillas et al. (2001). When the game is perturbed by trembles in the sense of Selten (1975) it continues to be a game of common interest, so any strategy profile maximizing the common utility is a Nash equilibrium. Therefore Γ always contains Kohlberg-Mertens stable sets. When $(m-1)(n-1)-1$ is even, so that the index of Γ is $+2$, Γ is essential in the sense of Kinoshita (1952), and Theorem 2 of Demichelis and Ritzberger (2003) implies that it is stable in the sense of Mertens (1989, 1991), which implies that it satisfies all the other concepts surveyed by Hillas et al. (2001). When $(m-1)(n-1)-1$ is odd, Γ is not essential in the sense of Kinoshita (1952) because its index is zero (Theorem 15.3 of McLennan (2016)). Balkenborg and Vermeulen conjecture that in this case Γ is not BR-stable (as defined in Hillas et al. (2001)) and therefore does not satisfy the concepts that are shown in Hillas et al. (2001) to be equivalent to BR-stability, and is not stable in the sense of Mertens because this concept is stronger. They establish that this is the case when $m = 2$ and n is odd, and when $n = 2$ and m is odd. The subject is quite intricate (as Hillas et al. (2001) explain, details of definitions often vary across papers) but the simple take away is that strategic stability concepts can strongly endorse sets such as $\{\rho\}$ which are dynamically unstable, and concepts stronger than Kohlberg-Mertens stability can disparage a component of the set of equilibria that is uniquely dynamically stable.

	L	M	R
L	$1, 1$	$0, -1$	$-1, 1$
M	$-1, 0$	$0, 0$	$-1, 0$
R	$1, -1$	$0, -1$	$-2, -2$

Figure 2a

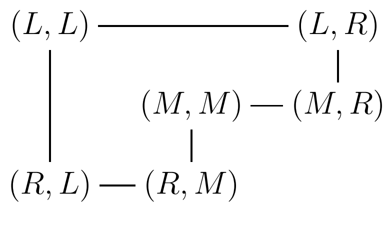

Figure 2b

FIGURE 2. A GAME WITH A CIRCULAR SET OF NASH EQUILIBRIA

One of the strongest predictions of the index +1 principle is that for models in which there is no component of the set of equilibria whose index agrees with its Euler characteristic, observed behavior will not be characterized by repetition of any one of the equilibria. The strategic form game in Figure 2a is from Appendix B of Kohlberg and Mertens (1986). Inspection reveals that there are six pure equilibria. When these are arranged in a circle, as shown in Figure 2b, it is easy to verify that mixtures of any two "adjacent" pure equilibria are mixed equilibria. For each agent and each pair of pure strategies for that agent, such mixtures evidently encompass all the equilibria in which that agent mixes strictly over those two pure strategies. There are no equilibria in which both agents mix over all three pure strategies (for each agent R is weakly dominated) so there are no other equilibria. Thus the set of Nash equilibria is homeomorphic to a circle. As always, its index is +1, but as we saw above, its Euler characteristic is zero. (Concretely the identity map of a circle is homotopic to a map (rotate the circle slightly) that has no fixed points.) Note that the instability predicted by the index +1 principle is, for this example, rather delicate, insofar as slight perturbations of the payoffs give games with strict pure equilibria, which are stable. (See Theorem 5 and the subsequent discussion.)

We now describe an application to general equilibrium theory. Fix a number of goods $\ell \geq 1$. Let $e = (1, \dots, 1) \in \mathbb{R}^\ell$, and let $H = \{\, p \in \mathbb{R}^\ell : \langle e, p \rangle = 1 \,\}$ and $L = \{\, p \in \mathbb{R}^\ell : \langle e, p \rangle = 0 \,\}$. Let $P = H \cap \mathbb{R}^\ell_{++}$ be the open price simplex. Let the aggregate excess demand function $\zeta : P \to \mathbb{R}^\ell$ satisfy:

(a) $\langle p, \zeta(p) \rangle = 0$ for all p (Walras' law).

(b) ζ is bounded below.

(c) $\|\zeta(p_n)\| \to \infty$ whenever $p_n \to p \in \overline{P} \setminus P$.

For $p \in P$ let $\tilde{\zeta}(p) = \zeta(p) - \langle e, \zeta(p) \rangle p$. Then $\tilde{\zeta}(p) \in L$, so $\tilde{\zeta}$ is a vector field on P. Since p and $\zeta(p)$ are orthogonal, $\tilde{\zeta}(p) = 0$ if and only if $\zeta(p) = 0$. If $\tilde{\zeta}$ is locally Lipschitz, then the associated dynamic system is a continuous time version of tatonnement. This dynamics is not invariant under changes in the units in which goods are measured. (Roughly, if we go from measuring milk in quarts to measuring it in pints, the price is halved and the numerical measure of excess demand is doubled, so the effective speed of price adjustment is quadrupled.) Thus there are many versions of tatonnement, with no apparent reason for preferring one to another. More generally, a *natural price dynamics* is a locally Lipschitz vector field $\nu : P \to L$ such that:

(a) ν is not outward pointing.

(b) For all $p \in P$, $\nu(p) = 0$ if and only if $\zeta(p) = 0$.

(c) $\langle \nu(p), \tilde{\zeta}(p) \rangle \geq 0$ for all $p \in P$.

Theorem 4. *If ν is a natural price dynamics and A is a compact set of equilibria that has a neighborhood containing no other equilibria, then $\text{ind}_{-\nu}(A) = \text{ind}_{-\tilde{\zeta}}(A)$. If, in addition, A is asymptotically stable for ν and an ANR, then $\text{ind}_{-\tilde{\zeta}}(A) = \chi(A)$.*

The proof is simple. Let $C \subset P$ be a compact neighborhood of A that contains no other equilibria. Then convex combination gives an index admissible homotopy between $\nu|_C$ and $\tilde{\zeta}|_C$. Therefore $\text{ind}_H(-\nu|_C) = \text{ind}_H(-\tilde{\zeta}|_C)$, and since C is arbitrary C, this establishes the first assertion. Of course the second is from Theorem 2.

Although it seems "natural" at first glance, from a mathematical point of view condition (c) is unnecessarily strong. The key idea is that $\nu|_C$ and $\tilde{\zeta}|_C$ are index admissible homotopic, which can be achieved in many ways. For example it would be enough to require that $\nu(p) \notin \{\, \alpha\tilde{\zeta}(p) : \alpha \leq 0 \,\}$ for all p such that $\zeta(p) \neq 0$. Similar remarks pertain to the definition of payoff consistent selection dynamics.

The processes of adjustment to equilibrium observed in reality, and in experimental settings, often have a stochastic aspect. For such processes one would expect analogues of the results above that assert that with high probability the process will spend very little time near an isolated equilibrium whose index is different from $+1$. Here "high probability" and "very little" can be made precise either by considering a sequence of models along which the stochastic aspect vanishes asymptotically, or by considering a single model in which the stochastic aspect vanishes asymptotically as time goes to infinity. These formulations are described, respectively, by the phrases "constant step size" and "decreasing step size."

Pemantle (1990) provides an elegantly formulated prototype for decreasing step size results. His method of proof (see p. 703) is to find a direction along which perturbations away from equilibrium will be positively reinforced. Because the process is bouncing around at least a little bit, it will necessarily arrive eventually in a part of the state space where this reinforcement will transport it some distance away with high probability. This method was significantly extended by Benaïm (1998) and Benaïm and Weibull (2003). Extensions of the main results to differential inclusions are given by Benaïm et al. (2005) (decreasing step size) and Roth and Sandholm (2013) (constant step size). Benaïm (1999) provides an expository overview of the innovations of Benaïm and his coauthors, and Pemantle (2007) and Sandholm (2010) give more recent surveys of this literature.

Examples of specific strategic adjustment processes that have been proposed by experimentalists include experience weighted attraction learning (Camerer and Ho (1999), Ho et al. (2007)) and individual evolutionary learning (Arifovic and Ledyard (2011), Arifovic and Ledyard (2015)). Sandholm (2010) provides a recent and comprehensive survey of the many processes considered the evolutionary game theory literature. Local Walrasian dynamics and local Marshallian dynamics (Champsaur and Cornet (1990), Bossaerts and Ledyard (2013)) are models of trading processes for general equilibrium exchange economies. Typically these processes can be perturbed stochastically (if they are not already stochastic) and then embedded in infinite horizon models in which the stochastic aspect vanishes asymptotically. (In the case of models of trading, one would also like to take limits along which the effective amount of out-of-equilibrium trade should go to zero.)

Although game theory and general equilibrium are central and prototypical, there are many other economic models in which equilibrium is a topological fixed point. Very often, if not always, it will be possible to define a compelling notion of a "natural" process of adjustment to equilibrium that is motivated by respect for the agents' efforts to increase their utilities. The intuitive ideas underlying the arguments in this section do not depend on the specific features of the models we studied, so they should be broadly and flexibly applicable. In this sense the results of this section are indicative but presumable far from exhaustive. My personal expectation is that the index +1 principle will hold for most (in some vague sense) models to which it might be applied. Nevertheless, the logic passing from the presumption of stability to the conclusion that the index of the equilibrium is +1 must be reconsidered in each instance. Once again, the index +1 principle is a general principle and not a law of economics.

4 Insensivity to Minor Details

Economic modelling requires strategic simplification. A model necessarily specifies only a few features of the world. The social scientist hopes that the selected features are the critical ones, and that their interaction in the analysis of the model sheds light on how economic outcomes are affected by their interaction in reality. In this section we consider one of the many ways that the modelling effort can go wrong: the solution concept makes misleading predictions because it is excessively sensitive to minor details of model specification. Most economists think about such issues rarely, if at all, but we have already mentioned how Ben-Porath and Dekel (1992) showed that certain solution concepts in the literature on refinements of Nash equilibrium gave different conclusions according to which of the two agents was modelled as having a capacity for self destruction that is always present in the world, but usually left out of economic models. A variety of reactions to this example can be found in the literature, but almost all game theorists would agree that it raises important issues that deserve careful consideration.

The index +1 principle refines whatever equilibrium concept it is applied to, so it is certainly fair to ask whether it might be similarly sensitive to insignificant details. Perhaps it is impossible to give a final and definitive answer, because there is no end of ways in which a model might be changed slightly. Nevertheless the results of this section provides considerable reassurance, and might serve as a prototype for other results with similar import.

We study two contexts in which there are two games, one of which is simple, and may be thought of as the social scientist's models, while the other is complex, and may be thought of either as a more complex model, or as the actual world. In the first context the simpler game is obtained by eliminating all pure strategies that are not best responses at any equilibrium in a connected component of the set of Nash equilibria of the complex game. We show that the component has the same index for the simple game and the complex game.

In the second context, for each agent there is a function mapping the set of pure strategies in the complex model onto the set of pure strategies of the simple game. The complex game is an ε-approximation of the simple game if, for every profile of strategies of the complex game, the payoffs in the complex game are within ε of the payoffs of the projection in the simple game. Holding the simple game fixed, the result asserts that for any clopen set of Nash equilibria of the simple game there is an $\varepsilon > 0$ such that if the complex game is an ε-approximation, the set of its equilibria that project onto the given set of equilibria has the same index.

We now present a general result concerning index theory that will be used in the proofs of both of these results. Its proof illustrates typical modes of reasoning based on the index axioms.

Lemma 1. *Suppose that X and Y are ANR's, $C \subset X$ and $D \subset Y$ are compact, $r : C \to D$ and $i : D \to C$ are continuous, and $r \circ i = \mathrm{Id}_D$. Let $F : C \to D$ be an upper hemicontinuous contractible valued correspondence. Then $i \circ F$ and $F \circ i$ are upper hemicontinuous contractible valued correspondences, and if $i \circ F$ and $F \circ i$ are index admissible, then*

$$\Lambda_X(i \circ F) = \Lambda_Y(F \circ i).$$

Proof. Automatically $F \circ i$ has compact, contractible values. Since i is continuous and F has compact values, $i \circ F$ has compact values. If $x \in C$, then $i|_{F(x)}$ and $r|_{i(F(x))}$ are inverse continuous bijections, so $i(F(x))$ is contractible because it is homeomorphic to $F(x)$. Thus $i \circ F$ has contractible values.

If $x \in C$ and $V \subset C$ is a neighborhood of $i(F(x))$, then $i^{-1}(V)$ is a neighborhood of $F(x)$, and the upper hemicontinuity of F gives a neighborhood U of x such that $F(x') \subset i^{-1}(V)$ and thus $i(F(x')) \subset V$ for all $x' \in U$. Thus $i \circ F$ is upper hemicontinuous.

If $y \in D$ and V is a neighborhood of $F(i(y))$, the upper hemicontinuity of F gives a neighborhood U of $i(y)$ such that $F(x) \subset V$ for all $x \in U$, and continuity implies that $i^{-1}(U)$ is a neighborhood of y. Thus $F \circ i$ is upper hemicontinuous.

We now show that for every open neighborhood $V \subset D \times D$ of $\mathrm{Gr}(F \circ i)$ there is a neighborhood $U \subset C \times C$ of $\mathrm{Gr}(i \circ F)$ such that $(i(D) \times i(D)) \cap U \subset (i \times i)(V)$. Aiming at a contradiction, suppose that every such U contained a point in $(i(D) \times i(D)) \setminus (i \times i)(V)$ (which is compact, because it is $(i \times i)((D \times D) \setminus V)$). A limit point of a sequence of such points is a point

$$(x, y) \in \mathrm{Gr}(i \circ F) \cap (i(D) \times i(D)) \setminus (i \times i)(V).$$

There are $w \in D$ such that $i(w) = x$, $z \in F(x)$ such that $i(z) = y$, and $z' \in D$ such that $i(z') = y$, and i is injective, so $z = z'$. Thus $(w, z) \in \mathrm{Gr}(F \circ i)$ and $(i(w), i(z)) = (x, y)$, so $(x, y) \in (i \times i)(\mathrm{Gr}(F \circ i)) \subset (i \times i)(V)$, which is a contradiction.

Now suppose that $i \circ F$ and $F \circ I$ are index admissible. Choose neighborhoods $U \subset C \times C$ of $\mathrm{Gr}(i \circ F)$ and $V \subset D \times D$ of $\mathrm{Gr}(F \circ i)$ as per Continuity. We just showed that by replacing U with a smaller neighborhood we may obtain $(i(D) \times i(D)) \cap U \subset (i \times i)(V)$. Let $U' = U \cap (\mathrm{Id}_C \times (i \circ r))^{-1}(U)$. Now $(i \circ r)|_{i(D)} = \mathrm{Id}_{i(D)}$ and $(\mathrm{Id}_C \times (i \circ r))|_{\mathrm{Gr}(i \circ F)} = \mathrm{Id}_{\mathrm{Gr}(i \circ F)}$. Since U is a neighborhood of $\mathrm{Gr}(i \circ F)$, so is U'.

Proposition 1 guarantees the existence of a continuous $f : C \to C$ with $\mathrm{Gr}(f) \subset U'$, and for such an f we have $\mathrm{Gr}(i \circ r \circ f) \subset U$. In addition

$$\mathrm{Gr}(r \circ f \circ i) = (r \times r)\mathrm{Gr}(i \circ r \circ f|_{i(D)}) = (r \times r)((i(D) \times i(D)) \cap \mathrm{Gr}(i \circ r \circ f))$$

$$\subset (r \times r)((i(D) \times i(D)) \cap U) \subset (r \times r)(i \times i)(V) = V.$$

Therefore

$$\Lambda_X(i \circ F) = \Lambda_X(i \circ r \circ f) = \Lambda_Y(r \circ f \circ i) = \Lambda_Y(F \circ i)$$

where the first and last equalities are from Continuity and the second is from Commutativity. □

Let $G = (S_1, \dots, S_n, u_1, \dots, u_n)$ be a strategic form game. We retain the notation developed in the last section: $N = \{1, \dots, n\}$, $S = S_1 \times \cdots \times S_n$, $\Sigma_i = \Delta(S_i)$, $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$, $B_i : \Sigma \to \Sigma_i$ is agent i's best response correspondence, and $B = B_1 \times \cdots \times B_n$. Let $\tilde{G} = (\tilde{S}_1, \dots, \tilde{S}_{\tilde{n}}, \tilde{u}_1, \dots, \tilde{u}_{\tilde{n}})$ be a second normal form game. Let $\tilde{N} = \{1, \dots, \tilde{n}\}$, and define $\tilde{S}$, $\tilde{\Sigma}_i$, $\tilde{\Sigma}$, $\tilde{B}_i$, and $\tilde{B}$ as above.

We first suppose that $\tilde{n} = n$, that $S_i \subset \tilde{S}_i$ for all i, and that $u_i(s) = \tilde{u}_i(s)$ for all $s \in S$, so that G is obtained from $\tilde{G}$ by eliminating some pure strategies. In the obvious way we regard each Σ_i as a subset of $\tilde{\Sigma}_i$.

Theorem 5. *Suppose that $C \subset \tilde{\Sigma}$ is compact and has no Nash equilibria of $\tilde{G}$ in its boundary, and that $\tilde{u}_i(\tilde{\sigma}_{-i}, \tilde{s}_i) < \tilde{u}_i(\tilde{\sigma})$ for all Nash equilibria $\tilde{\sigma}$ of $\tilde{G}$, all i, and all $\tilde{s}_i \in \tilde{S}_i \setminus S_i$. Let $D = C \cap \Sigma$. Then*

$$\Lambda_\Sigma(B|_D) = \Lambda_{\tilde{\Sigma}}(\tilde{B}|_C).$$

Proof. Additivity implies that we may replace C with any smaller compact neighborhood of the set of Nash equilibria that it contains, so we may assume that $\tilde{u}_i(\tilde{\sigma}_{-i}, \tilde{s}_i) < \tilde{u}_i(\tilde{\sigma})$ for all $\tilde{\sigma} \in C$, all i, and all $\tilde{s}_i \in \tilde{S}_i \setminus S_i$. Then the image of $\tilde{B}|_C$ is contained in Σ.

For each i let $\hat{\Sigma}_i = \tilde{\Sigma}_i \setminus \Delta(\tilde{S}_i \setminus S_i)$ if S_i is a proper subset of $\tilde{S}_i$, and otherwise let $\hat{\Sigma}_i = \tilde{\Sigma}_i = \Sigma_i$. Let $\hat{\Sigma} = \hat{\Sigma}_1 \times \cdots \times \hat{\Sigma}_n$. Any element of $\tilde{\Sigma}_i$ is $\alpha\sigma_i + (1-\alpha)\tau_i$ for a unique $\alpha \in [0,1]$ and some $\sigma_i \in \Sigma_i$ and $\tau_i \in \hat{\Sigma}_i$, and σ_i is unique if $\alpha > 0$. Let $r_i : \hat{\Sigma}_i \to \Sigma_i$ be the map $r_i(\alpha\sigma_i + (1-\alpha)\tau_i) = \sigma_i$, and let $r = (r_1, \ldots, r_n) : \hat{\Sigma} \to \Sigma$. Let $i : \Sigma \to \hat{\Sigma}$ be the inclusion. Of course D contains all the fixed points of $\tilde{B}|_C$, and Additivity implies that $\Lambda_{\tilde{\Sigma}}(\tilde{B}_C)$ is unaffected if we replace C with $C \cap r^{-1}(D)$, so we may assume that $r(C) \subset D$. If we set $F = r \circ \tilde{B}|_C$, then the claim follows from Lemma 1. □

Thus the index is unaffected by the inclusion or exclusion of strictly dominated strategies. Recall that a Nash equilibrium $\tilde{\sigma}$ of $\tilde{G}$ is *strict* if $\tilde{u}_i(\tilde{\sigma}_{-i}, \tilde{s}_i) < \tilde{u}_i(\tilde{\sigma})$ for all i and all $\tilde{s}_i \in \tilde{S}_i$ such that $\tilde{\sigma}_i(\tilde{s}_i) = 0$. Normalization implies that the index of the unique function from a singleton to itself is +1, so this result implies that the index of a strict pure equilibrium is +1. If all the Nash equilibria of $\tilde{G}$ are regular and strict, as is the case for generic payoffs (Harsanyi (1973)), then the index of each is either +1 of -1. Gul et al. (1993) used this and the fact that the sum of the indices is +1 to conclude that for generic $\tilde{G}$ the number of mixed equilibria is not less than the number of pure equilibria minus one.

We now assume that there are surjections $\theta : \tilde{N} \to N$ and $\pi_j : \tilde{S}_j \to S_{\theta(j)}$ for each $j \in \tilde{N}$. Abusing notation, for $i \in N$ let π_i also denote the map $\pi_i : \tilde{\Sigma} \to \Sigma_i$ given by

$$\pi_i(\tilde{\sigma})(s_i) = \sum_{j \in \theta^{-1}(i)} \frac{1}{|\theta^{-1}(i)|} \sum_{\tilde{s}_j \in \pi_j^{-1}(s_i)} \tilde{\sigma}_j(\tilde{s}_i),$$

and let $\pi : \tilde{\Sigma} \to \Sigma$ be the map $\pi(\tilde{\sigma}) = (\pi_1(\tilde{\sigma}), \ldots, \pi_n(\tilde{\sigma}))$. We say that $\tilde{G}$ is an ε-*approximation* of G if

$$\big|\tilde{u}_j(\tilde{s}) - u_{\theta(j)}\big(\pi_j(\tilde{s}_j), \pi_{-\theta(j)}(\tilde{s})\big)\big| < \varepsilon$$

for all $j \in \tilde{N}$ and $\tilde{s} \in \tilde{S}$. That is, an agent's payoff in the complicated game is within ε of the payoff of that agent's image in the simple game when she plays the image pure strategy and the mixed strategies of others are the population averages in the complicated game.

Theorem 6. *Let $D \subset \Sigma$ be compact. If the intersection of D with the set of Nash equilibria of G is contained in $\operatorname{int}(D)$, then there is an $\varepsilon > 0$ such that if $\tilde{G}$ is an ε-approximation of G and $C = \pi^{-1}(D)$, then $\tilde{B}|_C$ is index admissible and*

$$\Lambda_{\tilde{\Sigma}}(\tilde{B}|_C) = \Lambda_\Sigma(B|_D).$$

This result provides some assurance that the index +1 principle is unlikely to contradict itself. For example, it cannot be the case that only one equilibrium of $\tilde{G}$ with index -1 lies above an index +1 equilibrium of G.

Proof. If $\tilde{G}$ is an ε-approximation of G and $\tilde{\sigma}$ is a Nash equilibrium of $\tilde{G}$, then $\pi(\tilde{\sigma})$ is an ε-approximate equilibrium of G in the sense that each agent is achieving an expected utility that is within ε of the optimum. If, for arbitrarily small ε, there was an ε-approximate equilibrium of G in the topological boundary $\partial D = D \setminus \operatorname{int}(D)$ of D, then a limit point of a suitable sequence of such strategy vectors would be a Nash equilibrium in ∂D, contrary to assumption. Therefore there is an $\overline{\varepsilon}$ such that if $0 \le \varepsilon < \overline{\varepsilon}$ and $\tilde{G}$ is an ε-approximation of G, then $\tilde{G}$ has no Nash equilibria in $\pi^{-1}(\partial D)$. For such a $\tilde{G}$ continuity implies that $\pi^{-1}(\operatorname{int}(D))$ is open, so $\pi^{-1}(\partial D)$ contains the topological boundary of C, and consequently $\tilde{B}|_C$ is index admissible.

Fix a particular ε-approximation $\tilde{G} = (\tilde{S}_1, \ldots, \tilde{S}_{\tilde{n}}, \tilde{u}_1, \ldots, \tilde{u}_{\tilde{n}})$ where $0 \le \varepsilon < \overline{\varepsilon}$. For $t \in [0,1]$ let $\tilde{G}^t = (\tilde{S}_1, \ldots, \tilde{S}_{\tilde{n}}, \tilde{u}^t_1, \ldots, \tilde{u}^t_{\tilde{n}})$ where

$$\tilde{u}^t_j(\tilde{s}) = (1-t)\tilde{u}_j(\tilde{s}) + t u_{\theta(j)}(\pi_j(\tilde{s}_j), \pi_{-\theta(j)}(\tilde{s})),$$

and let $\tilde{B}^t$ be defined as above. Then $\tilde{G}^t$ is an $(1-t)\varepsilon$-approximation of G, so $\tilde{B}^t|_C$ is index admissible, and $t \mapsto \tilde{B}^t|_C$ is an IAH. By Homotopy it suffices to show that $\Lambda_{\tilde{\Sigma}}(\tilde{B}^0|_C) = \Lambda_\Sigma(B|_D)$, which is to say that it suffices to establish the claim when $\tilde{G}$ is a 0-approximation, which we now assume.

For each $j \in \tilde{N}$ fix a map $\hat{s}_j : S_{\theta(j)} \to \tilde{S}_j$ such that $\pi_j \circ \hat{s}_j = \mathrm{Id}_{S_{\theta(j)}}$, and let $\hat{S}_j$ be the image of $\hat{s}_j$. Let $\hat{\Sigma}_j$ be the set of $\tilde{\sigma}_j \in \tilde{\Sigma}_j$ such that $\tilde{\sigma}_j(\tilde{s}_j) = 0$ whenever $\tilde{s}_j \notin \hat{S}_j$. For $\tilde{\sigma} \in \tilde{\Sigma}$ let $\hat{B}_j(\tilde{\sigma}) = \tilde{B}(\tilde{\sigma}) \cap \hat{\Sigma}_j$, and let $\hat{B}(\tilde{\sigma}) = \hat{B}_1(\tilde{\sigma}) \times \cdots \times \hat{B}_{\tilde{n}}(\tilde{\sigma})$. There is an obvious homotopy between $\tilde{B}$ and $\hat{B}$ that restricts to an IAH between $\tilde{B}|_C$ and $\hat{B}|_C$, so $\Lambda_{\tilde{\Sigma}}(\tilde{B}|_C) = \Lambda_{\tilde{\Sigma}}(\hat{B}|_C)$.

Let $\iota : \Sigma \to \tilde{\Sigma}$ be the linear extension of the map $s \mapsto (\hat{s}_j(s_{\theta(j)}))$. Evidently $\pi \circ \iota = \mathrm{Id}_\Sigma$. Let $i = \iota|_D$ and $r = \pi|_C$. Then $B|_D = B|_D \circ r \circ i$ and $\hat{B}|_C = i \circ B|_D \circ r$, so Lemma 1 (with $F = B|_D \circ r$) implies that $\Lambda_{\tilde{\Sigma}}(\hat{B}|_C) = \Lambda_\Sigma(B|_D)$. □

5 Experimental Evidence

We now describe experimental research related to the index +1 principle, which is of two sorts. First, although there have been no studies in which the index +1 principle was itself the central issue, there have been a number of studies of games and markets with several equilibria, some of which have an index different from +1. Here we look for direct confirmation or violation of the index +1 principle. Second, many authors have studied the processes of adjustment to equilibrium in considerable detail, and we can ask whether the models they brought to this data, and the experimental findings, support the hypothesis that, in practice, processes of adjustment to equilibrium are natural in the senses considered in Section 3.

Before proceeding to this literature, we should first of all note that there are no anecdotal or historical examples in which unstable equilibria are or were self-reproducing, at least so far as I am aware. Such an example would be quite surprising and counterintuitive, not just for a mathematically trained social scientist, but also for ordinary people, which suggests that an expectation of stability is part of our instinctive or commonsensical understanding of how the world works.

In deciding what might constitute a test of the index +1 principle, it is first of all necessary to confront the fact that Nash equilibrium does a poor job of describing human behavior in all but the simplest settings. In part this is true even for decision problems, but in games the complexity of reasoning about the reasoning, and reliability, of other agents, quickly becomes overwhelming as the size of the game increases. For this reason a compelling counterexample to the index +1 principle would necessarily be quite simple. In the overview of literature below, some studies have been excluded on this basis. It is certainly possible that others that should have been included have, regrettably, been overlooked.

In a *coordination game* all agents have the same set of pure strategies, and maximal payoffs are attained if all agents play the same pure strategy, so for each agent, each pure strategy is the unique best response if all other agents are playing that pure strategy. The best known example is the battle of the sexes.

Consider a coordination game in which two agents each have the same three pure strategies. Each player receives one dollar if both players choose the same pure strategy, and she receives nothing if they fail to coordinate. This game has three pure equilibria, three equilibria in which each agent assigns equal probability to two of the pure strategies and no probability to the third, and a totally mixed equilibrium in which each agent assigns probability one third to each pure strategy. Each of the three pure equilibria is strict and therefore has index +1. Each of the three equilibria that mix over two pure strategies has the same index in the game obtained by eliminating the third pure strategy (by Theorem 5) which is -1 because the reduced game has three Nash equilibria, and the two pure equilibria each have index +1. Since the sum of the indices is +1, the totally mixed equilibrium has index +1. The totally mixed equilibrium is evidently not dynamically stable, which illustrates an important point: *the index +1 principle does not exhaust the consequences of dynamic stability.* (It is also an additional example of the divergence between strategic stability and dynamic stability.) More generally, for the coordination games considered in experimental studies to date only the pure equilibria are stable with respect to natural dynamics, and these studies test not only the index +1 principle, but also the stronger hypothesis that only pure equilibria are self-reproducing outcomes.

This literature already brings forth a concern that pervades the experimental evidence, as it relates to the index +1 principle: the number of rounds is often quite small, so a failure to converge does not imply that there will not eventually be convergence, nor can it be taken as strong evidence of convergence to a mixed equilibrium. In the experiments of Cox et al. (2001) testing the Jordan (1991) model of Bayesian learning there is frequent convergence to pure equilibria of the battle of the sexes when the same players are matched repeatedly, but not when players are rematched randomly. The number of rounds (15) seems insufficient to allow nonconvergence to be construed as evidence of convergence to the mixed equilibrium. In Cooper et al. (1990) seven cohorts of eleven players played seven different coordination games. Each cohort played 22 rounds in which 10 players were matched with each other and one player sat out. Each player played against each other player twice, but the orderings of the matchups were random. In each of the seven cohorts play converged to a pure Nash equilibrium, which was sometimes Pareto dominated, and the selected equilibrium seemed to be affected by varying the payoffs of a dominated pure strategy. In van Huyck et al. (1990) agents played a coordination game with a reward for the group's minimum effort and a lesser penalty, or no penalty, for individual effort. For games with large numbers of players play converged to the minimum effort equilibrium when individual effort was penalized and to the maximum effort equilibrium when it was not. When players were paired, either permanently or randomly in each period, and played the two player version with effort penalty, played converged mostly to the efficient equilibrium but sometimes to the minimum effort equilibrium. In all cases there was a strong tendency to converge to some pure Nash equilibrium.

We now describe two studies of games in which there are multiple equilibria. In neither of them do we see convergence to an equilibrium with an index other than +1.

Friedman (1996) (see also Bouchez and Friedman (2008)) is an experimental study of a variety of games, including some with multiple equilibria. The results are generally supportive of the hypothesis that play will not converge to unstable equilibria. (The authors suggest that apparent convergence to an unstable mixed equilibrium may actually be slow divergence away from it, which is consistent with the small run lengths (typically 10 or 16 rounds) and the limited information provided to the participants in several treatments.) This study included a single population Hawk-Dove-Bourgeois game with two equilibria, one of which is stable, so that the index of the other is zero. The combined data from this paper and subsequent experiments is described by Bouchez and Friedman (2008) as follows: "Loose (tight) convergence was found to some BE [behavioral equilibrium] in 41 (7) of 46 half-runs, loose (tight) convergence to the EE [evolutionary equilibrium] in 8 (3) half runs, and no loose or tight converge was found to the edge [index zero] equilibrium in spite of its large area."

Cox and Walker (1998) is an experimental study of two Cournot duopoly games, one of which has a unique equilibrium in which both firms' quantities are positive, while the other has three equilibria, two of which are monopolistic and have index +1, while the third, in which both firms have positive production, has index -1. In all but one experiment 20 subjects were divided into two groups, and in each of 30 market periods there was a random matching of the members of the two groups, with each pair playing the duopoly game. Play generally converged to one of the equilibria (even though the data was inconsistent with the particular learning models discussed in the paper) but the index -1 equilibrium was not observed. "The results of the experiments ... strongly suggest that the theoretical stability properties of a Nash equilibrium can serve as an effective 'refinement,' distinguishing equilibria that subjects can be expected to play from those that they generally will not play."

In a *signalling game* a Sender is informed of his *type* and chooses a *message*, after which a Receiver (who has prior beliefs concerning the type, but sees only the message) chooses an *action*. Payoffs depend jointly on the type, message and action. In the example[6] shown in Figure 3 the Sender is weak with probability 0.1 and strong with

[6] The numbers are taken from Cho and Kreps (1987). The story has been adjusted to avoid cultural references that might be incomprehensible now.

probability 0.9. The Receiver would like to attack if the Sender is weak and withdraw if the Sender is strong. The Sender's message is the choice of whether to listen to the blues, which he prefers if he is strong, or to classical music, which he prefers if he is weak. The Sender's utility is the sum of 2 or 0 utils, according to whether a fight is avoided, and 1 or 0 utils, according to whether he listens to his favorite music.

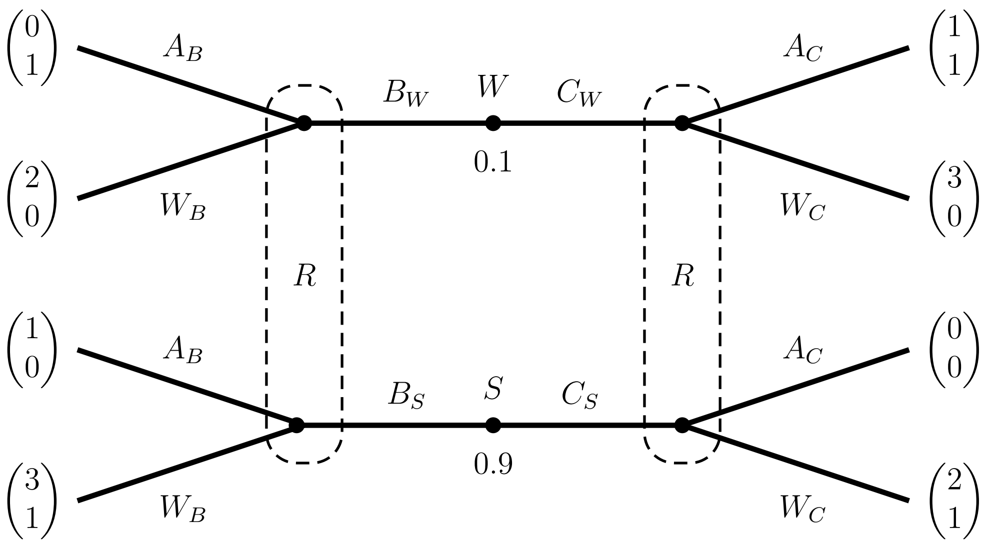

Figure 3: Cho and Kreps' Beer-Quiche Game

A sequential equilibrium for this game consists of mixed strategies for the two types of Senders and for the Receiver at each of the two information sets, together with beliefs at the two information sets. The behavior of the two Sender types must be optimal taking the Receiver's behavior as given, the behavior of the Receiver must be optimal taking the beliefs as given, and the beliefs must be given by Bayesian updating when it is well defined.

First consider sequential equilibria in which the strong Sender always listens to the blues. Bayesian updating requires that when she hears the blues, the Receiver believes that the Sender is strong with probability at least 0.9, so she will not attack. If the weak Sender sometimes listened to classical music, the Receiver would believe that the Sender was certainly weak when she heard classical music, and the weak Sender would be better off switching to the blues. Therefore the weak Sender always listens to the blues, the Receiver's beliefs after hearing classical music must justify attacking, and she must assign enough probability to attacking to deter the weak Sender. The set of such equilibria is

$$\{\, (B_W, B_S, W_B, \alpha A_C + (1-\alpha) W_C, 0.1W + 0.9S, \beta W + (1-\beta)S) :$$

$$\alpha = 1 \text{ and } \beta \geq 0.5, \text{ or } \alpha \geq 0.5 \text{ and } \beta = 0.5 \,\}.$$

(Here the components are the weak and strong Sender's strategies, the Receiver's strategies after hearing the blues and hearing classical music, and the Receiver's beliefs after hearing the blues and hearing classical music.)

Now consider sequential equilibria in which the strong Sender sometimes listens to classical music. The probability of an attack after classical music must be less than the probability of an attack after the blues, so the weak Sender always listens to classical music. Therefore the strong Sender always listens to classical music (otherwise Bayesian updating would assign all probability to the strong Sender after the blues) the Receiver's beliefs after hearing the blues must justify attacking, and she must assign enough probability to attacking to deter the strong Sender. The set of such equilibria is

$$\{\, (C_W, C_S, \gamma A_B + (1-\gamma)W_B, W_C, \delta W + (1-\delta)S, 0.1W + 0.9S) :$$

$$\gamma = 1 \text{ and } \delta \geq 0.5, \text{ or } \gamma \geq 0.5 \text{ and } \delta = 0.5 \,\}.$$

The second set vanishes if we perturb the best response correspondence by requiring both types of Sender to listen to each type of music with at least probability ε: the Receiver will play W_B unless the weak Sender is assigning more than probability ε to B_W, and any pair of mixed strategies for the Receiver that makes B_W a best response for the weak Sender will make B_S the only best response for the strong Sender. Since small perturbations of the best response correpondence eliminate the second set of equilibria, its index (as described in Section 2) is zero, so the index of the first set of sequential equilibria is +1.

Kohlberg and Mertens (1986), Cho and Kreps (1987), and Banks and Sobel (1987) propose a range of equilibrium refinements that select the first set of equilibria. In each case an existence theorem for the refinement is proved by showing that certain sequences of perturbations of the best response correspondence do not permit sequences of approximate equilibria that converge to an equilibrium violating the refinement. The points of view of the various pairs of authors are somewhat different, but in each case the motivation appeals to strategic intuition. In an equilibrium from the second set the weak Sender cannot possibly benefit by deviating to B_W, but the strong Sender can benefit from deviating to B_S if the Receiver then believes that the Sender is strong. Such a belief is well motivated by a consideration of the Sender's incentives, and the strong Sender will deviate if he expects the Receiver to reason along these lines. For this reason equilibria in the first (second) set are said to be *intuitive* (*unintuitive*).

Banks et al. (1994) use signalling games to test the refinements experimentally. The refinements can be ordered according to strength, and their experiments use a sequence of signalling games that discriminate between each solution concept and the next strongest refinement. In each case the equilibrium satisfying the stronger refinement has index +1. On the whole the data tend to support the refinements, but there is considerable out-of-equilibrium play. Even in the longest run lengths (20 rounds) there is little evidence of convergence. Although signalling games are perhaps quite simple in a mathematical sense, they certainly stress human cognition. In one paper in this literature the authors, to their credit, mention that a referee pointed out that their analysis of one of their games was incorrect. Other authors use software to check their analyses.

Brandts and Holt (1992, 1993) are able to induce the unintuitive equilibrium by manipulating the parameters of the game and the conditions of the experiment in certain ways. Their main conclusion is that equilibrium selection is strongly affected by the dynamics of adjustment. Paltrow and Schotter (1993) run the same experiments except that the subjects do not see the other player's payoff, with similar results. Anderson and Camerer (2000) replicate some of the results of Brandts and Holt (1992), but also extend the run length to 30 rounds. They found clear evidence that lengthening the run length leads to convergence in the treatment that induces the intuitive equilibrium, but in the treatment inducing the unintuitive equilibrium, the out-of-equilibrium message continued to be chosen at about the same rate as in the 9^{th} through 12^{th} periods, suggesting that substantially greater run lengths might eventually result in divergence from the unintuitive equilibrium, followed by convergence to the intuitive equilibrium.

It is important to note that if the unintuitive equilibrium is in fact an example of an equilibrium with index 0 that can be a self-reproducing outcome, it is possible precisely because the dynamic adjustment process leading away from that equilibrium can be shut down. The Receiver might explain this concretely as follows: "Perhaps a strong Sender sometimes imagines that their deviation sends a message about their type, but deviators are always ill informed or confused about the game, or the equilibrium in effect, or they simply had an accident. In practical experience such people tend to be weak. On those rare occasions when I hear the blues, my attacks succeed more often than not. That's how it's always been, and I don't expect things to change." In this sense this example reminds us that the index +1 principle is not a "law," and that its conclusion may fail in examples that do not conform to its underlying logic.

We now consider market examples. Gale (1963) presents an example of a symmetric two good exchange economy with three equilibria. The interior equilibrium is egalitarian but unstable (index -1) while the two boundary equilibria are stable (index +1) but highly inegalitarian. Crockett et al. (2011) implements this example experimentally, finding (contrary to the prior expectations of some of the authors) "robust evidence that prices in laboratory Gale economies resist the interior equilibrium and march upward or downward toward the corner equilibria."

The Walrasian (tatonnement) and Marshallian models of market adjustment in a partial equilibrium setting were pitted against each other experimentally by Plott and George (1992), Plott and Smith (1999), and Plott (2000). In the Walrasian model the variable being adjusted is price: if the amount demanded at the current price exceeds (is less than) the amount supplied, the price increases (decreases). In the Marshallian model the variable of equilibration is quantity: if the price at which consumers are willing to consume the current quantity is less (more) than the price at which suppliers are willing to provide it, then the quantity decreases (increases). An equilibrium at which the supply curve is upward sloping and the demand curve is downward sloping is stable for both dynamics. However, if the signs of the two curves' slopes are the same, then the models produce opposite predictions concerning stability.

Thus a setting that contrasts the Walrasian and Marshallian models is necessarily one in which the supply curve slopes down or the demand curve slopes up. Plott and George (1992) use externalities in production to induce downward sloping supply, Plott and Smith (1999) used externalities in consumption to induce upward sloping demand, and Plott (2000) used income effects to induce downward sloping (that is, backward bending) supply of labor. The experimental results in Plott and George (1992) and Plott and Smith (1999) strongly support the Marshallian model, but the data in Plott (2000) support the Walrasian model with at least equal strength. The models in these papers are not general equilibrium models, the equilibria are not fixed points, and in fact Marshallian dynamics is not even defined in a general equilibrium setting. Thus these studies are not close to being direct tests of the index $+1$ principle. Nevertheless, collectively they cast doubt on how the relationship between naive dynamics and stability is conventionally understood, and they raise many questions.

There is an extensive experimental literature that attempts to understand dynamic adjustment in games and markets, so we can ask what light it sheds on the question of whether adjustment dynamics are natural, in the sense that the agents in games adjust their strategies in directions that increase payoffs, and in markets price adjustment is not contrary to excess demand. Perhaps the first point to emphasize about this literature is that it seems to have never considered the possibility that dynamic adjustment might *not* be natural. I have not found any studies that claim that data exhibits unnatural dynamics, but there seem to be no studies that attempt to induce unnatural dynamics.

What we can hope to find in this literature is confirmation that actual adjustment does resemble natural processes such as best response dynamics and tatonnement. There are specific models such as Champsaur and Cornet (1990), Camerer and Ho (1999), Ho et al. (2007), Arifovic and Ledyard (2011), Bossaerts and Ledyard (2013), and Arifovic and Ledyard (2015) that have been advanced as candidates to fit experimental data, and, to a greater or lesser extent, some of these studies do achieve a good fit. (I am not competent to assess such claims in any detail.) Selten (1991) proposes a model of strategic adjustment in which each of two populations first computes best response dynamics of the opponent population, then adjusts in the direction of best response to this anticipation. Tang (2001) tests this experimentally, comparing a game that is stable with respect to the Selten model, but not for best response dynamics, with a game that is unstable for both dynamics. Play does not converge to the unique equilibrium in either case, but comes closer for the game that is stable for the Selten model.

Without committing to a particular model, we can ask whether experimental data exhibits the qualitative features predicted by models such as tatonnement. The two features we consider are nonconvergence when the unique equilibrium is unstable and cycling.

There is considerable experimental evidence that instability of naive (tatonnement or best response) dynamics can lead to nonconvergence. Anderson et al. (2004) and Hirota et al. (2005) are experimental implementations of the Scarf (1960) example of an exchange economy with a unique equilibrium that is unstable with respect to tatonnement. They produce extensive evidence that actual price dynamics in double oral auctions are well modelled by tatonnement. In particular, the unique equilibrium is not stable. Goeree and Lindsay (2012) replicate Anderson et al. (2004) and also show that the equilibrium is achieved in experiments using a different market mechanism in which agents submit demand schedules. Chen and Tang (1998), Healy (2006), and Van Essen et al. (2010) find that the Walker (1981) mechanism implementing Lindahl equilibrium does not converge to the unique equilibrium, at least within the number of periods allowed in the experiments. This mechanism was shown to be unstable by Kim (1987).

It is common for games, including even quite simple ones, to have best response dynamics that are cyclic. Cason et al. (2003) find clear cycles in the data of Cason and Friedman (2003). Xu and Wang (2011) conducted experiments in which two populations of 8 subjects played 300 rounds of the game Coyness and Philandering, with random rematching after each round, finding that population averages followed cyclic dynamics. Xu et al. (2014) analyze experimental data concerning 2×2 games from Selten and Chmura (2008), again finding evidence of cycling.

Until recently there had been little experimental work on Rock-Paper-Scissors with human subjects (Semmann et al. (2003) is perhaps the earliest study) but there have been several studies in recent years. In a generalized version one can vary the ratio of the benefit of winning (relative to a tie) to the loss from losing (again relative to a tie). As Hoffman et al. (2015) explain the unique Nash equilibrium is stable (unstable) for best response dynamics if the benefit of winning exceeds (is less than) the loss from losing. (See also Example 3.3.2 and pp. 351–356 of Sandholm (2010) and the illustrations on p. 759 of Sandholm (2015).) They find that aggregate behavior is further from the Nash equilibrium in the unstable case. Cason et al. (2010) study a variant with an additional strategy called Dumb that is never a best response. Like Hoffman et al. (2015), but unlike Semmann et al. (2003), they find no evidence of cycling even though the time average of a cycle provides a somewhat better description of aggregate behavior than the Nash equilibrium. Using different software that provides visual information to subjects and allows continuous time play, Cason et al. (2014) find clear and persistent cycles. Very recently Xu et al. (2013), Wang and Xu (2014), and Wang et al. (2014) produced persistent cycles in discrete time, and Frey and Goldstone (2013) found cyclic behavior in discrete time play of a variant of RPS.

Models of strategic adjustment are supported by data that respond in the expected manner to changes in factors that have been theoretically identified as relevant. Chen and Gazzale (2004) find faster and and tighter convergence to the unique Nash equilibrium when a parameter of a game is set in a range that makes the game supermodular.

Summing up, the experimental evidence supporting the index +1 principle seems quite strong. In games with multiple equilibria there is no evidence of convergence to equilibria with indices different from +1. Possibly in the signalling game of Figure 3 the index 0 equilibrium can persist if the system is induced to start there, but this is easily understood in terms of the underlying dynamics. Although dynamic adjustment processes such as tatonnement and best response dynamics are inconsistent with the rational expectations hypotheses, experimental data is in accord with such processes, at least roughly, and qualitative features predicted by these models have been found in many studies.

6 Samuelson's Correspondence Principle

The correspondence principle was described by Paul Samuelson in two articles (Samuelson (1941, 1942)) and his famous *Foundations of Economic Analysis* (Samuelson (1947)) after being stated informally by Hicks (1939). The idea can be illustrated in a two good exchange economy. Figure 4A shows the excess demand for the second good as a function of the second good's price, when the first good is the numeraire. There are three equilibria, two of which are stable relative to price dynamics that increase (decrease) the price of the second good when it is in excess demand (supply).

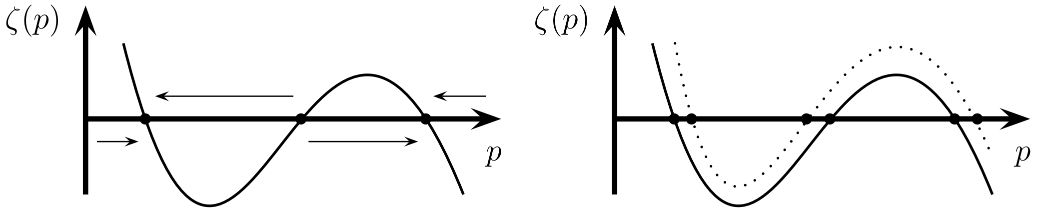

FIGURE 4A. EXCESS DEMAND AND DYNAMICS

FIGURE 4B. INCREASED DEMAND FOR THE SECOND GOOD

FIGURE 4. EXCESS DEMAND AND COMPARATIVE STATICS

Figure 4B shows the effect of changing a parameter in a way that increases demand for the second good. This has the expected effect of increasing the second good's equilibrium price for the two stable equilibria, but it leads to a price decrease in the unstable equilibrium. In a nutshell, Samuelson's understanding of the correspondence principle was that dynamic stability had implications for comparative statics.

In this example the correspondence principle combines three elements: a) equilibria that are unstable with respect to natural dynamics will not be observed; b) therefore excess demand is downward sloping at the equilibria that are empirically relevant; c) this allows us to sign certain comparative statics. The first two of these are the 1-dimensional case of the index $+1$ principle.

We now quickly review the mathematics of comparative statics. Consider a system of equations $0 = g(x, \alpha) \in \mathbb{R}^n$ where $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Here we are thinking of $g(\cdot, \alpha)$ as the function $\mathrm{Id} - f(\cdot, \alpha)$, where $f(\cdot, \alpha)$ is the function whose fixed points are the equilibria for parameter α. Fix an initial value α^* of the parameter and an equilibrium x^* for this parameter, let M be the matrix of partial derivatives $\frac{\partial g_i}{\partial x_j}(x^*, \alpha^*)$, and let Δ be its determinant. Differentiating the equation $g(x(\alpha), \alpha) = 0$ gives the equation

$$M \cdot \frac{dx}{d\alpha} + \frac{\partial g}{\partial \alpha} = 0$$

where $\frac{dx}{d\alpha}$ is the vector with entries $\frac{dx_j}{d\alpha}(x^*)$ and $\frac{\partial g}{\partial \alpha}$ is the vector with entries $\frac{\partial g_i}{\partial \alpha}(x^*, \alpha^*)$. Let Δ_{ij} be the (i, j)-cofactor of M, which is the determinant of the matrix obtained by eliminating the i^{th} row and the j^{th} column of M. Then the (i, j)-entry of the inverse of M is Δ_{ji}/Δ, so we obtain the formula

$$\frac{dx_j}{d\alpha} = -\frac{\sum_{i=1}^n \Delta_{ji} \frac{\partial g_i}{\partial \alpha}}{\Delta}.$$

In order to sign this quantity one needs to sign both the numerator and the denominator. The index +1 principle is the condition $\Delta > 0$. One way in which the numerator can be signed is that M is a (positive or negative) definite matrix, so that each Δ_{ij} has a known sign, and that the $\frac{\partial g_i}{\partial \alpha}$ all have the same sign. Another possibility is that for some i (most commonly $i = j$) it is known a priori that $\frac{\partial g_k}{\partial \alpha} = 0$ for all $k \neq i$, and that $\frac{\partial g_i}{\partial \alpha}$ and Δ_{ji} have definite signs. As the dimension n increases, the sorts of assumptions that have such implications become quite complex and restrictive. All of the concrete examples given by Samuelson are low dimensional. Basset et al. (1968) analyze many additional special cases, and provide an overview of related research.

Samuelson saw the correspondence principle as an initial insight gleaned from the development of a dynamic approach to economic analysis, whose relationship to static models was comparable to the relationship between static and dynamic analysis in physics: "An understanding of this principle is all the more important at a time when pure economic theory has undergone a revolution of thought—from statical to dynamical modes." (*Foundations of Economic Analysis*, p. 284).

Researchers in general equilibrium theory (e.g., Arrow and Hurwicz (1958); Arrow et al. (1959)) found some special cases in which some equilibria are necessarily stable with respect to Walrasian tatonnement dynamics, but examples developed by Scarf (1960) showed that this phenomenon is restricted to very small numbers of goods or agents. After summarizing this line of research, Arrow and Hahn expressed the following negative assessment (*General Competitive Analysis*, p. 321):

> Thus what the "correspondence principle" amounts to is this: Most of the restrictions on the form of the excess-demand functions that are at present known to be sufficient to insure global stability are also sufficient to allow certain exercises in comparing equilibria. It should be added that these same conditions also turn up in the discussion of the uniqueness of a competitive equilibrium. All these restrictions share the characteristic that they are not necessary for the task for which they were invented; they are only sufficient and this explains why the correspondence principle "isn't."

Over the last half century the correspondence principle has appeared only rather infrequently in theoretical economics. In addition to the work of Basset et al. (1968) already mentioned, a variant of the correspondence principle for certain international trade models is developed in Samuelson (1971), Bhagwati et al. (1987), and Kemp et al. (1990), and a version of the correspondence principle for dynamic optimization problems arises in the work of Burmeister and Long (1977), Brock (1977), and Magill and Scheinkman (1979). Perhaps the most important work is due to Echenique (2002, 2004) (see also Echenique (2008)) who works with games with strategic complementarities

(Topkis (1979); Vives (1990); Milgrom and Roberts (1990)) that satisfy very strong order theoretic conditions, but are otherwise technically quite general. This is part of the literature (surveyed by Amir (2005)) that now constitutes our most general understanding of monotone comparative statics.

In comparing Samuelson's perspective with ours, three broad points emerge.

Samuelson wrote during the heyday of logical positivism, which attempted to understand science as a collection of testable hypotheses. While economics clearly provided some large scale overview of its subject matter, the discipline was somewhat embarrassed to find itself with a paucity of concrete predictions that could be applied to available data sets. For this reason great conceptual importance was attached to the method of comparative statics. Since the time he wrote empirical methods in economics have become enormously more sophisticated, and game theory provides an entirely different set of testable hypotheses.

Although dynamic models have proliferated in economics, these are for the most part models of static equilibria that unfold over time. There is no evidence that Samuelson (at the time he wrote *Foundations of Economic Analysis*) understood the distinction between this sort of dynamic model and a model of dynamic adjustment to equilibrium. Samuelson's expectation that economic theory would develop through the interplay of dynamics and statics was, in this sense, largely unfulfilled.

The index +1 principle successfully generalizes parts a) and b) of the correspondence principle (as we describe it above) to multiple dimensions. Its utility in comparative statics may be rather slight, but it is applicable across the much broader range of economic models that are available now, with a variety of important consequences. In this sense Arrow and Hahn were excessively dismissive.

7 Concluding Remarks

Samuelson's vision of a theory of economic dynamics, in which each aspect of dynamic analysis would have a corresponding implication for static analysis, did not come to pass. Dynamic models have proliferated, but for the most part those that are thought to respect the agents' rationality are not models of strategic adjustment to equilibrium, but are instead models of essentially static equilibria that play out over time. Nevertheless, the index +1 principle is a successful multidimensional extension of the key properties of the particular examples Samuelson examined.

The index +1 principle is universally applicable to economic models in which equilibria are topological fixed points, because it is rooted in the axiomatic characterization of the fixed point index. As many of the arguments in this paper illustrate, the index axioms often allow simple and direct proofs of very general results. In an important sense the (quite lengthy) proof of Theorem 1 encompasses much of the technical nitty gritty that often burdens concrete discussions of the index, and the index axioms are a supple and powerful distillate.

The index +1 principle is well behaved with respect to inclusion/exclusion of strictly dominated strategies, and also with respect to comparison of a simple model with a complex reality that approximates it. These results suggest that the index +1 principle identifies a feature of economic reality, rather than an artifact of otherwise insignificant modelling choices, and they provide a prototype for more results in this direction.

The underlying justification of the index +1 principle is that equilibria that do not satisfy it are unstable with respect to *any* natural adjustment dynamics. Insofar as the only process of adjustment to equilibrium that is consistent with the rational expectations hypothesis is to go to the equilibrium immediately, the hypothesis that actual adjustment processes are natural is necessarily to some extent behavioral. Nevertheless, experimental evidence provides quite strong support for the index +1 principle, and for natural adjustment dynamics.

Thus the index +1 principle should be seen as a reliable refinement of equilibrium across the entire range of economic models in which equilibria are topological fixed points. Its significance in comparative statics is rather slight, because achieving definite signs for responses to changes in exogenous parameters requires many additional restrictions when there are several endogenous variables. But the toolbox of empirical strategies is much more varied than when Samuelson wrote, experiments allow data to be customized, and game theory has brought many new perspectives into economic analysis. The index +1 principle is, in a sense, only one bit of information, but it surely has many interesting and useful consequences.

References

Amir, R. (2005). Supermodularity and complementarity in economics: An elementary survey. *Southern Economic Journal*, 71:636–660.

Anderson, C. M. and Camerer, C. F. (2000). Experience-weighted attraction learning in sender-receiver signalling games. *Economic Theory*, 16:689–718.

Anderson, C. M., Plott, C., Shimomura, K.-I., and Granat, S. (2004). Global instability in experimental general equilibrium: The Scarf example. *Journal of Economic Theory*, 115:209–249.

Arifovic, J. and Ledyard, J. (2011). A behavioral model for mechanism design: Individual evolutionary learning. *Journal of Economic Behavior & Organization*, 78:374–395.

Arifovic, J. and Ledyard, J. (2015). Repeated Battle of Sexes: Experimental evidence and individual evolutionary learning. HSS Working Paper, Caltech.

Arrow, K. J., Block, H. D., and Hurwicz, L. (1959). On the stability of the competitive equilibrium, II. *Econometrica*, 27:82–109.

Arrow, K. J. and Hurwicz, L. (1958). On the stability of the competitive equilibrium, I. *Econometrica*, 26:522–552.

Balkenborg, D. and Vermeulen, D. (2016). Where strategic stability and evolutionary stability depart—a study of minimal diversity games. *Math. Oper. Res.*, 41:278–292.

Banks, J. S., Camerer, C., and Porter, D. (1994). An experimental analysis of Nash refinements in signalling games. *Games and Economic Behavior*, 6:1–31.

Banks, J. S. and Sobel, J. (1987). Equilibrium selection in signalling games. *Econometrica*, 55:647–661.

Basset, L., Maybee, J., and Quirk, J. (1968). Qualitative economics and the scope of the correspondence principle. *Econometrica*, 36:544–563.

Ben-Porath, E. and Dekel, E. (1992). Signaling future actions and the potential for sacrifice. *Journal of Economic Theory*, 57:36–51.

Benaïm, M. (1998). Recursive algorithms, urn processes and chaining number of chain recurrent sets. *Ergod. Th. and Dynam. Sys.*, 18:53–87.

Benaïm, M. (1999). Dynamics of stochastic approximation algorithms. In Azéma, J., Émery, M., Ledoux, M., and Yor, M., editors, *Séminaire de Probabilités XXXIII*, volume 1709 of *Lecture Notes in Mathematics*, pages 1–68. Springer, Berlin.

Benaïm, M., Hofbauer, J., and Sorin, S. (2005). Stochastic approximation and differential inclusions. *SIAM J. Control Optim.*, 44:328–348.

Benaïm, M. and Weibull, J. W. (2003). Deterministic approximation of stochastic evolution in games. *Econometrica*, 71:873–903.

Bhagwati, J. N., Brecher, R. A., and Hatta, T. (1987). The global correspondence principle: A generalization. *American Economic Review*, 77:124–132.

Blume, L. and Zame, W. (1993). The algebraic geometry of competitive equilibrium. In Neuefeind, W., editor, *Essays in General Equilibrium and International Trade: In Memorium Trout Rader*, pages 53–66. Springer-Verlag, New York.

Bochnak, J., Coste, M., and Roy, M.-F. (1987). *Géométrie Algébrique Réelle*. Springer-Verlag, New York.

Bossaerts, P. and Ledyard, J. (2013). Price formation in continuous double auctions; with implications for finance. HSS Working Paper, Caltech.

Bouchez, N. and Friedman, D. (2008). Equilibrium convergence in normal form games. In Plott, C. R. and Smith, V. L., editors, *Handbook of Experimental Economics Results, Volume 1*, pages 472–480. North Holland, Amsterdam.

Brandts, J. and Holt, C. A. (1992). An experimental test of equilibrium dominance in signalling games. *American Economic Review*, pages 1350–1365.

Brandts, J. and Holt, C. A. (1993). Adjustment patterns and equilibrium selection in experimental signalling games. *International Journal of Game Theory*, pages 279–302.

Brock, W. A. (1977). A revised version of Samuelson's correspondence principle: Applications of recent results on the asymptotic stability of optimal control to the problem of comparing long run equilibria. In Sonnenschein, H. F., editor, *Models of Economic Dynamics*, volume 264 of *Lecture Notes in Economics and Mathematical Systems*, pages 86–116. Springer-Verlag.

Browder, F. (1948). *The Topological Fixed Point Theory and its Applications to Functional Analysis*. PhD thesis, Princeton University.

Brown, J. and von Neumann, J. (1950). Solutions of games by differential equations. In *Contributions to the Theory of Games*, Annals of Mathematics Studies, no. 24, pages 73–79. Princeton University Press, Princeton, N. J.

Brown, R. (1971). *The Lefschetz Fixed Point Theorem*. Scott Foresman and Co., Glenview, IL.

Burmeister, E. and Long, N. V. (1977). On some unresolved questions in capital theory: An application of samuelson's correspondence principle. *Quarterly Journal of Economics*, 91:289–314.

Camerer, C. and Ho, T.-H. (1999). Experience weighted attraction learning in normal form games. *Econometrica*, 67:827–874.

Cason, T. N. and Friedman, D. (2003). Buyer search and price dispersion: A laboratory study. *Journal of Economic Theory*, 112:232–260.

Cason, T. N., Friedman, D., and Hopkins, E. (2010). Testing the TASP: An experimental investigation of learning in games with unstable equilibria. *Journal of Economic Theory*, 145(6):2309–2331.

Cason, T. N., Friedman, D., and Hopkins, E. (2014). Cycles and instability in a Rock-Paper-Scissors population game: A continuous time experiment. *Review of Economic Studies*, 81(1):112–136.

Cason, T. N., Friedman, D., and Wagener, F. (2003). The dynamics of price dispersion or Edgworth variations. *Journal of Economic Dynamics and Control*, 29:801–822.

Cellina, A. (1969). A theorem on the approximation of compact multi-valued mappings. *Rendiconti delle Sedute della Accademia Nazionale Dei Lincei*, 47:429–433.

Cellina, A. (1970). A further result on the approximation of set valued functions. *Rendiconti delle Sedute della Accademia Nazionale Dei Lincei*, 48:12–16.

Champsaur, P. and Cornet, B. (1990). Walrasian exchange processes. In Gabszewicz, J., Richard, J.-F., and Wolsey, L., editors, *Economic Decision Making: Games, Econometrics, and Optimization*, pages 1–13. Elsevier, Amsterdam.

Chen, Y. and Gazzale, R. (2004). When does learning in games generate convergence to Nash equilibria? The role of supermodularity in an experimental setting. *American Economic Review*, 94:1505–1535.

Chen, Y. and Tang, F.-F. (1998). Learning and incentive-compatible mechanisms for public goods provision: An experimental study. *Journal of Political Economy*, 106:633–662.

Cho, I.-K. and Kreps, D. M. (1987). Signalling games and stable equilibria. *Quart. J. Econom.*, 102:179–221.

Cooper, R. W., DeJong, D. V., Forsythe, R., and Ross, T. W. (1990). Selection criteria in coordination games: Some experimental results. *American Economic Review*, 80:218–233.

Cox, J. C., Shachat, J., and Walker, M. (2001). An experiment to evaluate Bayesian learning of Nash equilibrium play. *Games and Economic Behavior*, 34:11–33.

Cox, J. C. and Walker, M. (1998). Learning to play Cournot duopoly strategies. *Journal of Economic Behavior and Organization*, 36:141–161.

Cressman, R. (2003). *Evolutionary Dynamics and Extensive Form Games*. MIT Press, Cambrige MA.

Crockett, S., Opera, R., and Plott, C. (2011). Extreme Walrasian dynamics: The Gale example in the lab. *American Economic Review*, 101:3196–3220.

Demichelis, S. and Ritzberger, K. (2003). From evolutionary to strategic stability. *J. Econom. Theory*, 113:51–75.

Dierker, E. (1972). Two remarks on the number of equilibria of an economy. *Econometrica*, 40:951–953.

Dierker, E. (1974). *Topological Methods in Walrasian Economics*. Lecture Notes in Economics and Mathematical Systems, No. 72. Springer Verlag, Berlin-Heidelberg-New York.

Dixon, H. (1990). Equilibrium and explanation. In Creedy, J., editor, *The Foundations of Economic Thought*, pages 356–394. Blackwell, Oxford.

Dugundji, J. and Granas, A. (2003). *Fixed Point Theory*. Springer-Verlag, New York.

Echenique, F. (2002). Comparative statics by adaptive dynamics and the correspondence principle. *Econometrica*, 70:833–844.

Echenique, F. (2004). A weak correspondence principle for models with complementarities. *Journal of Mathematical Economics*, 40:145–152.

Echenique, F. (2008). The correspondence principle. In Durlauf, S. and Blume, L., editors, *The New Palgrave Dictionary of Economics (Second Edition)*. Palgrave Macmillan, New York.

Eilenberg, S. and Montgomery, D. (1946). Fixed-point theorems for multivalued transformations. *Amer. J. Math.*, 68:214–222.

Eraslan, H. and McLennan, A. (2011). Uniqueness of stationary equilibriun payoffs in coalitional bargaining. *Journal of Economic Theory*, 148:2195–2222.

Frey, S. and Goldstone, R. L. (2013). Cyclic game dynamics driven by iterated reasoning. *PLoS ONE*, 8:e56416.

Friedman, D. (1996). Equilibrium in evolutionary games: Some experimental results. *The Economic Journal*, 106:1–25.

Fudenberg, D. and Levine, D. (1998). *The Theory of Learning in Games*. MIT Press, Cambridge.

Gale, D. (1963). A note on global instability of competitive equilibrium. *Naval Logistics Research Quarterly*, 10:81–87.

Goeree, J. K. and Lindsay, L. (2012). Stabilizing the economy: Market design and general equilibrium. Working paper, University of Zurich.

Górniewicz, L. (2006). *Topological Fixed Point Theory of Multivalued Mappings*. Springer, The Netherlands, second edition.

Gul, F., Pearce, D., and Stachetti, E. (1993). A bound on the proportion of pure strategy equilibria in generic games. *Mathematics of Operations Research*, 18:548–552.

Harsanyi, J. C. (1973). Oddness of the number of equilibrium points: A new proof. *International Journal of Game Theory*, 2:235–250.

Healy, P. J. (2006). Learning dynamics for mechanism design: An experimental comparison of public goods mechanisms. *Journal of Economic Theory*, 129:114–149.

Hicks, J. R. (1939). *Value and Capital*. Clarendon Press, Oxford.

Hillas, J., Jansen, M., Potters, J., and Vermeulen, D. (2001). On the relations among some definitions of strategic stability. *Math. Oper. Res.*, 26:611–635.

Hirota, M., Hsu, M., Plott, C. R., and Rogers, B. W. (2005). Divergence, closed cycles and convergence in Scarf environments: Experiments in the dynamics of general equilibrium systems. California Institute of Technology, Humanities and Social Sciences Working Paper 1239.

Hirsch, M. W. (1976). *Differential topology*. Springer-Verlag, New York. Graduate Texts in Mathematics, No. 33.

Ho, T.-H., Camerer, C., and Chong, J.-K. (2007). Self-tuning experience weighted attraction learning in games. *Journal of Economic Theory*, 133:177–198.

Hofbauer, J. (1990). An index theorem for dissipative semiflows. *Rocky Mountain J. Math.*, 20:1017–1031.

Hofbauer, J. and Sigmund, K. (1998). *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge.

Hoffman, M., Suetens, S., Gneezy, U., and Nowak, M. A. (2015). An experimental investigation of evolutionary dynamics in the Rock-Paper-Scissors game. *Scientific reports*, 5.

Jordan, J. S. (1991). Bayesian learning in normal form games. *Games and Economic Behavior*, 3:60–81.

Kakutani, S. (1941). A generalization of Brouwer's fixed point theorem. *Duke Math. J.*, 8:457–459.

Kemp, M. C., Kimura, Y., and Tawada, M. (1990). Strategic experimentation with exponential bandits. *Economics Letters*, 34:1–4.

Kim, T. (1987). Stability problems in the implementation of Lindahl equilibrium. Doctoral Thesis, University of Minnesota.

Kinoshita, S. (1952). On essential components of the set of fixed points. *Osaka Math. J.*, 4:19–22.

Kinoshita, S. (1953). On some contractible continua without the fixed point property. *Fund. Math.*, 40:96–98.

Kohlberg, E. and Mertens, J.-F. (1986). On the strategic stability of equilibria. *Econometrica*, 54:1003–1038.

Kojima, M., Okada, A., and Shindoh, S. (1985). Strongly stable equilibrium points of n-person non-cooperative games. *Math. Oper. Res.*, 10:650–663.

Krasnosel'ski, M. A. and Zabreiko, P. P. (1984). *Geometric Methods of Nonlinear Analysis*. Springer-Berlin, Berlin.

Kreps, D. and Wilson, R. (1982). Sequential equilibrium. *Econometrica*, 50:863–894.

Magill, M. J. and Scheinkman, J. A. (1979). Stability of regular equilibria and the correspondence principle for symmetric variational problems. *International Economic Review*, 20:297–315.

Mas-Colell, A. (1974). A note on a theorem of F. Browder. *Math. Program.*, 6:229–233.

Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995). *Microeconomic Theory*. Oxford University Press, Oxford.

Maynard Smith, J. and Price, G. (1973). The logic of animal conflict. *Nature*, 246:15–18.

McLennan, A. (1989a). Consistent conditional systems in noncooperative game theory. *Internat. J. Game Theory*, 18:141–174.

McLennan, A. (1989b). The space of conditional systems is a ball. *Internat. J. Game Theory*, 18:125–139.

McLennan, A. (1991). Approxiation of contractible valued correspondences by functions. *J. Math. Econom.*, 20:591–598.

McLennan, A. (2016). *Advanced Fixed Point Theory for Economics*. Elsevier, New York.

Mertens, J.-F. (1989). Stable equilibria—a reformulation, part I: Definition and basic properties. *Math. Oper. Res.*, 14:575–625.

Mertens, J.-F. (1991). Stable equilibria—a reformulation, part II: Discussion of the definition and further results. *Math. Oper. Res.*, 16:694–753.

Milgrom, P. and Roberts, J. (1990). Rationalizability, learning and equilibrium in games with strategic complementarities. *Econometrica*, 58:1255–1277.

Myerson, R. (1986). Multistage games with communication. *Econometrica*, 54:323–358.

Nadzieja, T. (1990). Construction of a smooth Lyapunov function for an asymptotically stable set. *Czeckoslovak Math. J.*, 40:195–199.

Nash, J. (1950). *Non-cooperative Games*. PhD thesis, Mathematics Department, Princeton University.

O'Neill, B. (1953). Essential sets and fixed points. *American Journal of Mathematics*, 75:497–509.

Paltrow, Z. and Schotter, A. (1993). Does game theory predict well for the wrong reasons: An experimental investigation. C. V. Starr Center Research Report 93-46.

Pemantle, R. (1990). Nonconvergence to unstable points in urn models and stochastic approximations. *Annals of Probability*, 18:698–712.

Pemantle, R. (2007). A survey of random processes with reinforcement. *Probability Surveys*, 4:1–79.

Plott, C. (2000). Market stability: Backward-bending supply in a laboratory experimental market: 1999 Presidential Address Western Economic Association. *Economic Inquiry*, 38:1–18.

Plott, C. and George, G. (1992). Marshallian vs. Walrasian stability in an experimental market. *The Economic Journal*, 102:437–460.

Plott, C. and Smith, J. (1999). Instability of equilibria in experimental markets: Upward-sloping demands, externalities, and fad-like incentives. *Southern Economic Journal*, 65:405–426.

Rényi, A. (1970). *Foundations of Probability*. Holden-Day, San Francisco.

Ritzberger, K. (1994). The theory of normal form games from the differentiable viewpoint. *J. Internat. Game Theory*, 23:201–236.

Roth, G. and Sandholm, W. H. (2013). Stochastic approximation with constant step size and differential inclusions. *SIAM J. Control Optim.*, 51:525–555.

Samuelson, L. (1997). *Evolutionary Games and Equilibrium Selection*. MIT Press, Cambridge.

Samuelson, P. (1947). *Foundations of Economic Analysis*. Harvard University Press.

Samuelson, P. A. (1941). The stability of equilibrium: Comparative statics and dynamics. *Econometrica*, 9:97–120.

Samuelson, P. A. (1942). The stability of equilibrium: Linear and nonlinear systems. *Econometrica*, 10:1–25.

Samuelson, P. A. (1971). On the trail of conventional beliefs about the transfer problem. In et. al., J. N. B., editor, *Trade, Balance of Payments, and Growth: Papers in International Economics in Honor of Charles Kindleberger*, pages 327–351. North Holland, Amsterdam.

Sandholm, W. H. (2010). *Population Games and Evolutionary Dynamics*. MIT Press, Cambridge.

Sandholm, W. H. (2015). Population games and deterministic evolutionary dynamics. In Young, H. P. and Zamir, S., editors, *Handbook of Game Theory*, chapter 13. North-Holland, Amsterdam.

Scarf, H. (1960). Some examples of global instability of the competitive equilibrium. *Internat. Econom. Rev.*, 1:157–172.

Selten, R. (1975). Re-examination of the perfectness concept for equilibrium points of extensive games. *Internat. J. Game Theory*, 4:25–55.

Selten, R. (1991). Anticipatory learning in two-person games. In Selten, R., editor, *Game Equilibrium Models I*, pages 98–154. Springer, Berlin.

Selten, R. and Chmura, T. (2008). Stationary concepts for experimental 2×2 games. *American Economic Review*, 98:938–966.

Semmann, D., Krambeck, H.-J., and Milinski, M. (2003). Volunteering leads to Rock-Paper-Scissors dynamics in a public goods game. *Nature*, 425:390–393.

Shapley, L. S. (1974). A note on the Lemke-Howson algorithm. *Math. Programming Stud.*, 1:175–189. Pivoting and extensions.

Smirnov, G. V. (2002). *Introduction to the Theory of Differential Inclusions*. American Mathematical Society, Providence R.I.

Tang, F.-F. (2001). Anticipatory learning in two-person games: some experimental results. *Journal of Economic behavior & Organization*, 44(2):221–232.

Taylor, P. D. and Jonker, L. B. (1978). Evolutionarily stable strategies and game dynamics. *Math. Biosci.*, 40:145–156.

Topkis, D. M. (1979). Equlibrium points in nonzero-sum n-person submodular games. *SIAM Journal of Control and Optimization*, 17:773–787.

Van Essen, M., Lazzati, N., and Walker, M. (2010). Out-of-equilibrium performance of three Lindahl mechanisms: Experimental evidence. working paper, University of Arizona.

van Huyck, J. B., Battaio, R. C., and Beil, R. O. (1990). Tacit coordination games, strategic uncertainty, and coordination failure. *American Economic Review*, 80:234–248.

Vega-Redondo, F. (1996). *Evolution, Games, and Economic Behavior*. Oxford University Press, Oxford.

Vives, X. (1990). Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics*, 19:305–321.

Walker, M. (1981). A simple incentive compatible scheme for attaining Lindahl allocations. *Econometrica*, 49:65–71.

Wang, Z. and Xu, B. (2014). Incentive and stability in the Rock-Paper-Scissors game: an experimental investigation. *arXiv preprint arXiv:1407.1170*.

Wang, Z., Xu, B., and Zhou, H.-J. (2014). Social cycling and conditional responses in the Rock-Paper-Scissors game. *Scientific reports*, 4:5830.

Weibull, J. (1995). *Evolutionary Game Theory*. MIT Press, Cambridge.

Wilson, F. W. (1969). Smoothing derivatives of functions and applications. *Transactions of the American Mathematical Society*, 139:413–428.

Wu, W.-T. and Jiang, J.-H. (1962). Essential equilibrium points of n-person non-cooperative games. *Scientia Sinica*, 5:1307–1322.

Xu, B., Wang, S., and Wang, Z. (2014). Periodic frequencies of the cycles in 2×2 games: evidence from experimental economics. *The European Physical Journal B*, 87:46.

Xu, B. and Wang, Z. (2011). Evolutionary dynamical pattern of 'Coyness and Philandering': Evidence from experimental economics. In *Proceedings of the 8^{th} International Conference on Complex Systems*, volume 8.

Xu, B., Zhou, H.-J., and Wang, Z. (2013). Cycle frequency in standard Rock-Paper-Scissors games: Evidence from experimental economics. *Physica A*, 392:4997–5005.