Selected Topics in the Theory of Fixed Points*

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Abstract

We present a treatment for mathematical economists of three topics in the theory of fixed points: (a) the Lefschetz fixed point index; (b) the Lefschetz fixed point theorem; (c) the theory of essential sets of fixed points. Our treatment is geometric, based on elementary techniques, and largely self-contained. In particular there is no essential reference to algebraic topology. Within the chosen scope of the paper the results are at the level of generality of the mathematical literature. The development of these theories for correspondences is emphasized. It emerges that the solution concepts of Kohlberg and Mertens (1986) have a definite, though obscure, place in this theory.

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by

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1 Introduction

This paper presents three topics in the theory of fixed points:

(a) the Lefschetz fixed point index;

(b) the Lefschetz fixed point theorem;

(c) the theory of essential sets of fixed points.

In varying degrees the intent of this paper is to be simultaneously useful as an exposition, a reference, and a contribution.

The subjects discussed here are at this point quite old within mathematics, with extensive literatures. In spite of this, and in spite of the obvious relevance of fixed point theory to economics, the material presented here is not widely known or much used by economists. This is, I believe, largely due to the fact that these subjects had their origins in algebraic topology, a branch of mathematics with a high cost of entry that is not generally part of the background of mathematical economists. Here we develop the subject from the point of view of differential and point set topology. This approach is attractive both by virtue of its minimal prerequisites and because it emphasizes geometric concepts.

Within the chosen scope of this paper the cost of the restriction to nonalgebraic techniques is surprisingly slight. The main loss is that the property of the Lefschetz index known as normalization cannot be established. In other respects our results are as general as those found in the mathematical literature. This level of generality should be sufficient for the needs of mathematical economics to the extent that they can be foreseen at this point.

In fact our results concerning contractible valued correspondences appear to go beyond the existing mathematical literature. (It should be mentioned that the key lemma underlying these results is due to Mas-Colell (1974).) Another aspect of originality is the observation that the upper topology (cf. §2.1) provides a natural language for expressing many of the results.

The principal stimulus for a paper of this sort is the work of Kohlberg and Mertens (1986) (henceforth KM) on various notions of "stability" for noncooperative games. Their work came as a great surprise and has been very influential, but we will show how their concepts have a very definite, albeit obscure, place in a body of mathematics that had largely attained its final form prior to Selten (1975).

Our story, then, is about a large group of scholars failing to benefit from preexisting work by another large group of scholars over an extended period of time. To set the stage we begin with a potted history of the use of fixed point theory in economics. In §4, after the main results have been presented, we give a similar account of the evolution of fixed point theory in mathematics.

1.1 Fixed Point Theory in Economics

Variants of Brouwer's fixed point theorem are the key steps in the proofs of the two central existence results in mathematical economics (Nash (1951) and Arrow-Debreu (1954)). Naturally economists have been interested in fixed point theory since this became clear, and there have been many applications of generalizations of Brouwer's theorem. In particular, fixed point theorems for convex subsets of infinite dimensional spaces (Tychonov (1935), Fan (1952) and Glicksberg (1952)) have been applied many times. There have also been some applications of the Eilenberg-Montgomery (1946) fixed point theorem, a purely topological extension of Kakutani's (1941) theorem.

The "stability" of fixed points has been an issue from the beginning. At first (e.g. Samuelson (1948)) the question was posed in the sense of the stability of dynamical systems, but these investigations ultimately foundered on the inability to find models of dynamic convergence to equilibrium that were consistent with the rational expectations hypothesis and allowed nontrivial dynamics. A nonmartingale price process induces speculation against expected price changes, and a nontrivial trajectory in the space of mixed strategies of a game typically induces some agent to switch to a pure strategy. (Matching pennies is an adequate example.)

A more recent approach is to distinguish between equilibria that disappear with small changes in the parameters from those that persist, in a certain topological sense, under any perturbations of the equilibrium conditions. These questions were introduced into general equilibrium theory by Debreu (1970), and they have been investigated extensively using the methods of differential topology. Mas-Colell (1985) summarizes the work to date, stressing its culmination in index theorems. Balasko (1988) is another excellent extensive treatment.

From a general topological point of view the notion of a regular fixed point is quite special, but there was not a large effort devoted to extending the results in economics (especially index theory) to a framework based only on topology and convex analysis. One can argue that the smooth case is adequate by noting that regular economies are dense in the set of all economies, so analysis of regular economies is sufficient for many mathematical purposes. In addition, in certain senses regular economies are "typical." Finally, there is every reason to expect that econometric methodologies will be based on an assumption of regularity.

The situation in game theory is drastically different. The graph of the best response correspondence of a normal form game is a definite geometric object in a certain finite dimensional Euclidean space. Although the property of having only regular equilibria is "generic" in the space of normal form payoffs (Kreps and Wilson (1982)), this property may be nongeneric in the set of normal form payoffs derived from a given extensive form. The study of robustness of equilibria with respect to perturbations was initiated by Selten (1975) and has been very active since then.

These investigations were conducted without the benefit of the portions of the mathematical theory of fixed points laid out below. The consequences of this ignorance can clearly be traced in the discussion of an example due to Kohlberg.

[Insert Figure 1]

Here (R, r) is both a perfect (Selten (1975)) and proper (Myerson (1978)) equilibrium, but it is not thought to be reasonable. Rationality precludes M from being chosen, but it allows a quite sensible explanation of why L might be chosen, so the only belief for agent 2 consistent with the hypothesis that agent 1 is rational is the unit mass at L.

The inability to eliminate (R, r) was generally viewed as a serious defect of existing solution concepts. Several years elapsed before solution concepts without this defect were proposed (Pearce (1984), McLennan (1985), and KM, among others). Since it is more restrictive than these notions, the notion of an essential set of fixed points also resolves this example, so it certainly would have been investigated had there been any awareness of it.

The story of how economists became aware of the theory of essential sets is also illuminating. Stimulated by the first version of Kohlberg and Mertens' paper, Mas-Colell rediscovered methods due essentially to Kinoshita (1952) and pointed out that they could be used to establish several of Kohlberg and Mertens' results. This approach to the subject was widely discussed for over a year before Kinoshita's work was rediscovered.

Before moving on to the description of the mathematical theory we should also mention that there have been a variety of other applications of the ideas below in mathematical economics. Some of these are discussed at various points below, and at this point we would like to apologize to those authors whose work may have been unjustly ignored.

1.2 The Lefschetz Fixed Point Index

The Lefschetz fixed point index is the key to all three topics, so it is treated first in Section 2. If X is a compact metric space, Y is a subset of X, and $F : Y \to X$ is a function or correspondence, let $\mathcal{F}(F)$ be the set of fixed points of F. An *admissible pair* is a pair (F, U) where U is an open subset of X and $F : \bar{U} \to X$ is an upper hemicontinuous correspondence with $\mathcal{F}(F) \bigcap (\bar{U} - U) = \emptyset$. (We reserve the symbol ∂ for the boundary of ∂-manifolds.) A Lefschetz fixed point index is a function from a class of admissible pairs to the integers that satisfies certain axioms.

This type of index is a generalization of the fixed point index defined in differential topology (e.g. Guillemin and Pollack (1974, §3.4) or Appendix B below), and the geometric intuition for it is the same. Very roughly, the index of the pair (F, U) is the sum of the indices of the fixed points of F

in U, where the index of an individual fixed point is $+1$ or -1 according to whether $Id_X - F$ is orientation preserving or orientation reversing at the fixed point.

Index theory has been studied extensively in pure mathematics. The central results in this literature, called index theorems, state that for various classes of spaces and maps the index exists and is uniquely characterized by a system of axioms. In Section 2 (and related appendices) index theorems are proved at several levels of generality. We begin with extremely well behaved smooth functions. Using "bootstrap" arguments, we proceed through a series of generalizations, eventually attaining the level of generality of the mathematical literature as represented by Brown (1971).

Aside from the large scale architecture, the results and methods of proof in Section 2 are not new, but the material is still of some interest from the point of view of pure mathematics. We demonstrate the possibility of detaching index theory from an unintuitive fact of homological algebra, the Hopf (1929) trace theorem, thereby attaining an entirely nonalgebraic exposition. The Hopf trace theorem was the basis of the first proof of the Lefschetz fixed point theorem for general finite simplicial complexes[1] and it has been an

[1]The following notions are basic in topology but uncommon in economics. A (combinatoric) *simplicial complex* is a pair (V, S) where V is a set of *vertices* and S is a set of nonempty finite subsets of V with the property that $\sigma' \in S$ whenever $\emptyset \neq \sigma' \subset \sigma \in S$. Elements of S are called *simplices*. For $\sigma \in S$, let $|\sigma| = \{x \in \mathbf{R}^\sigma_+ \subset \mathbf{R}^V | \Sigma_{v \in \sigma}\ x_v = 1\}$. The space $X = \bigcup_{\sigma \in S} |\sigma|$ is the *canonical realization* of (V, S). If $h : X \to Y$ is a homeomorphism, then (V, S) and h constitute a *triangulation* of Y, while Y and h constitute a *realization* of (V, S). We say that Y is a (geometric) *simplicial complex,* and the sets $h(|\sigma|)$ are also called simplices. If $T \subset S$ with $\sigma' \in T$ whenever $\phi \neq \sigma' \subset \sigma \in T$, then we say that (V, T) is a subcomplex of (V, S) and $\bigcup_{\sigma \in T} h(|\sigma|)$ is a subcomplex of Y. The space X is compact if and only if S is finite. (Consider an open cover $\{X_\sigma\}_{\sigma \in S}$ where each X_σ is X with a point removed from the interior of $|\sigma|$ – *cf.* Borsuk (1967, p.72).) A famous and deep result is that every manifold with boundary is a simplicial complex - cf. Munkres (1963).

essential element of almost all treatments of the subject since then. The possibility of such a treatment has been noted by Brown (1970), but no author has found occasion to proceed as we have.

1.3 The Lefschetz Fixed Point Theorem

The axioms characterizing the index easily imply that F must have a fixed point in U if the index of the pair (F, U) is not zero. In particular, $F : X \to X$ must have a fixed point whenever the pair (F, X) has a nonzero index. This is the geometric content of the Lefschetz fixed point theorem, the subject of Section 3.

Traditionally the Lefschetz number of a continuous function $f : X \to X$ is defined by the formula $L_H(f) = \Sigma_{i=0}^{\infty} (-1)^i tr\,(H_i(f; \mathbf{Q}))$ (different fields of coefficients are possible). Those who know the relevant algebraic topology will understand this formula or know where to find it explained. Those who do not need only remember that $L_H(f)$ is an integer associated with the map f.

The celebrated Lefschetz fixed point theorem (henceforth LFPT) asserts that $\mathcal{F}(f)$ is nonempty whenever $L_H(f) \neq 0$. Here we define the Lefschetz number $L(f)$ to be the index of the pair (f, X), and we refer to $L_H(f)$ as the homological Lefschetz number. We may then view the LFPT as having two parts: (1) If $L(f) \neq 0$, then f has a fixed point; (2) $L_H(f) = L(f)$. The first assertion contains all the geometric information in principle (since the index axioms determine $L(f)$), and we prove it in great generality. In a certain precise sense this is the most general "topological" fixed point theorem possible.

The second assertion is a property of the index known as Normalization, and the principle loss in our avoidance of homology is the inability to discuss it. While in principle Normalization yields no additional geometric information, in many cases Normalization provides the only practical method of computing Lefschetz numbers (cf. Proposition 3.5).

For contractible[2] domains the LFPT reduces to Brouwer's fixed point theorem. Until recently the paper of Hart and Kuhn (1975) discussed at the end of §3 was, to my knowledge, the only case in which natural economic modeling led to a fixed point space that was not contractible. However, Duffie and Shafer's (1985) proof of generic equilibrium existence for economies with incomplete markets and "real" financial assets seems to require methods that go beyond Brouwer's theorem. (This is argued by Hirsch, Magill, and Mas-Colell, (1987).) The relationship between these techniques and the LFPT is described briefly in §3, and we recommend Hirsch, Magill, and Mas-Colell (1987) and Husseini, Lasry, and Magill (1987) to the interested reader.

1.4 The Theory of Essential Sets

As can be seen in Figure 2, there are different types of fixed points.

[Insert Figure 2]

The point A is robust with respect to perturbations of f : for any neighborhood U of A one can find a neighborhood W of $Gr(f)$[3] such that if g is continuous and its $Gr(g) \subset W$, then $\mathcal{F}(g) \bigcap U \neq \emptyset$. On the other hand we can find continuous functions arbitrarily near f that have no fixed points anywhere near B. We say that A is *essential* and B is *inessential.* These notions were introduced by Fort (1950).

[2] A topological space X is *contractible* if there is a continuous *contraction* $c : X \times [0, 1] \to X$ with $c(\cdot, 0) = Id_X$ and $c(\cdot, 1)$ a constant function.

[3] The graph of any function or correspondence F is denoted by $Gr(F)$.

Kinoshita (1952) and O'Neill (1953) pointed out that this type of robustness can be defined for compact sets $K \subset \mathcal{F}(f)$: K is *essential* for f if, for every neighborhood U of K, there is a neighborhood W of $Gr(f)$ such that every map g with $Gr(g) \subset W$ has a fixed point in U. If all nearby functions have fixed points then $\mathcal{F}(f)$ is essential (Lemma 4.2), but unless all fixed points are isolated one cannot guarantee the existence of an essential fixed point. For instance, the entire interval $\mathcal{F}(f) = [1/3,\ 2/3]$ is the only essential set for $f(x) = \max\{1/3,\ \min\{2/3,\ x\}\}$, $x \in [0,\ 1]$, since for any $x^* \in [1/3,\ 2/3]$ we can find g arbitrarily close to f with $\mathcal{F}(g) = \{x^*\}$.

As KM point out, this means that the use of this sort of robustness to defined game-theoretic solution concepts requires that the solution concept be set-valued. In such a solution concept it seems natural to think of the minimal essential sets as the analogues of singletons. In the settings we study we show that, like singletons, minimal essential sets are connected. For the special case of ∂-manifolds[4] we show that, if $\mathcal{F}(F)$ consists of finitely many connected components, then the minimal essential sets are precisely the essential components. KM also point out that the set of Nash equilibria of a game always consists of finitely many connected components.

As we will see, the Continuity axiom for the index implies that K is essential if $\Lambda(f,\ U) \neq 0$ for some neighborhood U of K with $\mathcal{F}(f) \bigcap \bar{U} = K$. This raises the question of whether the notion of a connected essential set is a significantly weaker concept than the notion of a connected set of fixed points with nonzero index. For manifolds the answer is no, as we show in Theorem 6. A version of this result is proved by O'Neill (1953), and it appears that this

[4]For the definition of a ∂-manifold (manifold with boundary) see Milnor (1965) or Hirsch (1976). In this paper a ∂-manifold may have an empty boundary, and the term 'manifold' is reserved for boundaryless manifolds.

result explains the neglect of essential sets in the subsequent mathematical literature.

Finally, we discuss in detail how the theory of essential sets can be used to prove existence theorems for the various concepts of stability introduced by KM. The results here also allow existence theorems for many other concepts derived using these techniques.

1.5 Originality and Organization

The possibility of passing from index theory for smooth functions to the topological index is mentioned by Brown (1970), and this general style of attack can be found in volume I of Zeidler's (1984) monumental work. The method of passing from the index for functions to an index for convex valued correspondences has parallels in the theory of the topological degree (Hukuhara (1967), Cellina (1969), Cellina and Lasota (1969), Ma (1972)). The application of this technique to contractible valued correspondences was initiated in McLennan (1989a).

An interesting novelty is the expression of many of the results in terms of the upper topology. To a certain extent this is simply a matter of terminology - familiar facts are reexpressed in terms of the new topological language - but our results strongly suggest that this topology is natural and useful for the discussion of fixed points. In particular, Lemma 2.10 allows us to move from the most special index to great generality with speed and ease.

Our results are more general than much of the literature insofar as we define the index and the notion of an essential set for admissible pairs (F, U) in which F may not have a continuous or upper hemicontinuous extension to all of X. Again, this additional generality is convenient for our purposes, but it does not involve novel geometric insights.

Of course the Lefschetz fixed point theorem and the index have extensive literatures, and the main results below are not new. Our approach draws on many sources, and the origins of some ideas are not completely clear to me. In some cases the precise details of our formulations are new, but the geometric ideas are well known, even if it is difficult to attribute them to a single author. Let me now apologize to those mathematicians whose work has been unjustly slighted.

The organization of the body of the paper has been described in detail above. Most of the more technical arguments have been placed in Appendices A-E. For ease of reference we have strived to make these appendices logically independent, but Appendix E depends at certain points on material in Appendix A, and Appendix D uses results developed in Appendix C.

2 The Lefschetz Fixed Point Index

2.1 Mathematical Preliminaries

We will need a few standard concepts of mathematics that may be unknown to economists, and in each instance we have attached an explanatory footnote to the first occurrence of the term (e.g. footnotes 1 and 2 above.). In addition two terminological points should be mentioned at the outset. First, we use the word "map" as a synonym for "continuous function," as is now standard in mathematics. For metric spaces X and Y we let $C(X, Y)$ be the set of maps $f : X \to Y$. Second, we use the term "correspondence," as is standard in economics, rather than any of the terms used in mathematics ("multi-valued mapping," "multivalued function," "multivalued transformation," and "set valued map" can be found in the references) to indicate a binary relation that assigns a non-empty subset of the range to each point in the domain.

We begin by defining a useful topology. For a metric space X let $\mathcal{H}(X)$ be the set of compact subsets of X. The *upper topology* on $\mathcal{H}(X)$ is the topology generated by the base of open sets of the form $\mathcal{O}_U - \{\mathcal{K} \in \mathcal{H}(X) | \mathcal{K} \subset U\}$, U open in X. A discussion of the properties of this topology can be found in Klein and Thompson (1984). The upper topology coincides with the Scott topology of the lattice of closed subsets of X when X is compact.

The topology of $\mathcal{H}(X)$ is weaker than the topology induced by the Hausdorff metric (e.g. Hildenbrand (1974, p. 16)) and clearly the upper topology is not a Hausdorff topology. Nonetheless we shall argue (Definition 2.1, Lemma 2.2, Proposition 2.3, Axiom I4, especially Proposition 2.10, Proposition 2.21, and Proposition 4.3) that it is the natural topology for the theory of fixed points of correspondences.

Definition 2.1: Let X and Y be metric spaces with X compact. A compact valued correspondence $F : X \to Y$ is *upper hemicontinuous* (u.h.c.) if the associated function $F : X \to \mathcal{H}(Y)$ is continuous. Let $\mathcal{K}(X, Y)$ be the space of u.h.c. compact valued correspondences $F : X \to Y$ endowed with the topology induced by the upper topology on graphs: a neighborhood base at F is given by the sets of the form $\{G \in \mathcal{K}(X, Y) \mid Gr(G) \subset U\}, U$ a neighborhood of $Gr(F)$. (It is an easy exercise to prove the compactness of the graph of an u.h.c. compact valued correspondence with compact domain.)

It is easily checked that this definition of upper hemicontinuity is equivalent to the usual definition (e.g. Hildenbrand (1974, p. 21)).

We do not distinguish between a function $x \mapsto f(x)$ and the associated correspondence $x \mapsto \{f(x)\}$. In the most important case it is also unnecessary to distinguish between the relative topology of $C(X, Y)$ in $\mathcal{K}(X, Y)$ and the standard topology on $C(X, Y)$.

Lemma 2.2: The relative topology induced by the inclusion $C(X, Y) \subset \mathcal{K}(X, Y)$ is the topology of uniform convergence.

Proof: Suppose $f \in A \subset C(X, Y)$ where A is open in the topology of uniform convergence. Then there is $\varepsilon > 0$ such that A contains all $g \in C(X, Y)$ with $Gr(g) \subset \{(x, y) \,|\, d(f(x), y) < \varepsilon\}$, so A is open in the relative upper topology.

Conversely, suppose that A is relatively open. Then there is W, a neighborhood of the graph of f, such that $\{g \in C(X, Y) \,|\, Gr(g) \subset W\} \subset A$. We claim that there is $\varepsilon > 0$ such that $\{(x, y) \,|\, d(f(x), y) < \varepsilon\} \subset W$; if not then one can construct a sequence $(x^n, y^n) \in (X \times Y) - W$ with $d(f(x^n), y^n) \to 0$. Taking a subsequence, let $x^n \to x$. Then $(x, f(x)) \in Gr(f)$ is a limit point of $(X \times Y) - W$, a contradiction.//

In general the relative topology of $C(X, Y) \subset \mathcal{K}(X, Y)$ is the strong C^0 topology (Hirsch (1976, p. 35)), but we will not need this fact.

If $X \subset Y$ then the *fixed point set* of a correspondence $F : X \to Y$ is $\mathcal{F}(F) = \{x \in X \,|\, x \in F(x)\}$. If $F \in \mathcal{K}(X, Y)$ then $Gr(F)$ is closed in $X \times Y$ (Hildenbrand (1974, p. 24)), so $\mathcal{F}(F)$ is closed in X. If X and Y are compact and $F^n \to F$ in $\mathcal{K}(X, Y)$, then $\{Gr(F^n)\}$ is eventually inside any neighborhood of $Gr(F)$, so $\{\mathcal{F}(F^n)\}$ is eventually inside any neighborhood of $\mathcal{F}(F)$. This is precisely

Proposition 2.3: Let Y and $X \subset Y$ be compact. Then $\mathcal{F} : \mathcal{K}(X, Y) \to$

$\mathcal{H}(X)$ is continuous.

2.2 The Index Axioms

In considering the axiomatic description of an index below the reader may wish to have the simplest example in mind. If $f : [0, 1] \to (0, 1)$ is C^1 with $\frac{df}{dt}(x) \neq 1$ for all $x \in \mathcal{F}(x)$, then the index of $x \in \mathcal{F}(f)$ is 1 if $\frac{df}{dt}(x) < 1$ and -1 otherwise, and the index of an open set $U \subset [0, 1]$ with no fixed points in $\bar{U} - U$ is the sum of the indices of the fixed points in U.

Definition 2.4: Let X be a compact metric space. An *admissible pair* for X is a pair (F, U) in which $U \subset X$ is open, $F \in \mathcal{K}(\bar{U}, X)$, and
$\mathcal{F}(F) \bigcap (\bar{U} - U) = \emptyset$.

We will discuss several indices, so we encompass them in a general framework.

Definition 2.5: An *index base* is a pair $(\mathcal{S}, \mathcal{P})$ with the following description: (a) $\mathcal{S}$ is a class of compact metric spaces; (b) for each $X \in \mathcal{S}$, $\mathcal{P}(X)$ is a set of admissible pairs (F, U) for X with the property that $(F|_{\bar{U}'}, U') \in \mathcal{P}(X)$ whenever $(F, U) \in \mathcal{P}(X)$, $U' \subset U$ is open, and $(F|_{\bar{U}'}, U')$ is also admissible; (c) $\mathcal{P} = \bigcup_{X \in \mathcal{S}} \mathcal{P}(X)$.

When no confusion is possible we write (F, U') rather than $(F|_{\bar{U}'}, U')$ when $F \in \mathcal{K}(\bar{U}, X)$ and $U' \subset U$ is open. If $U \subset X \in \mathcal{S}$ with U open, let $\mathcal{P}(X, U) = \{F \in \mathcal{K}(\bar{U}, X) \,|\, (F, U) \in \mathcal{P}(X)\}$.

Definition 2.6: A *Lefschetz fixed point index* (or simply an *index*) for $(\mathcal{S}, \mathcal{P})$

is a function $\Lambda : \mathcal{P} \to Z$ (the ring of integers) that satisfies the Index Axioms:

(I1) (Weak Normalization) If $c : X \to \{c \in X\}$ is a constant function with $(c,\, X) \in \mathcal{P}(X)$, then $\Lambda(c,\, X) = 1$.

(I2) (Additivity) If $U_1, \ldots,\; U_r$ are disjoint open subsets of U and $F \in \mathcal{P}(X,\, U)$ has no fixed points in $\bar{U} - (U_1 \bigcup \ldots \bigcup U_r)$, then $\Lambda(F,\, U) = \Sigma_i\, \Lambda(F,\, U_i)$.

(I3) (Homotopy) Suppose $h : \bar{U} \times [0,\, 1] \to X$ is a homotopy such that for all t, $h_t = h(\cdot,\, t)$ has no fixed points in $\bar{U} - U$, and $h_0,\, h_1 \in \mathcal{P}(X,\, U)$. Then $\Lambda(h_0,\, U) = \Lambda(h_1,\, U)$.

(I4) (Commutativity) If $f \in C(X,\, Y)$, $g \in C(Y,\, X)$, $(g \circ f,\, U) \in \mathcal{P}(X)$, and $(f \circ g,\, g^{-1}(U)) \in \mathcal{P}(Y)$, then $\Lambda(g \circ f,\, U) = \Lambda(f \circ g,\, g^{-1}(U))$.

(I5) (Continuity) The index is continuous in its first variable: for each $F \in \mathcal{P}(X,\, U)$, then there is a neighborhood A of F in $\mathcal{K}(\bar{U},\, X)$ such that $\Lambda(G,\, U) = \Lambda(F,\, U)$ for all $G \in \mathcal{P}(X,\, U) \bigcap A$.

(I6) (Multiplication) If $(F,\, U) \in \mathcal{P}(X)$, $(G,\, V) \in \mathcal{P}(Y)$ and $(F \times G,\, U \times V) \in \mathcal{P}(X \times Y)$, then $\Lambda(F \times G,\, U \times V) = \Lambda(F,\, U) \cdot \Lambda(G,\, V)$.

Remarks: (1) It is, perhaps, not immediately apparent that there is a connection between fixed points and the axioms for an index. Note, however, that if $F \in \mathcal{P}(X, U)$ has no fixed points in U, then Additivity implies that $\Lambda(F, U) = \Lambda(F, \emptyset) = \Lambda(F, \emptyset) + \Lambda(F, \emptyset) = 0$. In particular, if $F \in \mathcal{P}(X, X)$ and $\Lambda(F, X) \neq 0$, then F must have a fixed point. This result is discussed at length in §3.

(2) If $(F, U) \in \mathcal{P}$, $U' \subset U$ is open, and F has no fixed points in $\bar{U} - U'$, then Additivity implies that $\Lambda(F, U') = \Lambda(F, U)$. This means that we may think of the index as "really" being defined on pairs of the form (F, K) where $F \in \mathcal{P}(X, U)$ for some X and U and $K \subset \mathcal{F}(F)$ is clopen[5] in the relative topology of $\mathcal{F}(F)$. It is interesting to note that Additivity may be thought of as expressing two properties of the index, one corresponding to the case $r = 1$ and the other corresponding to the case $U_1 \bigcup \ldots \bigcup U_r = U$. These properties are called "excision" and "decomposition of domain" respectively.

(3) Most authors (e.g. Brown (1971)) define the index only on classes of pairs (F, U) in which F is defined on all of X, adding an additional condition called Localization which says that $\Lambda(F, U) = \Lambda(G, U)$ whenever $F|_{\bar{U}} = G|_{\bar{U}}$. Localization is inherent in our formulation, and we also have the possibility of defining the index on pairs (F, U) where $F : \bar{U} \to X$ has no extension to all of X in the relevant class of correspondences.

(4) In well behave settings Axioms (I3) and (I5) are equivalent: Continuity implies the constancy of the index along any homotopy, while Homotopy implies Continuity if the relevant class of correspondences is locally path

[5] Both closed and open.

connected.[6]

(5) Our axiom system is not logically minimal in another respect. In all of our theorems asserting the existence and uniqueness of an index, the uniqueness clause remains valid without Multiplication, and in the literature Multiplication is typically treated as a property of the index rather than an axiom. We have phrased the results below in such a way that the minimal set of axioms determining each index can easily be traced.

(6) Roughly speaking, the theory of the Lefschetz index is equivalent to the theory of the topological degree. The topological degree can be described by a system of axioms similar to the index axioms; we present such an axiom system and indicate the relationship at a low level of generality in Appendix A. For a more extensive description of degree theory we recommend Lloyd (1978). For a description of degree theory as a methodology for proving equilibrium existence and characterization results in economics we recommend Geanakoplos and Shafer (1989).

An *index theorem* is a result stating that some index base admits a unique index. The remainder of the section is devoted to the discussion of results of this type. We begin by establishing a very special result, after which we prove that the special index has a series of unique extensions to more general settings.

[6]A topological space is *locally path connected* if any neighborhood U of a point x contains a neighborhood V such that any pair of points in V can be connected by a path (continuous image of [0,1]) in U.

2.3 The Index for Smooth Functions

Our first index should be familiar to many readers from its applications in the theory of regular economies.

Definition 2.7: A *domain* is any nonempty compact set $X \subset \mathbf{R}^m$ that is the closure of its interior. Let $\mathcal{S}_\infty$ be the class of domains. For each domain X let $\mathcal{P}_\infty(X)$ be the set of admissible pairs (f, U) with $f \in C^\infty(\bar{U}, \text{int}\, X)$ and $Id_{\mathbf{R}^n} - Df(x)$ nonsingular for all $x \in \mathcal{F}(f)$. Let $\mathcal{P}_\infty = \bigcup_{X \in \mathcal{S}_\infty} \mathcal{P}_\infty(X)$.

Proposition 2.8: There is a unique index Λ_∞ defined on $\mathcal{P}_\infty$. For each domain X the restriction of Λ_∞ to $\mathcal{P}_\infty(X)$ is uniquely determined by Weak Normalization, Additivity, and Homotopy.

Proof: Appendix A.//

This index has the formula

$$\Lambda_\infty(f, U) = \Sigma_{x \in \mathcal{F}(f)} \ \text{sgn}|Id_{\mathbf{R}^n} - Df(x)|.$$

2.4 Extension by Continuity

One of our few claims to novelty is Proposition 2.10 below which gives conditions under which Continuity determines a unique extension of an index defined on a restricted set of admissible pairs to an index defined on a larger set of pairs. This result is the key step in the proofs of Proposition 2.12 and Theorems 2 and 3, and it is an important example of the utility of the upper topology.

Definition 2.9: A *subbase* of $(\mathcal{S}, \mathcal{P})$ is an index base $(\mathcal{S}, \mathcal{Q})$ with $\mathcal{Q} \subset \mathcal{P}$.

Proposition 2.10: Let $(\mathcal{S}, \mathcal{Q})$ be a subbase of $(\mathcal{S}, \mathcal{P})$, with the following properties:

(a) For all $U \subset X \in \mathcal{S}$ with U open, $\mathcal{Q}(X, U) \bigcap C(\bar{U}, X)$ is dense in $\mathcal{P}(X, U)$.

(b) For every neighborhood A of $F \in \mathcal{P}(X, U)$ there is a neighborhood $A' \subset A$ of F with the property that every pair of points in $A' \bigcap C(\bar{U}, X)$ can be connected by a path (i.e. a homotopy) in $A \bigcap C(\bar{U}, X)$.

(c) Identifying $c \in X$ and the constant function $x \mapsto c$, $\{c \in X \,|\, (c, X) \in \mathcal{Q}(X)\}$ is dense in X.

Then any index $\Lambda_{\mathcal{Q}} : \mathcal{Q} \to Z$ has a unique extension $\Lambda_{\mathcal{P}} : \mathcal{P} \to Z$ satisfying (I5), and this extension is an index, i.e. it also satisfies (I1) - (I4) and (I6).

Proof: Appendix B.//

2.5 An Index Theorem for Domains and Continuous Functions

Our first application of Proposition 2.10 is to extend the index above to all continuous functions with image in the interior of the domain.

Definition 2.11: Let $\mathcal{S}_0 = \mathcal{S}_\infty$. For each domain X let $\mathcal{P}_0(X)$ be the set of admissible pairs (f, U) with $f \in C(\bar{U}, \operatorname{int} X)$. Let $\mathcal{P}_0 = \bigcup_{X \in \mathcal{S}_0} \mathcal{P}_0(X)$.

Proposition 2.12: There is a unique index Λ_0 defined on $\mathcal{P}_0$. For each domain X the restriction of Λ_0 to $\mathcal{P}_0(X)$ is uniquely determined by Weak Normalization, Additivity, and Homotopy.

Proof: We apply Proposition 2.10 with $(\mathcal{S}, \mathcal{P}) = (\mathcal{S}_0, \mathcal{P}_0)$ and $(\mathcal{S}, \mathcal{Q}) = (\mathcal{S}_\infty, \mathcal{P}_\infty)$. By Hirsch (1976, Th. 2.6, p. 49) $C^\infty(\bar{U}, \operatorname{int} X)$ is dense in $C(\bar{U}, \operatorname{int} X)$, and Sard's theorem easily implies that $\mathcal{P}_\infty(X, U)$ is dense in $C^\infty(\bar{U}, \operatorname{int} X)$, so condition (a) of Proposition 2.10 holds. Condition (b) holds by virtue of the fact that for any $f \in C^\infty(\bar{U}, \operatorname{int} X)$, the image of $(1-t)f + tg$ is contained in the interior of X for all $g \in C^\infty(\bar{U}, \operatorname{int} X)$ sufficiently close to f. Recalling the definition of a domain, condition (c) holds because $\{c \in X \mid (c, X) \in \mathcal{P}_\infty(X)\} = \operatorname{int} X$.

Proposition 2.10 now implies the existence of an index Λ_0 that is uniquely determined by Λ_∞ and Continuity, and Proposition 2.8 states that Λ_∞ is uniquely determined by Weak Normalization, Additivity, and Homotopy. For reasons alluded to in the verification of (b), Continuity and Homotopy are equivalent in this context.//

2.6 Euclidean Neighborhood Retracts

Up to this point the Commutativity axiom has not played a visible role, but in our next extension it is the star of the show. We now define a class of spaces that is very well known in mathematics but may be unfamiliar to economists. These spaces are ideal for the application of Commutativity.

Definition 2.13: Let D be a metric space. A *retraction* of D onto $X \subset D$ is a map $r : D \to D$ with $r(D) = \mathcal{F}(r) = X$. Equivalently, a map $r : D \to X \subset D$ is a retraction if $r \circ i = \mathrm{Id}_X$ where $i : X \to D$ is the inclusion. A *Euclidean neighborhood retract* (ENR) is (a space homeomorphic to) a subset $X \subset \mathbf{R}^m$ for which there is a retraction $r : U \to X$ of a neighborhood $U \supset X$.

Dold (1980, §IV.8) surveys the point-set topology of ENR's. In particular Dold (1980, IV.8.5) is illuminating and useful:

Lemma 2.14: The property of being an ENR is intrinsic and does not depend on the embedding: if $Y \subset \mathbf{R}^n$ is homeomorphic to an ENR X, then Y is a retract of a neighborhood in $\mathbf{R}^n$.

A smooth compact ∂-manifold is an ENR. The easy Whitney embedding theorem implies that a compact ∂-manifold can be smoothly embedded in a Euclidean space. (The proof of Hirsch (1976, Theorem 3.5, p. 24) applies equally to ∂-manifolds.) A suitable retraction can now be constructed using the collaring theorem (Hirsch (1976, Theorem 6.1, p. 113)) and the tubular neighborhood theorem (Hirsch (1976, Theorem 6.3, p. 114)). That finite simplicial complexes are ENR's follows from Whitehead's regular neighborhood theorem (e.g. Hudson (1969, Theorem 2.10, p. 55)). An economic model that gave rise to a finite dimensional fixed point space not in this class would be very surprising indeed.

Definition 2.15: Let $\mathcal{S}_{\mathrm{ENR}}$ be the class of compact ENR's. For $X \in \mathcal{S}_{\mathrm{ENR}}$ let $\mathcal{P}_{\mathrm{ENR}}(X)$ be the set of admissible pairs (f, U) with $f \in C(\bar{U}, X)$, and let $\mathcal{P}_{\mathrm{ENR}} = \bigcup_{X \in \mathcal{S}_{\mathrm{ENR}}} \mathcal{P}_{\mathrm{ENR}}(X)$.

Proposition 2.16: There is a unique extension Λ_{ENR} of Λ_0 to $\mathcal{P}_{\mathrm{ENR}}$ that satisfies Commutativity, and this extension is an index.

Proof: Appendix B.//

Since compact ∂-manifolds are ENR's, among other things we have defined an index for admissible pairs (f, U) in which $M \subset U$ is a smooth compact ∂-manifold and $f \in C^\infty(\bar{U}, M)$ has only regular fixed points. For such pairs the index has the formula

$$\Lambda_{ENR}(f, U) = \Sigma_{x \in \mathcal{F}(f)} \operatorname{sgn}|\mathrm{Id}_{\mathrm{TM_x}} - \mathrm{Df}(x)|.$$

In fact this follows from Additivity, which allows us to consider each fixed point in isolation, Commutativity, which allows the imposition of a coordinate system, and the formula for Λ_∞ in §2.4.

It is natural to wonder about the relationship between the index for pairs (f, M), $f \in C^\infty(M, M)$, and the generalization of degree theory to such maps (Milnor (1965, §5) or Hirsch (1976, §5.1)). To see that the relationship, if there is one, is not obvious, consider that only the mod 2 degree (Milnor (1965, §4) or Guillemin and Pollack (1974, §2)) can be defined on an unorientable M, but the index is always defined. The question of whether $\mathrm{Id}_{\mathrm{TM_x}} - \mathrm{Df}(x)$ is orientation preserving at a fixed point x is meaningful even if M has no global orientation. At a more sophisticated level suppose that $f : M \to M$ where M is a compact orientable (boundaryless) n-manifold. Then $\deg(f) = tr(\mathcal{H}_n(f; Q))$ while $\Lambda_{ENR}(f, M) = \Sigma_i (-1)^i \, tr(\mathcal{H}_i(f; Q))$ (see §3.1).

2.7 Absolute Neighborhood Retracts

Proposition 2.16 is insufficiently general for applications in mathematical economics in two main respects. First, an ENR is necessarily a finite dimensional space, but in some economic applications the fixed point space is infinite dimensional. Second, for many economic applications it is necessary to extend index theory to correspondences. In this subsection we discuss the extension of index theory to infinite dimensional domains, and in the next two subsections we consider correspondences.

Definition 2.17: An *absolute neighborhood retract for metric spaces* (ANR) is a metric space X with the property that $\mathrm{i}(X)$ is a neighborhood retract in Y whenever $i : X \to Y$ is an embedding of X in another metric space Y.

The theory of ANR's is very extensive and is surveyed by Borsuk (1967). For us the most important property of ANR's is the following characterization (Borsuk (1967, IV. 3.1)):

Lemma 2.18: A metric space X is an ANR if it is homeomorphic to a retract of an open subset of a convex set in a locally convex linear space, and an ANR is necessarily homeomorphic to a retract of an open subset of a convex set in a normed linear space. (Thus every ENR is an ANR.)

This covers almost all spaces that have occurred as fixed point spaces in economics. In particular, ANR's include the space of probability measures on a compact space with the weak* topology and the unit ball, with the weak* topology, in the dual of a Banach space, since these spaces are convex. In addition these spaces are compact.[7]

Remark: I do not know the answer to the following interesting question. Let $\mathcal{H}(X)$ be the set of nonempty compact subsets of a compact metric space X with the topology induced by the Hausdorff metric. Is $\mathcal{H}(X)$ an ANR? What if X is an ANR, an ENR, or a ∂-manifold?

Definition 2.19: Let $\mathcal{S}_{\text{ANR}}$ be the class of compact ANR's. For each $X \in \mathcal{S}_{\text{ANR}}$ let $\mathcal{P}_{\text{ANR}}(X)$ be the set of admissible pairs (f, U) with $U \subset X$ and $f \in C(\bar{U}, X)$, and let $\mathcal{P}_{\text{ANR}} = \bigcup_{X \in \mathcal{S}_{\text{ANR}}} \mathcal{P}_{\text{ANR}}(X)$.

The following is our first maximally general index.

Theorem 1: There is a unique extension $\Lambda_{\text{ANR}} : \mathcal{P}_{\text{ANR}} \to Z$ of Λ_{ENR} that satisfies (I4), and this extension is an index.

The argument passing from Proposition 2.16 to Theorem 1 is long and technical. Since the main elements of it are given in Brown (1971, §V) (which does not presuppose advanced techniques), it is omitted here. Roughly, the idea is that ANR's can be approximated in an appropriate sense by finite simplicial complexes.

2.8 Contractible Valued Correspondences

Economic models naturally give rise to correspondences. This direction of generalization is less important for pure mathematics, but it is discussed in several papers. (E.g. Kakutani (1941), Eilenberg and Montgomery (1946),

[7] Cf. Dunford and Schwartz (1958, §V).

O'Neill (1957b), Granas (1959a), (1959b), Fuller (1961), Browder (1968), Mas-Colell (1974), McLennan (1989a).)

Ideally one would like a result asserting the existence and uniqueness of an index for admissible pairs (F, U) in which U is an open subset of a compact ANR X, $F \in \mathcal{K}(\bar{U}, X)$, and $F(x)$ is contractible for all $x \in \bar{U}$. Whether such a result is true appears to be an open question. In lieu of a final answer we present some of the best available results.

The approach considered here is to use Proposition 2.10 to extend the index of Theorem 1 to correspondences. (We cannot discuss the homological method developed by Eilenberg and Montgomery (1946).) The success of this method is a function of the available results concerning the approximation of correspondences by maps. For contractible valued correspondences the available results are restricted to polyhedra. (Recall, however, that every ∂-manifold is triangulable - cf. footnote 1.) In this connection it is interesting to note that Granas and Jaworowski (1959), who use algebraic methods to study acyclic valued correspondences, give results that are restricted to Euclidean domains.

Definition 2.20: For metric spaces X, Y let $\mathcal{K}_{ctr}(X, Y)$ be the set of $F \in \mathcal{K}(X, Y)$ such that, for all $x \in X$, $F(x)$ is contractible.

Proposition 2.21: Suppose that $J \subset \mathbf{R}^m$ is a finite simplicial complex and Y is a compact ENR, C is a subcomplex of J, Y is a compact ENR, and $F \in \mathcal{K}_{ctr}(J, Y)$. Then for every neighborhood W of $Gr(F)$ there is a neighborhood $W' \subset W \bigcap (C \times Y)$ of $Gr(F|_C)$ such that every continuous function $f' : C \to Y$ with $Gr(f') \subset W'$ has a continuous extension $f : J \to Y$ with $Gr(f) \subset W$.

Proof: McLennan (1987a).//

It should be mentioned that the proof is a straightforward adaptation of Mas-Colell's (1974) proof for the special case $Y \subset \mathbf{R}^n$, $F(x) \subset \operatorname{int} Y$ for all x, and $C = \emptyset$. In our terminology Mas-Colell's theorem has the following simple expression:

Corollary: If J is a finite simplicial complex and Y is a compact ENR, then $C(J, Y)$ is dense in $\mathcal{K}_{ctr}(J, Y)$.

In McLennan (1989c) the following generalization is established: if X and Y are ANR's with X compact, then $C(X, Y)$ is dense in $\mathcal{K}_{ctr}(X, Y)$. Unfortunately the techniques developed there do not quite suffice to establish the theory of the index or the theory of essential sets at this level of generality.

Definition 2.22: Let $\mathcal{S}_{ctr}$ be the class of finite simplicial complexes. For each $X \in \mathcal{S}_{ctr}$ let $\mathcal{P}_{ctr}(X)$ be the set of admissible pairs (F, U) with $F \in \mathcal{K}_{ctr}(\bar{U}, X)$. Let $\mathcal{P}_{ctr} = \bigcup_{X \in \mathcal{S}_{ctr}} \mathcal{P}_{ctr}(X)$.

Theorem 2: There is a unique extension $\Lambda_{ctr} : \mathcal{P}_{ctr} \to Z$ of Λ_{ENR} that satisfies (I5), and this extension is an index.

Proof: Appendix D.//

2.9 Convex Valued Correspondences

Convex valued correspondences with convex domains arise naturally in the two central models of economic theory, general economic equilibrium and

Nash equilibrium. A finite dimensional compact convex set is homeomorphic to a simplex and so is covered by Theorem 2. We now work out the infinite dimensional case.

Suppose that X is a metric space and Y is a subset of a locally convex space. We let $\mathcal{K}_{\text{con}}(X, Y)$ be the set of upper hemicontinuous $F : X \to Y$ whose values are compact and convex. As above, success in using Proposition 2.10 to extend the index is proportional to our ability to approximate elements of $\mathcal{K}_{\text{con}}(X, Y)$ with maps.

Definition 2.23: By a *convex ANR* we will henceforth mean a metrizable compact convex subset of locally convex topological vector space. (By Lemma 2.18 a convex ANR is indeed an ANR.) Let $\mathcal{S}_{\text{con}}$ be the class of convex ANR's. For each $X \in \mathcal{S}_{\text{con}}$ let $\mathcal{P}_{\text{con}}(X)$ be the set of admissible pairs (F, U) with $F \in \mathcal{K}_{\text{con}}(\bar{U}, X)$. Let $\mathcal{P}_{\text{con}} = \bigcup_{X \in \mathcal{S}_{\text{con}}} \mathcal{P}_{\text{con}}(X, Y)$.

Proposition 2.24: If X is a compact metric space and Y is a convex ANR, then $C(X, Y)$ is dense in $\mathcal{K}_{\text{con}}(X, Y)$.

Proof: Appendix C.//

Theorem 3: There is a unique extension $\Lambda_{\text{con}} : \mathcal{P}_{\text{con}} \to Z$ of the restriction of Λ_{ANR} to $\mathcal{P}_{\text{con}} \bigcap \mathcal{P}_{\text{ANR}}$ that satisfies (I5), and this extension is an index.

Proof Appendix C.//

It is not obvious that it is necessary that Y be convex here. Of course allowing convex valued correspondences with nonconvex ranges has no apparent usefulness in economics, but to outline the apparent limits of our methods we include a result in this direction.

Proposition 2.25: If X is a compact metric space and Y is a compact ANR embedded in a metrizable topological vector space, then $C(X, Y)$ is dense in $\mathcal{K}_{\text{con}}(X)$.

Proof: Appendix C.//

Of course Lemma 2.18 implies that an ANR embedded in a locally convex space can be embedded in a normed space, but the embedding need not be linear, so a correspondence that is convex valued in a given embedding may be only contractible valued in the new embedding.

3 The Index and Existence of Fixed Points

3.1 The Lefschetz Fixed Point Theorem

The Lefschetz fixed point theorem (LFPT) is one of the most important theorems of the twentieth century. It has Brouwer's fixed point theorem and the theory of the Euler characteristic, as well as many other results, as immediate corollaries. It is a triumph and guiding light of algebraic topology. Index theory is a refinement of it, and we now show how the geometric content of the LFPT can be derived from index theory.

For the most part our arguments depend only on the index axioms and apply equally to the settings of Theorems 1 - 3. In the remainder we therefore adopt the convention that X, $\mathcal{P}(X, X)$, and Λ may have any of the descriptions given in the hypotheses of Theorems 1 - 3:

(1) $X \in \mathcal{S}_{\mathrm{ANR}}$, $\mathcal{P}(X, X) = \mathcal{P}_{\mathrm{ANR}}(X, X)$ and $\Lambda = \Lambda_{\mathrm{ANR}}$;

(2) $X \in \mathcal{S}_{ctr}$, $\mathcal{P}(X, X) = \mathcal{P}_{ctr}(X, X)$, and $\Lambda = \Lambda_{ctr}$;

(3) $X \in \mathcal{S}_{\mathrm{con}}$, $\mathcal{P}(X, X) = \mathcal{P}_{\mathrm{con}}(X, X)$, and $\Lambda = \Lambda_{\mathrm{con}}$.

We associate an integer with each allowed map or correspondence as follows.

Definition 3.1: The *Lefschetz number* of $F \in \mathcal{P}(X, X)$ is $L(F) = \Lambda(F, X)$.

Theorem 4: If $L(F) \neq 0$, then F has a fixed point.

Proof: This follows from Additivity - cf. Remark (1) above.//

Our definition of the Lefschetz number is not the standard one, so Theorem 4 is not precisely the LFPT. The standard definition of the (homological) Lefschetz number of a map $f : X \to X$ is $L_H(f) = \Sigma_i \, (-1)^i \, tr(H_i(f; \mathbf{Q}))$, and the customary statement of the LFPT is that $L_H(f) \neq 0$ implies $\mathcal{F}(f) \neq \emptyset$. This result follows from Theorem 4 and the following important homological property of the index.

(I1′) (Normalization) $L(f) = L_H(f)$ for all $f \in \mathcal{P}(X, X) \bigcap C(X, X)$.

According to Brown (1971, p. 73), a relatively easy proof of Normalization is given in the Ph.D. thesis of D. McCord (1970), and of course the LFPT is treated in many books on algebraic topology, e.g. Dold (1980). Some authors have worked to extend the definition of the homological Lefschetz number to correspondences (Eilenberg and Montgomery (1946), O'Neill (1953), and Granas (1959a, 1959b)), and in general the corresponding generalization of Normalization holds.

For any X one has the identity function and the constant maps. In each case the LFPT has important consequences.

Definition 3.2: The *Euler characteristic* of a compact ANR X is $\chi(X) = L(Id_X)$.

Corollary 1: If $\chi(X) \neq 0$, then every map $f : X \to X$ homotopic to Id_X has a fixed point.

Combining Weak Normalization and Homotopy yields:

Corollary 2: If $f : X \to X$ is homotopic to a constant map, X a compact ANR, then f has a fixed point.

Any map $f : X \to X$ is homotopic to a constant map if X is contractible, so we obtain a quite general fixed point theorem of the type originated by Brouwer (1912).

Corollary 3: If X is a compact contractible ANR, then $L(F) = 1$ for any $F \in \mathcal{S}(X, X)$, hence $\chi(X) = 1$, and any such F has a fixed point.

It is natural to ask whether there is a converse to the LFPT. That is, if $L(f) = 0$ for a map $f : X \to X$, is there a map homotopic to f that has no fixed points? There are counterexamples (e.g. Brown (1971, §II.B)) in which X is a simplicial complex, but for a "nice" class of simplicial complexes that includes simplicial ∂-manifolds of dimension $n \geq 3$ the answer is yes (cf. Brown, (1971, Th. VIII.E.1) - this result was first proved for differentiable manifolds by Hopf (1929).) For homotopy classes of maps on these spaces the LFPT is the ultimate fixed point theorem, providing both necessary and sufficient conditions for all elements of the homotopy class to have fixed points.

3.2 Applications in General Equilibrium Theory

We now relate the ideas of §3.1 to the methods used to prove the existence of general economic equilibrium.

Definition 3.3: A *smooth aggregate excess demand function* is a C^∞ function $\zeta : \mathbf{R}^\ell_{++} \to \mathbf{R}^\ell$ that has the following properties:

(a) (Homogeneity of Degree Zero) $\zeta(\lambda p) = \zeta(p)$ for all $p \in \mathbf{R}^\ell_{++}$ and $\lambda > 0$;

(b) (Walras' Law) $p \cdot \zeta(p) = 0$ for all $p \in \mathbf{R}^\ell_{++}$;

(c) (Nonsatiation) If $\{p_j\}$ is a sequence in $\mathbf{R}^\ell_{++}$ with $p_j \to p \in (\mathbf{R}^\ell_{+} - \{0\}) - \mathbf{R}^\ell_{++}$, then $||\zeta(p_j)|| \to \infty$.

(d) (Boundedness) There is $\underline{z} \in \mathbf{R}^\ell$ such that $\zeta(p) > \underline{z}$ for all p.

We refer the reader to Debreu (1972) for a derivation of these properties of aggregate excess demand from assumptions concerning preferences and endowments.

Homogeneity implies that we may regard ζ as a function defined on rays out of the origin in $\mathbf{R}^\ell_{++}$, and in looking for equilibria we may restrict attention to any submanifold that intersects each such ray exactly once. Walras' Law implies that if we take the price space to be $S^\ell_{++} = \{p >> 0 \,|\, ||p|| = 1\}$, the positive part of the unit sphere, then we may regard ζ as a vector field on S^ℓ_{++}.

There are many technical details (cf. Mas-Colell (1985, §5.6)), but after they have been dealt with nonsatiation implies that one can find a compact $(\ell-1)$-dimensional ∂-manifold $M \subset S^{\ell-1}_{++}$ and a vector field ζ' on M with the following properties: (i) M is homeomorphic to $D^{\ell-1} = \{x \in \mathbf{R}^{\ell-1} \mid ||x|| \leq 1\}$; (ii) there is an open set U with $\bar{U} \subset M - \partial M$ that contains all zeros of ζ, ζ and ζ' agree on $\bar{U}$, and $(1-t)\zeta(p) + t\zeta'(p) \neq 0$ whenever $p \in M - \bar{U}$ and $0 \leq t \leq 1$; (iii) $\zeta(p)$ is inward-pointing at all points $p \in \partial M$. The existence of an equilibrium now follows if we can show that there is no nonvanishing tangent vector field on M that is inward-pointing on ∂M.

We now digress in order to explain the relationship between the fixed point index and the index of a zero of a vector field. We begin with the Euclidean case.

Definition 3.4: Let x be an isolated zero of a vector field $\zeta : U \to \mathbf{R}^n$ where $U \subset \mathbf{R}^n$ is open. The *index* $\mathrm{ind}_x(\zeta)$ of ζ at x is the degree (cf. Appendix A) of the map $\zeta_\varepsilon : S^{n-1} \to S^{n-1}$ given by $\zeta_\varepsilon(v) = \zeta(x + \varepsilon v)/||\zeta(x + \varepsilon v)||$ where ε is small enough that $x + \varepsilon D^n \subset U$. (The invariance of the degree under homotopy implies that the choice of ε does not matter.)

Consider the special case where x is a regular zero of ζ, i.e. $D\zeta(x)$ is nonsingular. Since $\zeta(y) = D\zeta(x)(y-x) + o(||y-x||)$, there is a neighborhood of x in which ζ is homotopic to the linear vector field $y \mapsto D\zeta(x)(y-x)$ by a homotopy (specifically convex combination) whose only zero at each time is x. Homotopy invariance of the degree implies that $\mathrm{ind}_x(\zeta)$ equals the index of its first order Taylor's approximation, and it also implies that

$\text{ind}_x(\zeta)$ depends only on the path component of $GL(n)$[8] that contains $D\zeta(x)$. Concrete calculation then yields the formula $\text{ind}_x(\zeta) = \text{sgn}|D\zeta(x)|$. In order to express $\text{ind}_x(\zeta)$ in terms of the Lefschetz index we introduce the following notation: if X is a metric space each of whose points has a neighborhood that is a compact ANR, $U \subset X$ is open, $f \in C(U, X)$, and x^* is an isolated fixed point of f, then let $\Lambda(f; x^*)$ denote $\Lambda_{\text{ANR}}(f, V)$ where V is a neighborhood of x such that $\bar{V} \bigcup f(\bar{V})$ is contained in a compact ANR and $\mathcal{F}(f|_{\bar{V}}) = x^*$. (Additivity implies that the definition does not depend on the choice of V.) Comparison of the formula above with the formula for the smooth fixed point index (§2.4) yields

$$\text{ind}_x(\zeta) = \Lambda(\text{Id}_{\bar{V}} - \zeta; x)$$

for sufficiently small neighborhoods V of x.

Now suppose that x is an isolated zero of a vector field ζ defined on a compact manifold M. Imposing a coordinate system allows us to define the index $\text{ind}_x(\zeta)$, and it is of course a standard result that $\text{ind}_x(\zeta)$ is independent of the coordinate system.

In order to make sense out of the equation above in a coordinate free framework it is necessary to replace $\text{Id}_{\bar{V}} - \zeta$ with the result of following the vector field $-\zeta$ for a small period of time. Let $\Phi : M \times \mathbf{R} \to M$ be the flow[9] associated with ζ. We find that $\text{ind}_x(\zeta) = \Lambda(\Phi_t; x)$ for *negative* t close to 0. Of course Φ_t is homotopic to Id_M for all t, and the formula for $\text{ind}_x(\zeta)$ yields $\text{ind}_x(-\zeta) = (-1)^n \text{ind}_x(\zeta)$, so summing over all the zeros of ζ yields the well known result $\chi(M) = -\chi(M) = 0$ if n is odd.

Finally, suppose that M is a compact ∂-manifold. Now the flow will not

[8] $GL(n)$ is the group of all nonsingular linear transformations from $\mathbf{R}^n$ to itself.

[9] That is, for each x, $\Phi(x, \cdot)$ is the unique solution of the ordinary differential equation $\partial[\Phi(x, t)]/\partial t = \zeta(\Phi(x, t))$ with $\Phi(x, 0) = x$. Standard results (e.g. Spivak (1979, §5)) show that the flow exists and is uniquely determined by this condition.

typically be defined on all of $M \times \mathbf{R}$. However, one can show without great difficulty that the flow is defined on $M \times (-\infty, 0]$ if ζ is outward pointing on ∂M. Combining the information above, we find that we have sketched a proof of the following famous result.

Poincare-Hopf Theorem: Let M be a compact ∂-manifold, and consider a vector field ζ that has regular zeros and is outward pointing on ∂M. Then the sum of the indices of ζ at its zeros is $\chi(M) = L(\Phi_t)$, $t \leq 0$.

In the case of excess demand it is $-\zeta$ that is outward pointing on ∂M, the boundary of the $(\ell - 1)$-dimensional ∂-manifold satisfying (i) - (iii). Since (i) implies that $\chi(M) = 1$, $-\zeta$ must have a zero, and if all zeros are regular then $\Sigma_{\zeta(p)=0} \operatorname{ind}_p(-\zeta) = 1$. For the excess demand vector field we have $\Sigma_{\zeta(p)=0} \operatorname{ind}_p(\zeta) = (-1)^{\ell+1}$, Dierker's (1972) result.

The Euler characteristic defined above is a special case of a more general construct. If $\xi = (E, M, \pi)$ is a smooth vector bundle[10] with M a compact manifold, then the Euler characteristic $\chi(\xi)$ of ξ is the self-intersection number[11] of the zero section. What we referred to above as the Euler characteristic of M is, in this broader terminology, the Euler characteristic of the tangent bundle of M.

In particular, if $\chi(\xi) \neq 0$ then every section of ξ has a zero. This result was used by Hirsch, Magill, and Mas-Colell (1987) to give an elegant proof of Duffie and Shafer's (1985) theorem asserting the generic existence of equilibria in economies with incomplete markets and "real" financial assets. Husseini, Lasry, and Magill (1987) use cohomology to explore this methodol-

[10] Cf. Hirsch (1976, §4).
[11] Cf. Guillemin and Pollack (1974, §3).

ogy in somewhat greater detail. Geanakoplos and Shafer (1987) give a general description of the methodology underlying Duffie and Shafer's original proof, which is based on degree theory.

I know of only one other instance in which a fixed point theorem for a noncontractible domain has been applied in general equilibrium theory. Hart and Kuhn (1975) use the following result to prove the general equilibrium existence theorem without the assumption of free disposal. This is a good example of a result in which Normalization provides the only practical method of computing Lefschetz numbers even though in principle they should be derivable from the index axioms alone.

Proposition 3.5: If $f : S^n \to S^n$ is a map without a fixed point, then there is a point p with $f(-p) = -f(p)$.

Proof: This is implied by the following facts: (a) if f has no fixed points then $L(f) = 0$; (b) if $f(-p) \neq -f(p)$ for all p, then $L(f)$ is odd. Of course (a) is Theorem 4. Whittlesey (1963, Th. 2) uses homology to prove that the Brouwer degree of f is even whenever f does not map antipodal points to antipodal points, and this implies (b) since, by the definitions of $L_H(f)$ and the Brouwer degree, $L_H(f) = 1 + (-1)^n tr(H_n(f)) = 1 + (-1)^n \deg(f)$.//

4 Robustness of Sets of Fixed Points

4.1 Definition

The theory of essential sets of fixed points studies the robustness of fixed points with respect to perturbations of the function or correspondence. As we will see in §4.4, the Lefschetz index subsumes the theory of essential sets to a very great extent, but the theory still has considerable independent interest for economics since it provides the point of comparison with the work of KM.

The notion of an essential fixed point was first defined for functions by Fort (1950). The following generalization is due to Kinoshita (1952) and O'Neill (1953).

Definition 4.1: A compact set $K \subset \mathcal{F}(F)$ is *essential* for $(F, U) \in \mathcal{P}(X)$ if, for every neighborhood V of K, there is a neighborhood $A \subset \mathcal{P}(X, U)$ of F such that $\mathcal{F}(G) \bigcap V \neq \emptyset$ for all $G \in A$. Let $\mathcal{E}ss(F) \subset \mathcal{H}(\mathcal{F}(F))$ be the set of essential subsets of $\mathcal{F}(F)$.

A theory of robustness with respect to perturbations must specify both the class of functions to which the theory applies and the space of perturbations with respect to which robustness is measured. It is natural to let these be the same space, and in the exposition we shall do so, but for many aspects of the theory this is not necessary. In the applications of interest $C(\bar{U}, X)$ is dense in $\mathcal{P}(X, U)$, so it makes no difference whether the space of perturbations is $C(\bar{U}, X)$ or some superset in $\mathcal{P}(X, U)$.

4.2 Essential Sets and the Half Hausdorff Topology

The following obvious fact is used repeatedly.

Lemma 4.2: If $(F, U) \in \mathcal{P}(X)$ and $K \subset L \subset \mathcal{F}(F)$ with K essential and L compact, then L is essential. Thus F has essential sets if and only if $\mathcal{F}(F)$ is essential.

We now present a final piece of evidence that the half Hausdorff topology is the "correct" topology for the fixed point theory of correspondences.

Proposition 4.3: For each $X \in \mathcal{S}$ and each open $U \subset X$, the set valued function $\mathcal{E}ss : \mathcal{P}(X, U) \to \mathcal{H}(U)$ is lower semicontinuous.

Proof: This is a rather fancy way of saying that if K is essential for $F \in \mathcal{P}(X, U)$, then for every neighborhood U of K one can find a neighborhood $A \subset \mathcal{P}(X, U)$ of F such that $\mathcal{F}(G) \bigcap \bar{U}$ is essential for each $G \in A$. Suppose that $G_n \to F$ with, say, $Gr(G_n) \subset \mathbf{B}(Gr(F; \varepsilon_n))$ for a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$, and each $\mathcal{F}(G_n) \bigcap \bar{U}$ is inessential for G_n. Then for any sequence $\delta_n \to 0$ we can find a correspondence $H_n \in \mathcal{P}(X, U)$ with $Gr(H_n) \subset \mathbf{B}(Gr(G_n; \delta_n))$ that has no fixed points in $\bar{U}$. But then $Gr(H_n) \subset \mathbf{B}(Gr(F; \delta_n + \varepsilon_n))$ and $\delta_n + \varepsilon_n \to 0$, a contradiction of the assumption that K is essential for F.//

This result is used by Allen (1985) to prove the existence, for a residual set of excess demands, of a weakly continuous global random selection from the Walrasian equilibrium correspondence. That is, for excess demands in a residual set there is a weakly continuous function from endowments to probability distributions on the set of equilibria. Green (1986) establishes a similar result for the Nash equilibrium correspondence using the Leray-Schauder degree. The relationship between the Leray-Schauder degree and the Lefschetz index generalizes the relationship displayed in Appendix A.

4.3 Minimal Essential Sets

Usually one is not interested in all essential sets. If one hopes to talk of a single "essential equilibrium" one apparently must mean a minimal essential set. In the simplest case when $\mathcal{F}(F)$ is finite, each fixed point is essential or inessential by itself, and an essential set is one that contains an essential fixed point. In general, if every essential set contains a minimal essential set, then an essential set is precisely a compact set of fixed points that contains a minimal essential set. In this subsection we show that every essential set contains a minimal essential sets, and with certain assumptions minimal essential sets, like singletons, are connected.

Lemma 4.4: If $X \in \mathcal{S}, (F, U) \in \mathcal{P}(X)$, and $\{K_j\}$ is a descending ($K_j \supset K_{j+1}$) sequence of essential sets for (F, U), then $K = \bigcap_j K_j$ is essential.

Proof: First note that K is nonempty and compact since each K_j has these properties (cf. Kelley (1955, p. 136)). Let U be a neighborhood of K. Clearly $K_j \subset U$ for large j, and since U is a neighborhood of K_j and K_j is essential, all correspondences sufficiently close to F have fixed points in U.//

Proposition 4.5: Every essential set contains a minimal essential set.

Proof: Let $\{\mathbf{B}(x_j; r_j)\}_{r=1,2,\ldots}$ be an enumeration of the set of open balls of rational radii around some countable dense subset of X. Let K_0 be an essential set, and define K_j inductively by letting $K_j = K_{j-1} - \mathbf{B}(x_j; r_j)$ if $K_{j-1} - \mathbf{B}(x_j; r_j)$ is essential and letting $K_j = K_{j-1}$ otherwise. Then each K_j is essential, so $K = \bigcap_j K_j$ is essential by Lemma 4.4.

If K is not minimal then $K - \mathbf{B}(x_j; r_j) \neq K$ is essential for some j, but then $K_{j-1} - \mathbf{B}(x_j; r_j)$ is essential and $K \subset K_j = K_{j-1} - \mathbf{B}(x_j; r_j)$, a contradiction.//

Theorem 5 is to be understood as referring equally to the three settings given by our interpretation of the symbols $X, \mathcal{P}(X)$, and Λ. This result is applied in Gale (1987a) and Gale (1987b).

Theorem 5: Minimal essential sets of pairs in $\mathcal{P}(X)$ are connected.

Proof: Appendix D.//

4.4 Essential Sets and the Index

In view of the Continuity axiom it is not surprising that the index and essential sets are closely related. We now show that if M is a compact ∂-manifold, $(\mathcal{F}, U) \in \mathcal{P}_{\mathrm{ENR}}(M)$ or $(F, U) \in \mathcal{P}_{con}(M)$ (where M is convex), and $K = \mathcal{F}(F)$ is connected, then K is essential if and only if $\Lambda(F, U) \neq 0$. This result is basically due to O'Neill (1953). It appears to account for the relative neglect of essential sets in subsequent literature.

Proposition 4.6: If $(F, U) \in \mathcal{P}(X)$ and $\Lambda(F, U) \neq 0$, then $\mathcal{F}(F)$ is essential.

Proof: A sufficiently small neighborhood A of F has the property that no $G \in A$ has any fixed points in $\bar{U} - U$, since, for instance, we could let A be the set of correspondences in $\mathcal{S}(X, X)$ whose graphs are contained in some neighborhood of the graph of F that does not intersect $\{(x, x) \mid x \in \bar{U} - U\}$. The Continuity axiom implies that we can find a smaller neighborhood A' on which $\Lambda(G, U) = \Lambda(F, U) \neq 0$ is constant. Of course this implies that all $G \in A'$ have fixed points in U.//

Proposition 4.6 has a converse for smooth manifolds, and in this setting the possibilities for $\mathcal{F}(g)$ when g is near F can be described in detail.

Proposition 4.7: Let M be a smooth compact n-dimensional ∂-manifold, $n \geq 2$. Suppose either that $(F, U) \in \mathcal{P}_{\text{ENR}}(M)$ (so that F is a function) or that M is convex and $(F, U) \in \mathcal{P}_{con}(M)$. Suppose that $\mathcal{F}(F) \subset \text{int}\, M$ is connected. Let $x_1, \ldots, x_k$ be distinct points in $\mathcal{F}(F)$, and let $r_1, \ldots, r_k \in \{-1, 1\}$ be integers with $\Sigma_i r_i = \Lambda_0(F, U)$. Then for every $\delta > 0$ there is $g \in C^\infty(\bar{U}, M)$ with $Gr(g) \subset \mathbf{B}(Gr(F); \delta)$, $\mathcal{F}(g) = \{x_1, \ldots, x_k\}$, and

$$\Lambda(g, x_i) = \lim_{\varepsilon \to o} \Lambda(g, \mathbf{B}(x_i; \varepsilon)) = r_i \ \ (i = 1, \ldots, k).$$

Proof: Appendix E.//

Remarks: (1) With additional work we could remove the restriction $r_i \in \{-1, 1\}$. I suspect that it is also possible to remove the restriction $\mathcal{F}(F) \subset \text{int}\, M$ and allow $(F, U) \in \mathcal{P}_{ctr}(M)$, but I have been unable to do so.

(2) In a technical sense Proposition 4.7 holds for the case $n = 0$, but it is not an interesting result. The case $n = 1$ is special since, roughly, indices of the fixed points must be -1, 0, or 1, and the nonzero indices must alternate, but at least it is still the case that if a connected set of fixed points is clopen in $\mathcal{F}(F)$ and has index zero, it is inessential.

In view of the intended applications to game theory it is fortunate that the following result does not require $\mathcal{F}(F) \subset \text{int}\, M$.

Theorem 6: Let C be a convex compact subset of a Euclidean space, and suppose that $(F, U) \in \mathcal{P}_{\rm con}(C)$ with $\mathcal{F}(F)$ connected. If $\Lambda(F, U) = 0$ then $\mathcal{F}(F)$ is inessential, and $\mathcal{F}(F)$ is the unique minimal essential set when $\Lambda(F, U) \neq 0$.

Proof: Appendix F.//

We also have an analogous result for functions on ∂-manifolds.

Theorem 7: Let M be a smooth compact n-dimensional ∂-manifold, and suppose that $(f, U) \in \mathcal{P}_{\rm ENR}(M)$ with $\mathcal{F}(f)$ connected. If $\Lambda(f, U) = 0$ then $\mathcal{F}(f)$ is inessential, and $\mathcal{F}(f)$ is the unique minimal essential set when $\Lambda(f, U) \neq 0$.

Proof: Appendix F.//

4.5 Applications to Game Theory

Finally we describe the niche in this theory occupied by the solution concepts of KM. For the most part their concepts fit into the following abstract framework.

Definition 4.8: Fix $(F, U) \in \mathcal{P}(X)$. Let $(A, *)$ be a pointed space[12], and let $Q : (A, *) \to (\mathcal{P}(X, U), F)$ be a pointed map. A compact set $K \subset \mathcal{F}(F)$ is *Q-stable* if, for every neighborhood $V \subset U$ of K, there is a neighborhood W of $*$ such that $\mathcal{F}(Q(a)) \bigcap V \neq \emptyset$ for all $a \in W$.

[12] A *pointed space* is a pair (X, x_0) where X is a topological space and x_0 is a distinguished point in x. A *pointed map* between pointed spaces (X, x_0) and (Y, y_0) is a map $f : X \to Y$ with $f(x_0) = y_0$.

We note some elementary properties of Q-stable sets.

Lemma 4.9: (1) An essential set is necessarily Q-stable. (2) If Q-stable sets exist then minimal Q-stable sets exist.

Proof: (1) Suppose that K is essential for F, and let $V \subset U$ be a neighborhood of K. Then there is a neighborhood $B \subset \mathcal{P}(X, U)$ of F such that $\mathcal{F}(G) \bigcap V \neq \emptyset$ for all $G \in B$, and $Q^{-1}(B)$ is an open neighborhood of a_0 since Q is continuous. (2) With suitable modifications the proof of Proposition 4.5 implies the desired result.//

Our notation for normal form games is standard. The set of *players* or *agents* is $I = \{1, \ldots, n\}$. For each i, S_i is a nonempty finite set of *pure strategies,* and $S = \prod_i S_i$ is the set of *pure strategy vectors.* The *utility function* or *payoff* is $u = \prod_i u_i : S \to \mathbf{R}^I$. A *mixed strategy* for i is a probability measure σ_i on S_i; let $\Delta(S_i)$ be the set of mixed strategies for i. The set of *mixed strategy vectors* is $\Sigma = \prod_i \Delta(S_i)$. We extend u to Σ by taking expectations with respect to the product measure:
$u(\sigma) = \int_S u \, d(\sigma_1 \times \ldots \times \sigma_n)$.

The *best response correspondence* $BR = \prod_i BR_i : \Sigma \to \Sigma$ is defined by

$$BR_i(\sigma) = \text{argmax}_{\tau_i \in \Delta(S_i)} \; u_i(\sigma \,|\, \tau_i)$$

where $\sigma \,|\tau_i$ is the mixed strategy vector obtained by replacing the i-component of σ with τ_i. Of course it is easy to show that $BR \in \mathcal{K}_{con}(\Sigma, \Sigma)$. By definition $\mathcal{F}(BR)$ is the set of Nash equilibria. It is useful to know that it has a simple structure.

Lemma 4.10: $\mathcal{F}(BR)$ consists of finitely many connected components.

Proof: The conditions defining Nash equilibria are polynomial equations and weak inequalities in u and σ, so $\mathcal{F}(BR)$ is a semialgebraic variety[13]. A theorem of Hironaka (1975) states that a semialgebraic variety can be triangulated, and since $\mathcal{F}(BR)$ is compact, the triangulation must be finite (cf. footnote 1). The claim now follows easily.//

The definitions below relate the solution concepts of KM to the ideas of this paper. In the case of the first concept what we present is a variant of KM's hyperstability that avoids the notion of equivalence between games.

Definition 4.11: Let $A_H = \mathbf{R}^{S \times I}$ be the space of possible payoffs (with the usual topology). The pointed map

$$Q_H : (A_H,\, u) \rightarrow (\mathcal{K}_{con}(\Sigma,\, \Sigma),\, BR)$$

is defined by $Q_H(u') = BR(u')$ where $BR(u')$ is the best response correspondence for the payoff u'. A compact set $K \subset \mathcal{F}(BR)$ is *semihyperstable* if it is Q_H-stable and minimal with respect to this property.

Definition 4.12: Let A_F be the space whose elements are $* = (\Delta(S_1), \ldots,\, \Delta(S_n))$ and all vectors $P = (P_1, \ldots,\, P_n)$ in which each P_i is a nonempty convex polyhedron in the interior of $\Delta(S_i)$. Endow A_F with the topology induced by the Hausdorff metric (e.g. Hildenbrand (1974, p. 16)). Define the pointed map

$$Q_F : (A_F,\, *) \rightarrow (\mathcal{K}_{con}(\Sigma,\, \Sigma),\, BR)$$

[13] A *semialgebraic variety* is the solution set of a finite system of polynomial equations and inequalities.

by setting $Q_F(P) = BR_P$ where $BR_P = \prod_i BR_{P_i} : \Sigma \to \Sigma$ is defined by

$$BR_{P_i}(\sigma) = \operatorname{argmax}_{\tau_i \varepsilon P_i} \; u_i(\sigma \mid \tau_i).$$

A compact set $K \subset \mathcal{F}(BR)$ is *fully stable* if it is Q_F-stable and minimal with respect to this property.

A *tremble* is a vector $\varepsilon = (\varepsilon_i)$ of functions $\varepsilon_i : S_i \to (0, 1]$ with the property that $\Sigma_{s_i \,\varepsilon\, S_i} \; \varepsilon_i(s_i) \leq 1$. A tremble induces a vector of polyhedra $P(\varepsilon) = (P_1(\varepsilon), \ldots, P_n(\varepsilon))$ where

$$P_i(\varepsilon) = \{s_i \in \Delta(S_i) \mid \sigma_i(s_i) \geq \varepsilon_i(s_i) \,\text{for all}\, s_i \in S_i\}.$$

Definition 4.13: Let A_S be the subspace of A_F consisting of $*$ and the vectors of polyhedra induced by trembles (with the subspace topology). Let Q_S be the restriction of Q_F to A_S. A compact set $K \subset \mathcal{F}(BR)$ is *stable* if it is Q_S-stable and minimal with respect to this property.

Remarks: (1) Of course Lemma 4.9 implies existence results for these concepts.

(2) An example due to Gul (KM, p. 1027-8) shows that stable sets need not be connected. Variants of these concepts can be defined by requiring sets that are minimal with respect to the property of being both connected and Q-stable.

It is evident that, in addition to the concepts introduced by KM, many other solution concepts can be defined using Q-stability. Moreover, the notions of minimal essential set (or essential component), component of positive index, and component of odd index, all generate solution concepts for which there are existence results. Mertens (1988) explains in detail how cohomology can be used to define subtle variants of these concepts, and Hillas (1989) has defined yet another notion of stability with many attractive properties. We do not wish to enter into a discussion of the relative philosophical merits of the various possibilities. Instead we simply note that essential sets and the index are likely to play an important role in the mathematical analysis of any solution concept of this type.

At the level of connected components the notion of an essential set is more restrictive that Q-stability: an essential set is Q-stable for any Q. This is particularly important in connection with Kreps and Wilson's (1982, p. 881) result that for generic extensive form payoffs the set of Nash equilibrium paths[14] of an extensive game is finite. The map from equilibria to paths is continuous, so the preimage of an isolated path is a union of components of the set of equilibria. Thus (in the generic case) essential sets and the index provide maximally restrictive refinements for paths.

McLennan (1987b) shows that the set of sequential equilibria (Kreps and Wilson (1982)) can be represented as the set of fixed points of a contractible valued correspondence defined on a space homeomorphic to a disk. Thus the index, essential sets, and the Q-stability can be used to define refinements of the sequential equilibrium concept. Our inability to extend Theorem 6 to contractible valued correspondences is particularly unfortunate from this point of view.

[14] The *path* of a mixed strategy vector of the normal form (or the agent normal form) of an extensive form game is the induced probability distribution on terminal nodes. See Kreps and Wilson (1982).

4.6 Historical Remarks

Lefschetz announced his result in 1923 and published a long paper on the subject in 1926, but he did not succeed in showing that the LFPT held for all finite simplicial complexes. This was first accomplished by Hopf (1928), and the basis of the argument was a fact of homological algebra known as the Hopf trace theorem that implies Normalization for simplicial maps[15]. This technical result has been a basic element of virtually all treatments of the subject since then.

The possibility of localizing the Lefschetz number, thereby defining an index, was pointed out by Hopf (1929) and developed by Leray (1945) and Browder (1948). The first axiomatic characterization of the index was given by O'Neill (1953). There are several differences between his axioms and the ones given in §2.

First, neither Continuity nor Multiplication appear in O'Neill's axiom system. As we mentioned in the Remarks following Definition 2.6, Continuity is implied by Homotopy in most settings and Multiplication is usually regarded as a property of the index rather than an axiom.

Second, his index is defined only on pairs (f, U) with f defined on all of X, so Localization (Remark (3)) was one of O'Neill's axioms. In this respect the domain of his index is smaller than the classes of pairs we consider.

Third, O'Neill's axiom system has Normalization in place of Weak Normalization. As late as 1970 one can find a well informed author (Faddell (1970, p. 11)) under the impression that the presence of Normalization in the axiom system made the index inherently homological.

[15]Roughly a map $f : X \to Y$ between simplicial complexes is *simplicial* if there is a triangulation of $X \times Y$ compatible with the given triangulations of X and Y in which $Gr(f)$ is a subcomplex (union of simplices). See Hudson (1969, §I) for details.

At about that time Brown (1970) showed that the uniqueness of the index could be obtained by elementary methods from an axiom system with Weak Normalization in place of Normalization. His methods are similar to those employed in §2, and he remarked that they could be used to construct an index, thereby proving existence as well, but he dismisses the possibility as of little use in view of Dold's (1965) elegant homological definition. Of course we find the possibility of developing the geometric content of index theory by purely geometric methods quite interesting in itself, both conceptually and historically, even if the principle practical benefit is to circumvent a pedagogical problem peculiar to economics.

Brown (1971) surveys the theory of the Lefschetz index and related work. This book is the source of many of our historical remarks. A sample of more recent work can be found in Faddell and Fournier (1981).

Appendix A: Smooth Index Theory

The purpose of this appendix is to prove Proposition 2.8. In order to highlight the relationship between the index and the degree we begin with a parallel treatment of degree theory.

Definition B.1: Let $X \subset \mathbf{R}^m$ be a domain. A *regular pair* for X is a pair (f, U) in which $U \subset X$ is open and $f : \bar{U} \to \mathbf{R}^m$ is a C^∞ map with $f^{-1}(0) \bigcap (\bar{U} - U) = \emptyset$ that has 0 as a regular value. Let $\mathcal{G}_\infty(X)$ be the set of regular pairs for X.

Proposition B.2: There is a unique function deg: $\mathcal{G}_\infty(X) \to Z$ with the following properties:

(D1) deg $(-y + \mathrm{Id}_X, X) = 1$ if $y \in \mathrm{int}\, X$.

(D2) If $(f, U) \in \mathcal{G}_\infty(X)$ and $U_1, \ldots, U_r$ are disjoint open subsets of U with $f^{-1}(0) \subset U_1 \bigcup \ldots \bigcup U_r$, then deg $(f, U) = \Sigma_i \deg(f|_{\bar{U}_i}, U_i)$.

(D3) If $U \subset X$ is open, h, $\bar{U} \times [0, 1] \to \mathbf{R}^m$ is a (not necessarily smooth) homotopy with $h^{-1}(0) \subset U \times [0, 1]$, and $(h_0, U), (h_1, U) \in \mathcal{G}_\infty(X)$, then deg $(h_0, U) = \deg(h_1, U)$.

Proof: Fix $(f, U) \in \mathcal{G}_\infty(X)$. Since 0 is a regular value of f, $f^{-1}(0)$ is discrete (by the inverse function theorem) and thus finite, so we may define

$$\deg(f, U) = \sum_{x \in f^{-1}(0)} \operatorname{sgn} |\, Df(x) \,|.$$

This formula immediately implies (D1) and (D2).

If the hypotheses of (D3) are satisfied then, using simple but tedious technical constructions, one can find a compact ∂-manifold $W \subset \bar{U} \times [0, 1]$ that contains $h^{-1}(0)$ in its interior relative to $\bar{U} \times [0, 1]$. The conclusion of (D3) now follows from Hirsch (1976, Lemma 1.2, p. 123).

We now prove uniqueness. By (D2), the degree is completely determined by its behavior on individual zeros. Using (D3) one can show, first, that the degree of a zero is determined by the derivative of the function at the zero. Thus the degree is determined by its values on $GL(m)$, the set of nonsingular linear transformations from $\mathbf{R}^m$ to itself. Second, (D3) implies that the degree is constant on the path components of $GL(m)$. Using any of a variety of techniques (e.g. the Gram-Schmidt process) one can show that there are two components. Condition (D1) determines the degree on one component, and any example of a homotopy between a function with no zeros and a function with two zeros show that the degree of a linear transformation in the other component must be -1.//

Clearly (D3) implies that the degree is continuous in the sense of the Continuity axiom. The degree is also multiplicative:

Lemma B.3: If $(f, U) \in \mathcal{G}_\infty(X)$ and $(g, V) \in \mathcal{G}_\infty(Y)$, then $(f \times g, U \times V) \in \mathcal{G}_\infty(X \times Y)$ and $\deg(f \times g, U \times V) = \deg(f, U) \cdot \deg(g, V)$.

Proof: This follows from (D2) and the formula used to define the degree at the beginning of the proof above.//

The argument below translates these results into the language of index theory.

Proof of Proposition 2.8: If $(f, U) \in \mathcal{P}_\infty(X)$ then $(\mathrm{Id}_{\bar{U}} - f, U) \in \mathcal{G}_\infty(X)$, so we may define

$$\Lambda_\infty(f, U) = \deg(\mathrm{Id}_{\bar{U}} - f, U).$$

Obviously Weak Normalization, Additivity, and Homotopy follow from this formula and (D1) - (D3). Moreover, the argument used to prove the uniqueness of the degree clearly works equally well to show that these three properties determine Λ_∞ uniquely. As we pointed out above, in this context Continuity is implied by Homotopy. Clearly Multiplication follows from Lemma B.3. Finally, Commutativity is obtained by combining the formula for the index with the following fact of linear algebra. This result can be found in Jacobson (1953, pp. 103 - 106), but we present a proof anyway since the matter is not widely understood and our proof is shorter and, perhaps, more elegant than Jacobson's.

Lemma B.4: Suppose $K \in L(V, W)$ and $L \in L(W, V)$ where V and W are vector spaces over an arbitrary field of dimension m and n respectively, $m \leq n$. Then the characteristic polynomials κ_{KL} and κ_{LK} of KL and LK are related by the equation $\kappa_{KL}(\lambda) = \lambda^{n-m}\, \kappa_{LK}(\lambda)$. In particular,

$$\kappa_{LK}(1) = |\,\mathrm{Id}_V - LK\,| = |\,\mathrm{Id}_W - KL\,| = \kappa_{KL}(1).$$

Proof: It is possible to represent V as a direct sum $V = V_1 + V_2 + V_3 + V_4$ where $V_1 = \mathrm{im}\, L \bigcap \ker K$, $V_1 + V_2 = \mathrm{im}\, L$, and $V_1 + V_3 = \ker K$. Similarly, let

$W = W_1 + W_2 + W_3 + W_4$ where $W_1 = \operatorname{im} K \bigcap \ker L$, $W_1 + W_2 = \operatorname{im} K$, and $W_1 + W_3 = \ker L$. With suitably chosen bases the matrices of K and L have the forms

$$\begin{bmatrix} 0 & K_{12} & 0 & K_{14} \\ 0 & K_{22} & 0 & K_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & L_{12} & 0 & L_{14} \\ 0 & L_{22} & 0 & L_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A little computation shows that κ_{KL} has the formula

$$\kappa_{KL}(\lambda) = \begin{vmatrix} \lambda I & -K_{12}L_{22} & 0 & -K_{12}L_{24} \\ 0 & \lambda I - K_{22}L_{22} & 0 & -K_{22}L_{24} \\ 0 & 0 & \lambda I & 0 \\ 0 & 0 & 0 & \lambda I \end{vmatrix}$$

Consider a permutation of σ of $\{1, \ldots, n\}$. If the term corresponding to σ in the standard expansion of this determinant is nonzero, then $\sigma(i) = i$ for all i corresponding to the first block of columns and the last two blocks of rows. This reduces the proof to the special case $V_2 = V$ and $W_2 = W$, i.e. K and L are isomorphisms. But now the claim follows from the following simple computation

$$|\,\lambda \mathrm{Id}_V - LK\,| = |\,L^{-1}\,| \cdot |\,\lambda \mathrm{Id}_V - LK\,| \cdot |\,L\,| =$$

$$|\,L^{-1}(\lambda \mathrm{Id}_V - LK)L\,| = |\,\lambda \mathrm{Id}_W - KL\,|.//$$

Appendix B: Extension Arguments

This appendix contains two arguments that have a common outline. In each case the idea is to use some property of the index to define an extension to a larger set of admissible pairs, show that the definition of the extension is unambiguous, and verify that the new index satisfies the other index axioms. It is not necessary to absorb the first proof before reading the second.

Proof of Proposition 2.10: Consider $X \in \mathcal{S}$, an open $U \subset X$, and $F \in \mathcal{P}(X, U)$. Then

$$A = \{G \in \mathcal{K}(\bar{U}, X) \,|\, Gr(G) \subset \bar{U} \times X - (\bar{U} - U) \times (\bar{U} - U)\}$$

is a neighborhood of F in $\mathcal{K}(\bar{U}, X)$, and (b) guarantees the existence of a smaller neighborhood B with the property that every pair of points in $B \bigcap C(\bar{U}, X)$ is connected by a path in $A \bigcap C(\bar{U}, X)$. A path in $C(\bar{U}, X)$ is a homotopy of its endpoints, so (I3) implies that $\Lambda_{\mathcal{Q}}$ is constant on $B \bigcap C(\bar{U}, X) \bigcap \mathcal{Q}(X, U)$ with, say, constant value λ. By (a) the set $C(\bar{U}, X) \bigcap \mathcal{Q}(X, U)$ has F as a limit point, so a continuous extension $\Lambda_{\mathcal{P}}$ of $\Lambda_{\mathcal{Q}}$ must have $\Lambda_{\mathcal{P}}(F, U) = \lambda$. The value λ is assigned to all $G \in B \bigcap \mathcal{Q}(X, U)$, so the extension $\Lambda_{\mathcal{P}}$ defined in this way is in fact continuous.

Weak Normalization for $\Lambda_{\mathcal{P}}$ follows from Weak Normalization for $\Lambda_{\mathcal{Q}}$ and Continuity since $\{c \in X \,|\, (c, X) \in \mathcal{Q}(X)\}$ is dense in X by (c).

Again, a homotopy is precisely a continuous path in the space of continuous functions, so Homotopy for $\Lambda_{\mathcal{P}}$ follows from (I5).

Suppose that $F \in \mathcal{P}(X, U)$, $U_1, \ldots, U_r$ are disjoint open subsets of U, and F has no fixed points in $C = \bar{U} - (U_1 \bigcup \ldots \bigcup U_r)$. For each i we can find a neighborhood B_i of $F|_{\bar{U}_i}$ in which $\Lambda_{\mathcal{P}}(\cdot, U_i) : B_i \bigcap \mathcal{P}(X, U_i) \to Z$ is constant, and in general the function $G \mapsto G|_{D_2}$ from $\mathcal{K}(D_1, X)$ to $\mathcal{K}(D_2, X)$ is continuous whenever $D_2 \subset D_1$ (exercise), so there is a neighborhood B of F such that $G|_{\bar{U}i} \in B_i$ for all $G \in B$ and all i. Replacing B with

$$B \bigcap \{G \in \mathcal{K}(\bar{U}, X) \,|\, Gr(G) \subset (X \times X) - (C \times C)\},$$

if necessary, we may assume that no $G \in B$ has any fixed points in C, and our argument above shows that we may replace B by a smaller neighborhood in which $\Lambda_{\mathcal{P}}$ is constant. Assumption (a) implies the existence of $f \in B \bigcap \mathcal{Q}(X, U) \bigcap C(\bar{U}, X)$. Additivity for $\Lambda_{\mathcal{Q}}$ and the constancy of $\Lambda_{\mathcal{P}}$ on the sets B and B_i now yield the computation

$$\Lambda_{\mathcal{P}}(F, U) = \Lambda_{\mathcal{Q}}(f, U) = \Sigma_i \Lambda_{\mathcal{Q}}(f, U_i) = \Sigma_i \Lambda_{\mathcal{P}}(F, U_i).$$

Thus $\Lambda_{\mathcal{P}}$ satisfies Additivity.

It remains only to verify Commutativity and Multiplication (assuming that $\Lambda_{\mathcal{Q}}$ satisfies (I6)). As with Additivity, the idea in each case is to find suitable approximations, after which the desired properties follow from simple computations. For Commutativity the computation is

$$\begin{aligned} \Lambda_{\mathcal{P}}(g \circ f, U) &= \Lambda_{\mathcal{Q}}(g' \circ f', U) = \Lambda_{\mathcal{Q}}(f' \circ g', g'^{-1}(U)) \\ &= \Lambda_{\mathcal{Q}}(f' \circ g', V) = \Lambda_{\mathcal{P}}(f \circ g, V) = \Lambda_{\mathcal{P}}(f \circ g, g^{-1}(U)), \end{aligned}$$

where V is open with $\mathcal{F}(f \circ g) \bigcap \overline{g^{-1}(U)} \subset V$ and $\bar{V} \subset g^{-1}(U)$, and f' and g' are close enough to f and g to guarantee that $\mathcal{F}(f' \circ g') \bigcap \overline{g'^{-1}(U)} \subset V$ and $\bar{V} \subset g'^{-1}(U)$. For Multiplication the computation is:

$$\begin{aligned} \Lambda_{\mathcal{P}}(F \times G, U \times V) &= \Lambda_{\mathcal{Q}}(f \times g, U \times V) \\ &= \Lambda_{\mathcal{Q}}(f, U) \cdot \Lambda_{\mathcal{Q}}(g, V) = \Lambda_{\mathcal{P}}(F, U) \cdot \Lambda_{\mathcal{P}}(G, V). \end{aligned}$$

Since the proofs that suitable approximations exist are similar to the one given above in connection with Additivity, the details are left to the reader. (In the case of Commutativity one should begin by showing that composition is a continuous function from $C(X, Y) \times C(Y, Z)$ to $C(X, Z)$ if X, Y and Z are compact metric spaces.)//

Proof of Proposition 2.16: Fix $X \in \mathcal{S}_{ENR}$. We fix an embedding $X \subset \mathbf{R}^m$ and a retraction $r : D \to X$ where D is a compact domain and int D is a neighborhood of X. (Clearly it is always possible to obtain this situation.) Let $i : X \to D$ be the corresponding inclusion.

Suppose $(f, U) \in \mathcal{P}_{ENR}(X)$. Then $(i \circ f \circ r, r^{-1}(U)) \in \mathcal{P}_0(D)$. Any extension Λ_{ENR} of Λ_0 satisfying Commutativity must have

$$\Lambda_{ENR}(f, U) = \Lambda_0(i \circ f \circ r, r^{-1}(U)),$$

so the extension is unique if it exists. Suppose $s : E \to X$ is another retraction of a neighboring domain onto (a homeomorphic image of) X with corresponding inclusion $j : X \to E$. Then the Commutativity property of Λ_0 implies that

$$\begin{aligned}
\Lambda_0(i \circ f \circ r, r^{-1}(U)) &= \Lambda_0(i \circ s \circ j \circ f \circ r, r^{-1}(U)) \;\; (\text{since}\, s \circ j = \mathrm{Id}_X) \\
&= \Lambda_0(j \circ f \circ r \circ i \circ s, (i \circ s)^{-1}(r^{-1}(U))) \\
&\qquad \text{(by Commutativity)} \\
&= \Lambda_0(j \circ f \circ s, s^{-1}(U)).
\end{aligned}$$

Thus Commutativity defines the extension Λ_{ENR} unambiguously.

It remains only to show that Λ_{ENR} satisfies the index axioms.

If $c : X \to X$ is a constant function then $i \circ c \circ r$ is constant, so Weak Normalization for Λ_0 implies Weak Normalization for Λ_{ENR}.

If $U_1, \ldots, U_k$ are pairwise disjoint open subsets of an open set $U \subset X$, then $r^{-1}(U_1), \ldots, r^{-1}(U_k)$ are pairwise disjoint open subsets of $r^{-1}(U)$. If $f : X \to X$ has no fixed points in $\bar{U} - \bigcup_{j=1}^{k} U_j$, then $i \circ f \circ r$ has no fixed points in $c\ell(r^{-1}(U)) - \bigcup_{j=1}^{k} r^{-1}(U_j)$. Thus Additivity for Λ_0 implies Additivity for Λ_{ENR}.

If $h : \bar{U} \times [0, 1] \to X$ is a homotopy such that $\mathcal{F}(h_t) \bigcap (\bar{U} - U) = \emptyset$ for all t, then $(t, x) \mapsto i(h(t, r(x)))$ is a homotopy of $i \circ h_0 \circ r$ and $i \circ h_1 \circ r$ such that $\mathcal{F}(i \circ h_t \circ r) \bigcap (\bar{U} - U) = \emptyset$ for all t. Thus Homotopy for Λ_0 implies Homotopy for Λ_{ENR}.

It is easy to show that the map $f \mapsto i \circ f \circ r$ from $C(\bar{U}, X)$ to $C(c\ell(r^{-1}(U)), D)$ is continuous, so Continuity for Λ_0 implies Continuity for Λ_{ENR}.

Let $r : D \to X$, $i : X \to D$, $s : E \to Y$, and $j : Y \to E$ be as above. If $f : X \to Y$ and $g : Y \to X$ are continuous and $U \subset X$ is open with $\mathcal{F}(g \circ f) \bigcap (\bar{U} - U) = \emptyset$, then

$$\begin{aligned}
\Lambda_{ENR}(g \circ f, U) &= \Lambda_0(i \circ g \circ f \circ r, r^{-1}(U)) \\
&= \Lambda_0(i \circ g \circ s \circ j \circ f \circ r, r^{-1}(U)) \quad (\text{since } s \circ j = \mathrm{Id}_Y) \\
&= \Lambda_0(j \circ f \circ r \circ i \circ g \circ s, (i \circ g \circ s)^{-1}(r^{-1}(U))) \\
&\qquad (\text{by Commutativity}) \\
&= \Lambda_0(j \circ f \circ g \circ s, s^{-1}(g^{-1}(U))) \quad (\text{since } r \circ i = \mathrm{Id}_X) \\
&= \Lambda_{ENR}(f \circ g, g^{-1}(U)).
\end{aligned}$$

This establishes Commutativity for Λ_{ENR}.

In addition $D \times E$ is a compact domain and a neighborhood of $X \times Y$, and $r \times s : D \times E \to X \times Y$ is a retraction. Multiplication for Λ_0 implies Multiplication for Λ_{ENR} by the following computation:

$$\begin{aligned}
\Lambda_{ENR}(f\times g,\, U\times V) &= \Lambda_0((i\times j)\circ(f\times g)\circ(r\times s),\,(r\times s)^{-1}(U\times V))\\
&= \Lambda_0((i\circ f\circ r)\times(j\circ g\circ s),\, r^{-1}(U)\times s^{-1}(V))\\
&= \Lambda_0(i\circ f\circ r,\, r^{-1}(U))\cdot\Lambda_0(j\circ g\circ s,\, s^{-1}(V))\\
&= \Lambda_{ENR}(f,\,U)\cdot\Lambda_{ENR}(g,\,V).
\end{aligned}$$

The proof of Proposition 2.16 is complete.//

Appendix C: Approximations of Convex Valued Correspondences

Our first goal is the proof of Proposition 2.24, after which the proof of Theorem 3 is immediate and Proposition 2.25 is a straightforward extension. Finally we will develop some approximations results required in the proof of Theorem 5.

We fix a compact metric space X, a convex ANR Y, and $F\in\mathcal{K}_{\text{con}}(X,\,Y)$. We begin by showing that every neighborhood of $Gr(F)$ contains an open neighborhood that is the graph of a convex valued correspondence.

Lemma C.1: Suppose $W\subset X\times Y$ is a neighborhood of $Gr(F)$. Then for each x one can find open sets U_x and V_x with $\{x\}\times F(x)\subset U_x\times V_x\subset W$, V_x convex, and $Gr(F|_{\bar{U}_x})\subset\bar{U}_x\times V_x$.

Proof: Fix x. One definition of the product topology implies that for each $y\in F(x)$ there are open sets U_y and $V_y\subset Y$ with $(x,\,y)\in U_y\times V_y\subset W$. The local convexity of the space containing Y implies that there is a convex neighborhood of the origin B_y with $(y+B_y)\bigcap Y\subset V_y$. Choose $y_1,\ldots,y_r$

such that $\{y_j + (1/2)B_{y_j}\}$ is an open cover of $F(x)$, let $U_x = \bigcap_j U_{y_j}$, and let $V_x = (F(x) + (1/2)\bigcap_j B_{y_j}) \bigcap Y$. If $(x', y') \in U_x \times V_x$ then there is $y \in F(x)$ such that $y' - y \in (1/2)\bigcap_j B_{y_j}$, and there is j such that $y \in \{y_j\} + (1/2)B_{y_j}$, so $y' \in \{y_j\} + B_{y_j}$, and $(x', y') \in U_y \times V_y \subset W$.

The upper hemicontinuity of F states that $\{x' \in U_x \,|\, F(x') \subset V_x\}$ is an open neighborhood of x, so U_x can be replaced by an open neighborhood of x whose closure is contained in this set.//

Lemma C.2: Every neighborhood $W \subset X \times Y$ of $Gr(F)$ contains an open neighborhood W' with $W'(x) = \{y \,|\, (x, y) \in V\}$ convex for all x.

Proof: For each x let U_x and V_x be as in Lemma C.1. Choose $x_1, \ldots, x_r$ such that $\{U_{x_j} \times V_{x_j}\}$ is an open cover of $Gr(F)$. It is easily verified that $W' = \bigcup_{x \in X} (\{x\} \times \bigcap_{x \in \bar{U}_j} V_{x_j})$ has the desired properties.//

Convex combinations of convex valued correspondences can be defined naturally, and in fact we can generalize this operation. For $G_1, G_2 \in \mathcal{K}_{\text{con}}(X, Y)$ and a map $\alpha : X \to [0, 1]$ we let

$$[(1-\alpha)G_1 + \alpha G_2](x) = \{(1-\alpha(x))y_1 + \alpha(x)y_2 \,|\, y_1 \in G_1(x),\, y_2 \in G_2(x)\}.$$

The proof that $(1-\alpha)G_1 + \alpha G_2$ is upper hemicontinuous is left as an exercise, and it is clearly convex valued, so $(1-\alpha)G_1 + \alpha G_2 \in \mathcal{K}_{\text{con}}(X, Y)$. By convexity of a set in $\mathcal{K}_{\text{con}}(X, Y)$ we will always mean that the set is convex in the strong sense of being closed with respect to these generalized convex combinations.

Now suppose that W' is as in Lemma C.2, that is, $W' \supset Gr(F)$ is open and each $W'(x)$ is convex. If $Gr(G_1)$, $Gr(G_2) \subset W'$ then $Gr((1-\alpha)G_1 + \alpha G_2) \subset W'$, obviously, so Lemma C.2 implies:

Lemma C.3: Every neighborhood of F in $\mathcal{K}_{\text{con}}(X, Y)$ contains a convex neighborhood.

The proof that $C(X, Y)$ is dense in $\mathcal{K}_{\text{con}}(X, Y)$ employs a technical construction.

Definition C.4: A *configuration* is a finite collection of quadruples $\{(U_j, \varphi_j, x_j, y_j)\}_{j=1,\ldots,r}$ satisfying the following description: (a) $\{U_j\}$ is an open cover of X; (b) $\{\varphi_j : X \to [0, 1]\}$ is a partition of unity[16] subordinate to $\{U_j\}$; (c) for each j, $x_j \in U_j$ and $y_j \in Y$. We call $\Sigma_j \varphi_j y_j : X \to Y$ the function *generated* by the configuration.

Lemma C.5: For every neighborhood W of $Gr(F)$ there is a smaller neighborhood W' and $\gamma > 0$ such that $Gr(\Sigma_j \varphi_j y_j) \subset W$ for all configurations $\{(U_j, \varphi_j, x_j, y_j)\}$ with diam $U_j < \gamma$ and $(x_j, y_j) \in W'$ for all j.

Proof: Since X and Y are metric spaces there is a descending sequence $W^1 \supset W^2 \supset \ldots$ of neighborhoods of $Gr(F)$ such that for any neighborhood W of $Gr(F)$, $W^n \subset W$ for sufficiently large n. If the claim is false then there is a neighborhood W of $Gr(F)$ for which we can find:

[16]If $\{U_\alpha\}$ is an open cover of a topological space X, a *partition of unity* subordinate to $\{U_\alpha\}$ is a collection of maps $\{\mu_\alpha : X \to [0, 1]\}$ such that supp $\mu_\alpha \subset c\ell(\mu_\alpha^{-1}((0, 1])) \subset U_\alpha$ and $\Sigma_\alpha \mu_\alpha = 1$. See Kelley (1955, p. 171) and Hirsch (1976, §2.2) for theorems guaranteeing the existence of partitions of unity.

(a) a sequence $\{\gamma^n > 0\}$ converging to 0;

(b) for each n a configuration $\{(U_j^n,\, \varphi_j^n,\, x_j^n,\, y_j^n)\}_{j=1,\ldots,J_n}$ with diam $U_j^n < \gamma^n$ and $(x_j^n,\, y_j^n) \in W^n$;

(c) a sequence $\{x^n \in X\}$ with $(x^n,\, y^n) \notin W$ where $y^n = \Sigma_j\, \varphi_j^n(x^n) y_j^n$.

Taking subsequences, let $x = \lim x^n$ and let $y = \lim y^n$.

Let U_x and V_x be as in Lemma C.1, that is, V_x is a convex neighborhood of $F(x)$ and $Gr(F|_{\bar{U}_x}) \subset \bar{U}_x \times V_x \subset W$. Note that $(\bar{U}_x \times V_x) \bigcup ((X - \bar{U}_x) \times Y)$ is a neighborhood of $Gr(F)$, so for large n it must be the case that $y_j^n \in V_x$ whenever $\varphi_j^n(x^n) > 0$. This means that $y^n \in V_x$ and $(x^n,\, y^n) \in U_x \times V_x \subset W$, a contradiction.//

Proof of Proposition 2.24: In view of Lemma C.5 it is only necessary to show that configurations with the indicated properties exist. The proof of this fact is an easy exercise in view of the theorem guaranteeing the existence of partitions of unity.//

Proof of Theorem 3: We apply Proposition 2.10 with $(\mathcal{S},\, \mathcal{Q}) = (\mathcal{S}_{\text{con}},\, \mathcal{P}_{\text{con}} \bigcap \mathcal{P}_{\text{ANR}})$ and $(\mathcal{S},\, \mathcal{P}) = (\mathcal{S}_{\text{con}},\, \mathcal{P}_{\text{con}})$. Conditions (a) and (b) of Proposition 2.10 follows from Proposition 2.24 and Lemma C.3 respectively. Condition (c) is obvious.//

Proof of Proposition 2.25: Recall that the space of equivalence classes of Cauchy sequences in a normed linear space, where the equivalence relation is "is eventually arbitrarily close to," is a Banach space. Thus no generality is lost in assuming that Y is embedded in a Banach space. A theorem of Mazur (Dunford and Schwartz (1958, p. 416)) states that $\overline{co}(Y)$, the closed convex hull of Y, is compact whenever Y is a compact subset of a Banach space. Lemma 2.18 now implies that $\overline{co}(Y)$ is a convex ANR.

Since Y is an ANR, it is a retract of a neighborhood in the normed space containing Y. Let U be the intersection of this neighborhood with $\overline{co}(Y)$, and let $r : U \to Y$ be the restriction of the retraction to U.

Note that the operator $r\,\circ : C(X, U) \to C(X, Y)$, $f \mapsto r \circ f$, is continuous, and in fact it is easily seen to be a retraction. Concretely, if W is open in $X \times Y$, then $(\mathrm{Id}_X \times r)^{-1}(W)$ is open in $X \times U$.

Now fix $F \in \mathcal{K}_{\mathrm{con}}(X, Y)$ and a neighborhood W of $Gr(F)$ in $X \times Y$. It suffices to prove the existence of $f \in C(X, U)$ with $Gr(f) \subset (\mathrm{Id}_X \times r)^{-1}(W)$. But $\mathcal{K}_{\mathrm{con}}(X, Y) \subset \mathcal{K}_{\mathrm{con}}(X, \overline{co}(Y))$ and Proposition 2.24 implies that $C(X, \overline{co}(Y))$ is dense in $\mathcal{K}_{\mathrm{con}}(X, \overline{co}(Y))$. Of course $C(X, U)$ is open in $C(X, \overline{co}(Y))$ since $X \times U$ is open in $X \times \overline{co}(Y)$, and $X \times U$ is a neighborhood of $Gr(F)$, so $C(X, U)$ is dense at F. This completes the proof of Proposition 2.25.//

In Appendix D we need the technical results below which discuss extending, in an approximate sense, a pregiven approximation on a compact subset of the domain.

Definition C.6: Suppose $D \subset U \subset X$ with X a compact metric space, U open, and D compact. Suppose Y is a convex ANR. Let $f : D \to Y$ be a map and $W \subset X \times Y$ an open set containing $Gr(f)$. We say that f is *approximately extendable to* $\bar{U}$ *in* W if for every neighborhood W'' of $Gr(f)$

there is a map $f' : \bar{U} \to Y$ with $Gr(f') \subset W$ and $Gr(f'|_D) \subset W''$. (See Figure 3)

[Insert Figure 3]

Definition C.7: Suppose $D \subset X$ with X a compact metric space and D compact. Suppose that Y is another metric space. Finally, suppose that for each open $U \subset X$ we are given $\mathcal{J}(\bar{U}, Y) \subset \mathcal{K}(\bar{U}, Y)$. We say that D *admits approximate extensions of approximations* of correspondences in $\mathcal{J}(X, Y)$ (or, more precisely, $\{\mathcal{J}(\bar{U}, Y)(U \subset X) \text{ is open}\}$) if for every open U with $D \subset U \subset X$, every $F \in \mathcal{J}(\bar{U}, Y)$, and every neighborhood W of $Gr(F)$ there is a neighborhood W' of $Gr(F|_D)$ such that every map $f : D \to X$ with $Gr(f) \subset W'$ is approximately extendable to $\bar{U}$ in W.

Proposition C.8: Suppose $D \subset X$ with X a compact metric space and D compact. Suppose Y is a convex ANR. Then D admits approximate extensions of approximations of correspondences in $\mathcal{K}_{\text{con}}(X, Y)$.

Proof: We fix an open neighborhood U of D, $F \in \mathcal{K}_{\text{con}}(\bar{U}, Y)$, and a neighborhood $W \subset U \times Y$ of $Gr(F)$. We must show that there is a neighborhood W' of $Gr(F|_D)$ such that every map $f : D \to Y$ with $Gr(f) \subset W'$ is approximately extendable to $\bar{U}$ in W. Replacing W with a smaller neighborhood only makes this more difficult, so Lemma C.2 allows us to assume that $W(x)$ is convex for all x.

With this assumption we claim that every map $f : D \to Y$ with $Gr(f) \subset W$ is approximately extendable to $\bar{U}$ in W. Fix such an f and a neighborhood $W'' \subset W$ of $Gr(f)$.

Fix a sequence $\{\gamma^n > 0\}$ converging to 0. For each n the theorem guaranteeing the existence of a partition of unity allows us to choose a configuration $\{(U_j^n, \varphi_j^n, x_j^n, y_j^n)\}_{j=1,\ldots,J^n}$ with the following properties:

(a) $\text{diam}(U_j^n) < \gamma^n$;
(b) there is I^n such that $U_j^n \bigcap D \neq \emptyset$ if and only if $j \leq I^n$;
(c) $x_j^n \in D$ and $y_j^n = f(x_j^n)$, $j = 1, \ldots, I^n$;
(d) $y_j^n \in F(x_j^n)$, $j = I^n + 1, \ldots, J^n$.

Let f^n be the function generated by $\{(U_j^n, \varphi_j^n, x_j^n, y_j^n)\}_{j=1,\ldots,J^n}$. Lemma C.5 implies that $Gr(f^n|_D)$ is eventually inside W''. Let $G \in \mathcal{K}_{\text{con}}(\bar{U}, Y)$ be given be $G(X) = \overline{co}(F(x) \bigcup \{f(x)\})$, $x \in D$, and $G(x) = F(x)$, $x \notin D$. Since each $W(x)$ is convex, $Gr(G) \subset W$, and applying Lemma C.5 to G now shows that $Gr(f^n)$ is eventually inside W. This shows that f is approximately extendable to $\bar{U}$ in W, so the proof is complete.//

Proposition C.9: Suppose $D \subset X$ with X a compact metric space and D compact. Suppose Y is a compact ANR embedded in a normed space. Then D admits approximate extensions of approximations of correspondences in $\mathcal{K}_{\text{con}}(X, Y)$.

Proof: We sketch a proof using the ideas developed above. Recall from the proof of Proposition 2.25 that no generality is lost in assuming that Y is embedded in a Banach space, and this implies that $\overline{co}(Y)$ is a convex ANR. Proposition C.8 implies that D admits approximate extensions of approximations of correspondences in $\mathcal{K}_{\text{con}}(X, \overline{co}(Y))$. For $F \in \mathcal{K}_{\text{con}}(X, \overline{co}(Y))$ the concrete meaning of this is roughly that approximations of $F|_D$ in $C(D, \overline{co}(Y))$

have approximate extensions in $C(X, \overline{co}(Y))$ that approximate F. The argument by retraction used in the proof of Proposition 2.25 can now be used to show that approximations of $F|_D$ in $C(D, Y)$ have approximate extensions in $C(X, Y)$ that approximate F, and this is the desired result.//

An arbitrary ANR Y can be embedded in a normed space, but the embedding effects which correspondences are in $\mathcal{K}_{\text{con}}(X, Y)$. But of course the choice of embedding has no effect on the set $C(X, Y)$ or its topology, so Proposition C.9 has the following important corollary.

Proposition C.10: Suppose $D \subset X$ with X a compact metric space and D compact. Suppose Y is a compact ANR. Then D admits approximate extensions of approximations of maps in $C(X, Y)$.

Appendix D: Proof of Theorem 5

The gist of the argument establishing that minimal essential sets are connected is as follows. Suppose $F \in \mathcal{P}(X, U)$, and let K be a minimal essential set for F. If K is not connected then it is possible to write $K = K_1 \bigcup K_2$ where K_1 and K_2 are compact, nonempty, and disjoint. Since K is minimal, both K_1 and K_2 are inessential, so there are disjoint neighborhoods $U_1 \supset K_1$ and $U_2 \supset K_2$ and arbitrarily close approximations $f_1, f_2 \in C(\bar{U}, X)$ of F such that f_i has no fixed points in $\bar{U}_i$, $i = 1, 2$. Let $f : K \to X$ be given by f_1 on $\bar{U}_1$ and f_2 on $\bar{U}_2$. If f (or some nearby function) necessarily has an extension to all of X, then we have contradicted the assumption that K is essential.

The delicate step is showing the existence of the desired extension f. It is here that the refined approximation properties introduced in Appendix C come into play.

Definition D.1: Suppose $D \subset X$ with X a compact metric space and D compact. Suppose Y is a convex ANR. Suppose that for each open $U \subset X$ we are given $\mathcal{J}(\bar{U}, Y) \subset \mathcal{K}(\bar{U}, Y)$. We say that the system of correspondences $\{\mathcal{J}(\bar{U}, Y)\}$ has the *approximation extension property*[17] (AEP) if, whenever $K \subset U \subset X$ with K compact and U open, there is a compact D with $K \subset D \subset U$ that admits approximate extensions of approximations of correspondences in $\mathcal{J}(X, Y)$.

We now give a precise formulation of the argument above in which the AEP is used to give the necessary extension.

Proposition D.2: Suppose that for all U, $\mathcal{P}(X, U) \subset \mathcal{J}(\bar{U}, X)$, where $\{\mathcal{J}(\bar{U}, X)\}$ is a system of correspondences that has the AEP. Then minimal essential sets of elements of $\mathcal{P}(X)$ are connected.

Proof: Fix $X \in \mathcal{S}$ and $(F, U) \in \mathcal{P}(X)$, and let K be a minimal essential set for F. Suppose that $K = K_1 \bigcup K_2$ where K_1 and K_2 are nonempty, compact, and disjoint. Since K is minimal, neither K_1 nor K_2 is essential. For $i = 1, 2$ let U_i be a neighborhood of K_i such that one can find elements of $\mathcal{P}(X, U)$ arbitrarily close to F that have no fixed points in U_i. Without loss of generality we may assume that U_1 and U_2 are disjoint. Let $U_0 = U_1 \bigcup U_2$. For

[17]Aside from the fact that they both deal with extension problems, there is no close relationship between the AEP and the similarly titled homotopy extension property (e.g. Dugundji (1966)).

$i = 1, 2$ let V_i be a neighborhood of K_i with $\bar{V}_i \subset U_i$. Let $V_0 = V_1 \bigcup V_2$. Since $\{\mathcal{J}(\bar{U}, X)\}$ has the AEP, there is a compact set D with $V_0 \subset D \subset U_0$ that admits approximate extensions of approximations. Let $D_i = D \bigcap U_i$, $i = 1, 2$. Fix an open neighborhood W of $Gr(F)$. Choose $f_i \in C(\bar{U}, X)$ with $Gr(f_i) \subset W$ and $\mathcal{F}(f_i) \bigcap U_i = \emptyset$, $i = 1, 2$. Let $f : D \to X$ be given by f_i on D_i. Then f has no fixed points, there are maps arbitrarily close to f that extend to X in W, and if such a map is sufficiently close it cannot have any fixed points in D. This contradicts the assumption that K is essential.//

The assertion of Theorem 5 follows in our three interpretations from the following three results.

Proposition D.3: If X is a compact ANR, then $\{C(\bar{U}, X)\}$ has the AEP.

Proof: This follows from Proposition C.10.//

Proposition D.4: If X is a finite simplicial complex, then $\{\mathcal{K}_{ctr}(\bar{U}, X)\}$ has the AEP.

Proof: Suppose that $K \subset U \subset X$ with K compact and U open. Using barycentric subdivision (e.g. Border (1985, p. 21)) one can construct a subcomplex D (of a subdivision of X) with $K \subset D \subset U$, so it suffices to show that any subcomplex D admits approximate extensions of approximations of correspondences in $\mathcal{K}_{ctr}(D, X)$. In fact the conclusion of Proposition 2.21 (roughly "D admits exact extensions of approximations") is somewhat stronger than this.//

Proposition D.5: If X is a convex ANR, then $\{\mathcal{K}_{con}(\bar{U}, X)\}$ has the AEP.

Proof: This follows from Proposition C.8.//

Appendix E: The Proof of Proposition 4.7

In short the proof of Proposition 4.7 is a matter of proving the existence of a nicely behaved approximation of F and then modifying it to have the right fixed points with the right indices. We now "prepare" M by adding useful geometric structure.

Our argument is explicitly geometric: we impose a Riemannian metric[18] and exploit the properties of geodesics. By the easy Whitney embedding theorem we may assume that M is embedded in $\mathbf{R}^\ell$ for some sufficiently large ℓ. (With small changes the proof of Hirsch (1976, Th. 3.4, p. 23) applies to ∂-manifolds.) The tangent manifold TM of M is now a concrete subset of $M \times \mathbf{R}^\ell$, and it inherits a Riemannian metric from the standard inner product on $\mathbf{R}^\ell$.

We will always assume that M is connected. For each pair of points $x, y \in M$ there is now at least one shortest path from x to y in M; let $d(x, y)$ be the length of such a path. Obviously $d : M \times M \to \mathbf{R}_+$ is a metric. We also let d denote the induced metric on $C(K, M)$ for any compact space $K : d(f, g) = \max d(f(z), g(z))$.

We now give a casual description of the exponential map of differential geometry. Everything we have to say is geometrically intuitive and can be

[18] Let TM^2 be the vector bundle over M whose fibre at each x is $TM_x \times TM_x$. A *Riemannian metric* is a smooth function $< \cdot, \cdot >: TM^2 \to \mathbf{R}_+$ whose restriction to each fibre is an inner product. See Spivak (1979, §9).

derived rigorously from the results in Spivak (1979, §9) without great difficulty. In the first part of the discussion we will assume that $\partial M = \emptyset$ in order to avoid complications.

With this assumption we may define a *geodesic* to be a curve $\gamma : [a, b] \to M$ of constant speed $||\gamma'(t)||$ such that if $a \leq s < t \leq b$ with $t - s$ sufficiently small, then the image of $\gamma|_{[s,\,t]}$ is a shortest path from $\gamma(s)$ to $\gamma(t)$. The theory of the Euler equation allows geodesics to be characterized as the solution of ordinary differential equations. Since M is boundaryless and compact, the standard results concerning such equations allow one to show that for each $x \in M$ and $v \in TM_x$, there is a unique geodesic $\gamma_{x,v} : \mathbf{R} \to M$ with $\gamma_{x,v}(0) = x$ and $\gamma'_{x,v}(0) = v$, and that $\gamma_{x,v}(t)$ is a smooth function of (x, v, t). We define the *exponential map* $\exp : TM \to M$ by setting $\exp(x, v) = \gamma_{x,v}(1)$, and we let $\exp_x$ denote the restriction of exp to $TM_x : \exp_x(v) = \exp(x, v)$.

The conditions defining $\exp_x$ imply that $D(\exp_x)(0) = \mathrm{Id}_{TM_x}$, so each $\exp_x$ is a diffeomorphism in a neighborhood of $0 \in TM_x$. (Here and below this equation is to be understood in terms of the natural identification $T(TM_x)_0 = TM_x$.) In fact we need a stronger result.

Proposition E.1: There is $\kappa > 0$ such that for each $x \in M$, the restriction of $\exp_x$ to $\{v \in TM_x \,|\, ||v|| < \kappa\}$ is a diffeomorphism.

Proof: Consider the map $H : TM \to M \times M$ given by $H(x, v) = (x, \exp(x, v))$. Identifying $T(TM)_{(x,0)}$ with $TM_x \times TM_x$ in the natural way, the matrix of $DH(x, 0)$ consists of four $n \times n$ blocks where three blocks are Id_{TM_x} and the fourth is 0, so $DH(x, 0)$ is nonsingular. Thus H is diffeomorphism in a neighborhood of $(x, 0)$ by the inverse function theorem. The desired result now follows from the compactness of M.//

The situation is obviously much more complicated when $\partial M \neq \emptyset$, but for our purposes the following properties of the exponential map are sufficient.

Proposition E.2: Suppose $K \subset \operatorname{int} M$ is compact. Then for $\kappa > 0$ sufficiently small there is a unique map $\exp : \{(x, v) \in TM \mid x \in K$ and $||v|| < \kappa\} \to M$ that has the following properties:

(a) For all $x \in M$, $\exp_x(0) = x$ and $D(\exp_x)(0) = \mathrm{Id}_{TM_x}$.
(b) For all $(x, v) \in TM$, the curve $\gamma_{x,v}(t) = \exp(x, tv)$, $\kappa/\|v\| < t < \kappa/\|v\|$, is a geodesic.

Moreover, for each $x \in K$ the restriction $\exp_x : \{v \in TM_x | \, ||v|| < \kappa\} \to M$ is a diffeomorphism onto its image.

Consider $x \in K$ and $v \in TM_x$ with $||v|| < \kappa$. Since the speed $||\gamma'_{x,v}(t)||$ of $\gamma_{x,v}$ is constant with value $||v||$, this is also the length of the path traced by $\gamma_{x,v}$ between time 0 and time 1. Any shortest path between x and $\exp(x, v)$ can be parametrized as a geodesic, and if there were a shorter path we would have $\exp(x, v) = \exp(x, w)$ for some w with $||w|| < \kappa$, a contradiction of Proposition E.2. This argument proves the equation

$$d(x, \exp(x, v)) = ||v|| \;\; (x \in K, \, ||v|| < \kappa).$$

These results allow one to define a bijection between smooth functions $g : K \to M$ with $d(g, \mathrm{Id}_K) < \kappa$ and smooth tangent vector fields $\zeta : K \to TM$, $\zeta(x) \in TM_x$, with $\max ||\zeta(x)|| < \kappa$. Specifically, let the vector field $\Phi_K(g)$ and the function $\Psi_K(\zeta)$ be defined by the equations

$$\Phi_K(g)(x) = -\exp_x^{-1}(g(x)) \text{ and } \Psi_K(\zeta)(x) = \exp_x(-\zeta(x)).$$

The signs are required here in order to equate the two indices.

Lemma E.3: Suppose $x \in \operatorname{int} K$ in a regular zero of ζ. Then

$$\Lambda_{ENR}(\Psi_K(\zeta),\, x) = \operatorname{ind}_x(\zeta).$$

Proof: With the natural identification $T(TM)_{(x,0)} = TM_x \times TM_x$ we can decompose $D(\exp)(x,\, 0)$ into its "partials," but of course Proposition E.2 implies that these partials are both Id_{TM_x}. Applying the chain rule to the formula $\Psi_K(\zeta)(x') = \exp(x',\, -\zeta(x'))$ now yields

$$D(\Psi_K(\zeta))(x) = \mathrm{Id}_{TM_x} - D\zeta(x) \;\text{ or }\; D\zeta(x) = \mathrm{Id}_{TM_x} - D(\Psi_K(\zeta))(x).$$

In the second equation the determinant of the left hand side is $\operatorname{ind}_x(\zeta)$, as we saw in §3.2, while the determinant of the right hand side is
$\Lambda_{ENR}(\Psi_K(\zeta),\, x)$.//

With these generalities out of the way we proceed to the central step of the proof. The statement of the following result summarizes the remaining work in this appendix. Conclusions (1) and (2) are the geometric operations of addition and erasure of pairs of fixed points of opposite indices used to bring about the properties asserted in Proposition 4.7. The "geometric" part of the argument is the proof that (A) implies (1) and (2). The "topological" part of the argument is to show that the hypotheses of Proposition 4.7 imply the existence, for any $\delta > 0$, of $g : \bar{U} \to M$ with $Gr(g) \subset \mathbf{B}(Gr(F);\, \delta)$ and a connected neighborhood V of $\mathcal{F}(F)$, such that $(g|_{\bar{V}},\, V)$ is as in (A).

Proposition E.4: Let $U \subset M$ be open with $\bar{U} \subset \operatorname{int} M$. Then there exists $\kappa > 0$ such that (A) implies (1) and (2).

(A) $(g, V) \in \mathcal{P}_\infty(M)$, where $V \subset U$ is connected and $d(g, \mathrm{Id}_{\bar{V}}) < \kappa$.

(1) For any points $x^+, x^- \in V - \mathcal{F}(g)$ there is a smooth $g' : \bar{V} \to M$ with $d(g', \mathrm{Id}_{\bar{V}}) \leq \kappa$ that agrees with g in a neighborhood of $(\bar{V} - V) \bigcup \mathcal{F}(g)$ and has $\mathcal{F}(g') = \mathcal{F}(g) \bigcup \{x^+, x^-\}$ where x^+ and x^- are regular fixed points with $\Lambda(g'; x^+) = 1$ and $\Lambda(g'; x^-) = -1$.

(2) For any fixed points $x^+, x^- \in \mathcal{F}(g)$ with $\Lambda(g; x^+) = 1$ and $\Lambda(g; x^-) = -1$ there is a smooth $g' : \bar{V} \to M$ with $d(g', \mathrm{Id}_{\bar{V}}) \leq \kappa$ that agrees with g in a neighborhood of $(\bar{V} - V) \bigcup \mathcal{F}(g) - \{x^+, x^-\}$ and has $\mathcal{F}(g') = \mathcal{F}(g) - \{x^+, x^-\}$.

For the time being our goal is the proof of Proposition E.4. The following result states the geometric basis of our argument.

Proposition E.5: Let M be a connected oriented compact n-dimensional ∂-manifold. Then a map $\eta : \partial M \to S^{n-1}$ has a continuous extension $M \to S^{n-1}$ if and only if deg $\eta = 0$.

Proof: This is Hirsch (1976, Th. 1.8, pp. 126).//

Remark: Our treatment of degree theory in Appendix A is inadequate for this result in the following respects. One must define the degree of a smooth map $f : M \to N$ over a regular value $y \in N$, where M and N are compact oriented manifolds, and show that the degree so defined is independent of y. This definition of $\deg(f)$ must then be extended to continuous functions

$f : M \to N$ by approximation. Of course these tasks are conceptually straightforward; for further detail we recommend Milnor (1965, §5) and Lloyd (1978).

The specific applications of Proposition E.5 are the following two lemmas which will allow us to add and delete pairs of fixed points in a concrete geometric setting as per (1) and (2).

Lemma E.6: Suppose $\zeta : D^n \to \mathbf{R}^n - \{0\}$ is a nonvanishing vector field on the unit disk in $\mathbf{R}^n$. Let y^+ and y^- be points in int $D^n = D^n - S^{n-1}$. Then there is a vector field $\zeta' : D^n \to \mathbf{R}^n$ that agrees with ζ on S^{n-1} and whose only two zeros are y^+ and y^-, both of which are regular, with $\text{ind}_{y^+}(\zeta') = 1$ and $\text{ind}_{y^-}(\zeta') = -1$.

Proof: Let $\eta : S^{n-1} \to S^{n-1}$ be the normalization of ζ on $S^{n-1} : \eta(y) = \zeta(y)/||\zeta(y)||$. Proposition E.5 implies that $\deg(\eta) = 0$. Let D^+ and D^- be disjoint closed disks in $D^n - S^{n-1}$ centered at y^+ and y^- respectively. Let $S^+ = \partial D^+$ and let $S^- = \partial D^-$. Let $L^+, L^- \in L(\mathbf{R}^n, \mathbf{R}^n)$ be nonsingular linear transformations with $\text{ind}_0(L^+) = 1$ and $\text{ind}_0(L^-) = -1$. We start the construction of ζ' by setting $\zeta'(y) = L^+(y - y^+)$ for $y \in D^+$, $\zeta'(y) = L^-(y - y^-)$ for $y \in D^-$, and $\zeta'(y) = \zeta(y)$ for $y \in S^{n-1}$.

Let $\eta' : S^{n-1} \bigcup S^+ \bigcup S^- \to S^{n-1}$ be the extension of η obtained by normalizing ζ' on $S^+ \bigcup S^-$. The definition of the index of a zero of a vector field implies that $\deg(\eta' \mid S^+) = 1$, where S^+ has the orientation induced by regarding S^+ as the boundary of D^+. The inclusion $S^+ \subset \partial(D^n - \text{int } D^+ - \text{int } D^-) = S^{n-1} \bigcup S^+ \bigcup S^-$ induces the opposite orientation, but in any event it is clear that with this induced orientation we have $\deg(\eta') = 0$, so Proposition E.5

implies the existence of an extension $\eta'' : D^n - \text{int } D^+ - \text{int } D^- \to S^{n-1}$. Using obvious methods one can combine ζ' and η'' to obtain the desired extension of ζ'.//

Lemma E.7: Suppose $\zeta : D^n \to \mathbf{R}^n$ is a vector field whose only zeros are $y^+, y^- \in \text{int } D^n$, both of which are regular, with $\text{ind}_{y^+}(\zeta) = 1$ and $\text{ind}_{y^-}(\zeta) = -1$. Then there is a nonvanishing vector field $\zeta' : D^n \to \mathbf{R}^n - \{0\}$ that agrees with ζ on S^{n-1}.

The proof is similar to the proof of Lemma E.6 and is omitted.

Proof of Proposition E.4: Fix an open set U with $\bar{U} \subset \text{int } M$, and let κ be as in Proposition E.2 for $\bar{U}$. We wish to show that (A) implies (1), so fix $(g, V) \in \mathcal{P}_\infty(M)$ with $V \subset U$ connected and $d(g, \text{Id}_{\bar{V}}) < \kappa$.

Fix $x^+, x^- \in V - \mathcal{F}(g)$. Since V is open and connected, and since g has only regular fixed points, $\mathcal{F}(g)$ is finite and $V - \mathcal{F}(g)$ must also be path connected. (This is the only place where we use the assumption $n \geq 2$.) Combining the existence of a smooth path between x^+ and x^- with the tubular neighborhood theorem (Hirsch (1976, Th. 6.3, p. 114)) quickly yields the existence of a (diffeomorphic image of) a disk in $V - \mathcal{F}(g)$ that contains x^+ and x^-. Let $\alpha : D^n \to V$ and $\beta : \alpha(D^n) \to D^n$ be inverse diffeomorphisms, and let α_* and β_* be the induced transformations of vector fields: $\alpha_*(\zeta)(x) = D\alpha(\beta(x))\zeta(\beta(x))$, and similarly for β. Of course α_* and β_* are inverse transformations.

Consider the vector field $\beta_*(\Phi_{\bar{U}}(g)) : D^n \to \mathbf{R}^n - \{0\}$. Lemma E.6 guarantees the existence of a vector field $\zeta : D^n \to \mathbf{R}^n$ that agrees with $\beta_*(\Phi_{\bar{U}}(g))$ on a neighborhood of S^{n-1} and whose two zeros $\beta(x^+)$ and $\beta(x^-)$ are regular with $\text{ind}_{x^+}(\zeta) = 1$ and $\text{ind}_{x^-}(\zeta) = -1$. Since smooth functions are dense in the space of continuous functions we may assume that ζ is smooth, and obvious constructions allow us to assume that ζ agrees with $\beta_*(\Phi_{\bar{U}}(g))$ in a neighborhood of S^{n-1}.

The obvious temptation is to define g' by replacing the restriction of g to $\alpha(D^n)$ with $\Psi_{\bar{U}}(\alpha_*(\zeta))$. The problem is that $\max ||\alpha_*(\zeta)(x)|| > \kappa$ is possible. To circumvent this difficulty we instead consider $\Psi_{\bar{U}}(s \cdot \alpha_*(\zeta))$ where $s : \alpha(D^n) \to (0, 1]$ is a smooth function that is constant with value 1 in a neighborhood of S^{n-1} and is constant with value less that $\kappa / \max ||\alpha_*(\zeta(x))||$ outside the neighborhood of S^{n-1} on which ζ and $\beta_*(\Phi_{\bar{U}}(g))$ agree.

Define $g' : \bar{V} \to M$ by replacing g on $\alpha(D^n)$ with $\Psi_{\bar{U}}(s \cdot \alpha_*(\zeta))$. There is a neighborhood of $\partial\alpha(D^n) = \alpha(S^{n-1})$ on which ζ agrees with $\beta_*(\Phi_{\bar{U}}(g))$ and s is identically 1, and this implies that $\alpha_*(\zeta) = \Phi_{\bar{U}}(g)$ and $\Psi_{\bar{U}}(s \cdot \alpha_*(\zeta)) = g$. In particular g' is smooth.

Of course we have been careful to set things up so that

$$\begin{aligned} \Lambda_{ENR}(g', x^+) &= \Lambda_{ENR}(\Psi_{\bar{U}}(s \cdot \alpha_*(\zeta)), x^+) = \text{ind}_{x^+}(s \cdot \alpha_*(\zeta)) \\ &= \text{sgn}\,|\,D(s \cdot \alpha_*(\zeta))(x^+)\,| = \text{sgn}\,|\,D(\alpha_*(\zeta))(x^+)\,| \\ &= \text{sgn}\,|\,D\zeta(\beta(x^+))\,| = \text{ind}_{\beta(x^+)}(\zeta) = 1. \end{aligned}$$

Here the second equality is from Lemma E.3, the fourth equality is due to the fact that s is a positive constant in a neighborhood of x^+, and the fifth equality follows from noting that in the coordinate system given by α and β, $D(\alpha_*(\zeta))(x^+)$ and $D\zeta(\beta(x^+))$ have the same matrix. Similarly $\Lambda_{ENR}(g', x^-) = -1$. This completes the proof that (A) implies (1).

The proof that (A) implies (2) is quite similar, naturally, applying Lemma E.7 instead of Lemma E.6, so we regard the proof of Proposition E.4 as complete.//

We now turn to the two results showing that in the two cases considered by Proposition 4.7 we can find $g \in C^\infty(\bar{U}, M)$ arbitrarily near F such that $(g|_{\bar{V}}, V)$ is as in (A) of Proposition E.4 for some connected neighborhood V of $\mathcal{F}(F)$. These results conclude the proof of Proposition 4.7.

Proposition E.8: Suppose $(f, U) \in \mathcal{P}_{ENR}(M)$ and $\mathcal{F}(f)$ is connected. For any positive numbers κ and δ one can find V, a connected neighborhood of $\mathcal{F}(f)$, and a smooth function $g : \bar{U} \to M$, all of whose fixed points are regular, with $\mathcal{F}(g) \subset V$, $d(g|_{\bar{V}}, \mathrm{Id}_{\bar{V}}) \leq \kappa$, and $Gr(g) \subset \mathbf{B}(Gr(f); \delta)$.

Proof: Let V be the component of the open set $\{x \mid d(x, f(x)) \leq \kappa/2\}$ that contains $\mathcal{F}(f)$. By Hirsch (1976, Th. 2.1, p. 74) there is a smooth function $g : \bar{U} \to M$ all of whose fixed points are regular with $d(g, f) < \kappa$ and $Gr(g) \subset \mathbf{B}(Gr(f); \delta)$. By choosing g sufficiently close to f we can force $\mathcal{F}(g) \subset V$.//

Proposition E.9: Suppose M is convex, $(F, U) \in \mathcal{P}_{con}(M)$, and $\mathcal{F}(F)$ is connected. Then for any positive numbers κ and δ one can find V, a connected neighborhood of $\mathcal{F}(F)$, and a smooth function $g : \bar{U} \to M$, all of whose fixed points are regular, with $\mathcal{F}(g) \subset V$, $d(g, \mathrm{Id}_{\bar{V}}) \leq \kappa$, and $Gr(g) \subset \mathbf{B}(Gr(F); \delta)$.

Proof: For $k = 1, 2, \ldots$ let $s_k : \bar{U} \to [0, 1]$ be a smooth function with $s_k(x) = 0$ for all $x \in \bar{U} - \mathbf{B}(\mathcal{F}(F); 1/k)$ and $s_k(x) = 1$ for all x in a neighborhood of $\mathcal{F}(F)$. Define $F_k \in \mathcal{K}_{con}(\bar{U}, M)$ by $F_k(x) = (1 - s_k(x))F(x) + s_k(x)\{x\}$. If it is not the case that $F_k \to F$, then there is $\varepsilon > 0$ and a sequence $(x_k, y_k) \in Gr(F_k) - \mathbf{B}(Gr(F); \varepsilon)$, but it is easy to show that any limit point of $\{(x_k, y_k)\}$ is in $Gr(F)$. Thus $Gr(F_k) \subset \mathbf{B}(Gr(F); \delta/2)$ for large k.

Fixing such a k, choose $\varepsilon > 0$ such that $d(\varepsilon F(x) + (1-\varepsilon)\{x\}, x) < \kappa/2$ for all $x \in \bar{U}$, and let V be the connected component of $s_k^{-1}((1-\varepsilon, 1])$ that contains $\mathcal{F}(F)$. Proposition 2.24 allows F_k to be approximated with arbitrary accuracy, and standard results (e.g. Hirsch (1976, Th. 2.1, p. 74) allow us to choose approximations that are smooth and have only regular fixed points. Sufficiently close approximations of this sort clearly have the desired properties.//

Appendix F: The Proofs of Theorems 6 and 7

The arguments given below are logically independent but entirely parllel. To obtain a firm understanding of either proof one should read both.

Proof of Theorem 6: Without loss of generality we may assume that $C \subset \mathbf{R}^n$ with $0 \in \operatorname{int} C$. Fix $\delta > 0$ and $x^* \in \mathcal{F}(F)$. (If $\mathcal{F}(F) = \emptyset$ then the proof is trivial.) If $\Lambda(F, U) = 0$ then our goal is to construct a map $f : \bar{U} \to C$ with $Gr(f) \subset \mathbf{B}(Gr(F); \delta)$ and $\mathcal{F}(f) = \emptyset$. If $\Lambda(F, U) \neq 0$ then $\mathcal{F}(F)$ is essential, and in order to show that no proper compact subset is essential we construct a map $f : \bar{U} \to C$ with $Gr(f) \subset \mathbf{B}(Gr(F); \delta)$ all of whose fixed points are in the δ-ball around x^*.

Let V be an open subset of ∂C with $\bar{V} \subset U$ and $\mathcal{F}(F) \bigcap \partial C \subset V$. For sufficiently small $\varepsilon > 0$ we have

$$\{\alpha x | x \in \bar{V} \text{ and } 1 - \varepsilon < \alpha \leq 1\} \subset U \text{ and}$$

$$\mathcal{F}(F) \bigcap \{\alpha x | x \in \bar{V} \text{ and } 1 - \varepsilon \leq \alpha \leq 1\} \subset \{\alpha x | x \in V \text{ and } 1 - \varepsilon \leq \alpha \leq 1\}.$$

Fixing such an ε, it is clearly possible to construct U_1, U_2, open subsets of C, with the following properties: $\alpha x' \in U_1$ for all $\alpha \in (1-\varepsilon, 1]$ whenever

$0 \neq x \in U_1$, x' is the point in ∂C on the ray from 0 through x, and $\|x\| > (1-\varepsilon)\|x'\|$; $\bar{U}_1 \bigcap (\bar{U} - U) = \emptyset$; $U_1 \bigcup U_2 = U$; $\mathcal{F}(F) \bigcap \bar{U}_2 = \emptyset$. Replacing δ with a smaller number, we may assume that $\mathbf{B}(\mathcal{F}(F);\, \delta) \bigcap \bar{U}_2 = \emptyset$.

Since $U_1 \bigcup U_2 = U$ and $\bar{U}_1 \bigcap (\bar{U}-U) = \emptyset$, any point in $\bar{U}-U$ must be in the interior, relative to $\bar{U}$, of $\bar{U}_2$. Thus U_1 and the interior of $\bar{U}_2$ relative to $\bar{U}$ constitute an open cover of $\bar{U}$, so the theorem guaranteeing a partition of unity implies the existence of a map $\theta : \bar{U} \to [0,\, 1]$ with $\theta(x) = 0$ for all $x \in \bar{U} - U_1$ and $\theta(x) = 1$ for all $x \in \bar{U} - U_2$. Fix $\beta \in (0,\, \delta)$ small enough that $\mathcal{F}(f) = \emptyset$ whenever $f : \bar{U}_2 \to C$ is a map with $Gr(f) \subset \mathbf{B}(Gr(F);\, \beta)$. Fix $\gamma \in (0,\, \beta)$ small enough that the graph of the map $x \mapsto \theta(x)f_1(x) + (1-\theta(x))f_2(x)$ is contained in $\mathbf{B}(Gr(F);\, \beta)$ whenever $f_1 : \bar{U}_1 \to C$ and $f_2 : \bar{U}_2 \to C$ are maps with $Gr(f_1)$, $Gr(f_2) \subset \mathbf{B}(Gr(F);\, \gamma)$. (The existence of such a γ can easily be established by supposing that no such γ exists, taking appropriate sequences, using compactness to extract convergent subsequences, and deriving a contradiction.) Proposition 2.25 implies the existence of a map $f_2 : \bar{U}_2 \to C$ with $Gr(f_2) \subset \mathbf{B}(Gr(F);\, \gamma)$. If $\Lambda(F,\, U) = 0$ it now suffices to construct a map $f_1 : \bar{U}_1 \to C$ with $Gr(f_1) \subset \mathbf{B}(Gr(F);\, \gamma)$ and $\mathcal{F}(f_1) = \emptyset$, while if $\Lambda(F,\, U) \neq 0$ it suffices to construct a map $f_1 : \bar{U}_1 \to C$ with $Gr(f_1) \subset \mathbf{B}(Gr(F);\, \gamma)$ all of whose fixed points are in the γ-ball around x^*.

Let C' be C together with the set of $x \in \mathbf{R}^n - C$ such that if x' is the point in ∂C on the ray from 0 through x, then $\|x - x'\| < \gamma/3$. We now construct a smooth compact convex ∂-manifold M with $C \subset \operatorname{int} M$ and $M \subset C'$. Let $\kappa > 0$ be small enough that $x \in C'$ whenever $d(x,\, C) \leq \kappa$, where $d(x,\, C) = \min_{x' \in C} \|x - x'\|$. Let $q : \mathbf{R}^n \to [0,\, 1]$ be the map $q(x) = \min\{0,\, 1 - d(x,\, C)/\kappa\}$. Observe that q is concave on $\{x | d(x,\, C) \leq \kappa\}$ since C is convex. Let $\varphi : \mathbf{R}^n \to \mathbf{R}_+$ be a C^∞ function with support contained in $\mathbf{B}(0;\, \kappa/4)$ and $\int_{\mathbf{R}^n} \varphi(z)\, dz = 1$ (cf. Hirsch (1976, §2.2)). Define the convolution $q_\varphi : \mathbf{R}^n \to [0, 1]$ by

$$q_\varphi(x) = \int_{\mathbf{R}}^n q(x-z)\varphi(z)\,dz = \int_{\mathbf{R}^n} q(z)\varphi(x-z)\,dz.$$

Differentiation under the integral sign shows that q_φ is C^∞. Since q is concave on $\{x|d(x, C) \le \kappa\}$, q_φ is concave on $\{x|d(x, C) \le \frac{3}{4}\kappa\}$. If $q_\varphi(x) > \frac{1}{2}\kappa$ then $d(x, C) < \frac{3}{4}\kappa$, so that $x \in C'$, and if $q_\varphi(x) < \frac{3}{4}\kappa$ then $x \notin C$. Sard's theorem implies the existence of a regular value $c \in (\frac{1}{2}\kappa, \frac{3}{4}\kappa)$ of q_φ. Let $M = q_\varphi^{-1}([c, 1])$. Then M is convex, and regularity and the implicit function theorem implies that ∂M is a smooth manifold, so M is a smooth ∂-manifold.

Let $r : M \to C$ be the retraction given by letting $r(x) = x$ if $x \in C$ and letting $r(x)$ be the point in ∂C on the ray from 0 through x if $x \in M - C$. Let $i : C \to M$ be the inclusion. Note that our construction of U_1 implies that $r^{-1}(\bar{U}_1)$ is the closure of $r^{-1}(U_1)$. Proposition 2.25 implies the existence of maps $f : \bar{U}_1 \to C$ with graphs arbitrarily close to the graph of F, and by making the graph of f close to the graph of F we can insure that the graph of $f \circ r : r^{-1}(\bar{U}_1) \to C$ is arbitrarily close to the graph of $F \circ r : r^{-1}(\bar{U}_1) \to C$. For such an f the index axioms yield

$$\begin{aligned}\Lambda(F, U) &= \Lambda(F, U_1) = \Lambda(f, U_1) = \Lambda(r \circ i \circ f, U_1) \\ &= \Lambda(i \circ f \circ r, r^{-1}(U_1)) = \Lambda(i \circ F \circ r, r^{-1}(U_1)).\end{aligned}$$

Now assume that $n \ge 2$. (The case $n = 1$ is elementary and left to the reader.) If $\Lambda(F, U) = 0$ then Proposition 4.7 implies the existence of a map $g : r^{-1}(\bar{U}_1) \to M$ with $Gr(g) \subset \mathbf{B}(Gr(F \circ r|_{r^{-1}(\bar{U}_1)}); \gamma/4)$ and $\mathcal{F}(g) = \emptyset$. If $\Lambda(F, U) \neq \emptyset$ there is a minor technicality insofar as it may be the case that $\Lambda(F, U) \notin \{-1, 0, 1\}$ even though $\mathcal{F}(F) = \{x^*\}$ is a singleton. In this case we can modify F without disturbing our hypotheses by replacing $F(x)$ with the convex hull of $\{x\} \bigcup F(x)$ for all x in a small convex closed neighborhood of x^*. Combining this trick with Proposition 4.7, if $\Lambda(F, U) \neq 0$ there is a map $g : r^{-1}(\bar{U}_1) \to M$ with $Gr(g) \subset \mathbf{B}(Gr(F \circ r|_{r^{-1}(\bar{U}_1)}); \gamma/4)$ all of whose fixed points are in the $\frac{3}{4}\gamma$-ball around x^*.

Let $h : M \to C$ be defined as follows: if $x \in \partial C$, then $h(ax) = ax$ for all $a \in [0, 1-\varepsilon]$, and h maps the line segment between $(1-\varepsilon)x$ and the point in ∂M on the ray from 0 through x linearly onto the line segment between $(1-\varepsilon)x$ and x. Clearly h is a homeomorphism. The construction of M implies that $\|h(x) - x\| < \gamma/4$ for all $x \in M$. The construction of U_1 implies that $h(r^{-1}(\bar{U}_1)) = \bar{U}_1$. We may therefore let $f_1 = h \circ g \circ h^{-1} : \bar{U}_1 \to C$. Clearly $\mathcal{F}(f_1) = h(\mathcal{F}(g))$, so the fixed points of f_1 (in the case where there are any) are contained in the γ-ball around x^*. If $(x, y) \subset Gr(f_1)$ then $(h^{-1}(x), h^{-1}(y)) \in Gr(g)$, and there is a point $(x', y') \in Gr(F \circ r|_{r^{-1}(\bar{U}_1)})$ in the $\gamma/4$ ball around (x', y'), and of course $(r(x'), y') \in Gr(F)$. We now have

$$\begin{aligned} \|x - r(x')\| \leq \ & \|x - h^{-1}(x)\| + \|h^{-1}(x) - x'\| \\ & + \|x' - r(x')\| \leq \tfrac{1}{2}\gamma + \|h^{-1}(x) - x'\| \end{aligned}$$

and

$$\|y - y'\| \leq \|y - h^{-1}(y)\| + \|h^{-1}(y) - y'\| \leq \gamma/4 + \|h^{-1}(y) - y'\|.$$

Since $\|h^{-1}(x) - x'\| + \|h^{-1}(y) - y'\| < \gamma/4$, we have shown that $Gr(f_1) \subset \mathbf{B}(Gr(F); \gamma)$. This completes the proof.//

Proof of Theorem 7: Fix $\delta > 0$ and $x^* \in \mathcal{F}(f)$. (If $\mathcal{F}(f) = \emptyset$ then the proof is trivial.) If $\Lambda(f, U) = 0$ then our goal is to construct a map $f' : \bar{U} \to C$ with $Gr(f') \subset \mathbf{B}(Gr(f); \delta)$ and $\mathcal{F}(f') = \emptyset$. If $\Lambda(F, U) \neq 0$ then $\mathcal{F}(f)$ is essential, and in order to show that no proper compact subset is essential we construct a map $f' : \bar{U} \to M$ with $Gr(f') \subset \mathbf{B}(Gr(f); \delta)$ all of whose fixed points are in the δ-ball around x^*.

The collaring theorem (Hirsch (1976, Th. 6.1, p. 113) states that there is a smooth map $\psi : \partial M \times [0, 1) \to M$ with $\psi(x, 0) = x$ for all $x \in \partial M$ that is a diffeomorphism between $\partial M \times [0, 1)$ and a neighborhood of ∂M in M. Let V be an open subset of ∂M with $\bar{V} \subset U$ and $\mathcal{F}(f) \bigcap \partial M \subset V$. For sufficiently small $\varepsilon > 0$ we have $\psi(\bar{V} \times [0, \varepsilon)) \subset U$ and $\mathcal{F}(f) \bigcap \psi(\bar{V} \times [0, \varepsilon]) \subset \psi(V \times [0, \varepsilon])$. Fix such an ε. Let U' be an open subset of U with $\mathcal{F}(f) \bigcup \psi(\bar{V} \times [0, \varepsilon]) \subset U'$. Let

$$U_1 = \psi(V \times [0, \varepsilon]) \cup (U' \cap (M - \psi(\partial M \times [0, \varepsilon]))).$$

Clearly U_1 is open: $\psi(V \times [0, \varepsilon))$ is open, $\psi(\partial M \times [0, \varepsilon])$ is closed so that $U' \bigcap (M - \psi(\partial M \times [0, \varepsilon]))$ is open, and if $x \in V$ and ρ is small enough that $\mathbf{B}(\psi(x, \varepsilon); \rho) \subset U'$ and $\mathbf{B}(\psi(x, \varepsilon); \rho) \bigcap \psi(\partial M \times [0, \varepsilon]) \subset \psi(V \times [0, \varepsilon])$, then $\mathbf{B}(\psi(x, \varepsilon); \rho) \subset U_1$. Let U_2 be an open subset of U with $U_1 \bigcup U_2 = U$ and $\bar{U}_2 \bigcap \mathcal{F}(f) = \emptyset$. The key properties of U_1 and U_2 guaranteed by this construction are as follows: $\psi(\{x\} \times [0, \varepsilon)) \subset U_1$ whenever $\psi(\{x\} \times [0, \varepsilon)) \bigcap U_1 \neq \emptyset$; $\bar{U}_1 \bigcap (\bar{U} - U) = \emptyset$; $U_1 \bigcup U_2 = U$; $\mathcal{F}(f) \bigcap \bar{U}_2 = \emptyset$.

Let a triangulation of M be given (recall footnote 1). After sufficiently fine subdivision (e.g. Dold (1980, pp. 40-41)) one can find subcomplexes K and L such that $U_1 - U_2 \subset \operatorname{int} K$, $M - U_1 \subset \operatorname{int} L$, and $K \bigcap L = \emptyset$. Denote $K - \operatorname{int} K$ and $L - \operatorname{int} L$ by ∂K and ∂L respectively. Then ∂K and ∂L are also subcomplexes: if, for instance, $x \in \partial K$ and σ is the simplex containing x in its interior, then there is an open simplex τ not in K whose closure contains x, and the definition of a simplicial complex implies that the closure of τ contains σ, so $\sigma \subset \partial K$. Let $J = M - (\operatorname{int} K \bigcup \operatorname{int} L)$. Then J is the closure of $M - (K \bigcup L)$, hence the closure of a union of (open) simplices, so it too is a subcomplex, and of course ∂K and ∂L are subcomplexes of J. Note that $\mathcal{F}(f) \bigcap J = \emptyset$ since $\mathcal{F}(f) \subset U_1 - U_2 \subset \text{int}\, K$. Without loss of generality we may assume that δ is small enough that $\mathcal{F}(f'') = \emptyset$ whenever $f'' : J \to M$ is a map with $Gr(f'') \subset \mathbf{B}(Gr(f|_J);\, \delta)$.

Proposition 2.21 implies the existence of a number $\delta_1 > 0$ small enough that if $f_1 : \partial K \bigcup \partial L \to M$ with $Gr(f_1) \subset \mathbf{B}(Gr(f|_{\partial K \bigcup \partial L});\, \delta_1)$, then there is $f_2 : J \to M$ with $Gr(f_2) \subset \mathbf{B}(Gr(f|_J);\, \delta)$ and $f_2|_{\partial K \bigcup \partial L} = f_1$. An easy compactness argument implies that there is some $\gamma > 0$ small enough that if $f'' : \bar{U}_1 \to M$ with $Gr(f'') \subset \mathbf{B}(Gr(f|_{\bar{U}_1});\, \gamma)$, then $Gr(f''|_{\partial K \bigcup \partial L}) \subset \mathbf{B}(Gr(f|_{\partial K \bigcup \partial L});\, \delta_1)$. We now claim that it suffices to construct a map $f'' : \bar{U}_1 \to M$ with $Gr(f'') \subset \mathbf{B}(Gr(f|_{\bar{U}_1});\, \gamma)$ and $\mathcal{F}(f) = \emptyset$ if $\Lambda(f,\, U) = 0$ or $\mathcal{F}(f) \subset \mathbf{B}(x^*;\, \delta)$ if $\Lambda(f,\, U) \neq 0$. For suppose that such a map is given. Let $f_1 : \partial K \bigcup \partial L$ be given by f'' on ∂K and by f on ∂L. Let $f_2 : J \to M$ be an extension of f_1 with $Gr(f_2) \subset \mathbf{B}(Gr(f|_J);\, \delta)$. A satisfactory $f' : \bar{U} \to M$ is now given by letting $f'|_K = f''|_K$, $f'|_J = f_2$, and $f'|_{L \bigcap \bar{U}} = f|_{L \bigcap \bar{U}}$.

Fix a number $\kappa \in (0,\, \varepsilon)$ small enough that $d(\psi(x,\, s),\, \psi(x,\, t)) < \gamma/3$ for all $x \in \partial M$ and all $s,\, t \in [0,\, \kappa]$. (Here d may be any metric on M.) Let $M' = M - \psi(\partial M \times [0,\, \kappa/2))$. Define a retraction $r : M \to M'$ by setting $r(x) = x$ if $x \in M'$ and setting $r(\psi(x,\, t)) = \psi(x,\, \kappa/2)$ for $x \in \partial M$ and $0 \le t < \kappa/2$. Let $i : M' \to M$ be the inclusion. Define a homeomorphism $h : M \to M'$ by setting $h(x) = x$ if $x \notin \psi(\partial M \times [0,\, \kappa))$ and setting $h(\psi(x,\, t)) = \psi(x,\, (t+\kappa)/2)$ for $x \in \partial M$ and $0 \le t < \kappa$.

Observe that our construction of U_1 implies that $r(\bar{U}_1) = \bar{U}_1 \bigcap M' = h(\bar{U}_1)$, so the maps $i \circ h \circ f \circ h^{-1} \circ r|_{\bar{U}_1}$ and $f \circ h^{-1} \circ r \circ i \circ h|_{\bar{U}_1}$ are defined. In addition our construction of $\bar{U}_1$ implies that $(i \circ h)^{-1}(U_1) = U_1$ and that $(i \circ h)^{-1}(\bar{U}_1) = \bar{U}_1$, so that in particular $\bar{U}_1$ is the closure of $(i \circ h)^{-1}(U_1)$.

Commutativity now yields

$$\Lambda(i \circ h \circ f \circ h^{-1} \circ r|_{\bar{U}_1}, U_1) =$$

$$\Lambda(f \circ h^{-1} \circ r \circ i \circ h|_{\bar{U}_1}, (i \circ h)^{-1}(U_1)) = \Lambda(f, U_1),$$

and Additivity implies that $\Lambda(f, U_1) = \Lambda(f, U)$.

If x is a fixed point of $i \circ h \circ f \circ h^{-1} \circ r|_{\bar{U}_1}$, then $x \in M'$ since x is an image of i, so $r(x) = x$ and $h^{-1}(x)$ is a fixed point of f. Conversely $h(x)$ is a fixed point of $i \circ h \circ f \circ h^{-1} \circ r|_{\bar{U}_1}$ whenever x is a fixed point of f. Thus $\mathcal{F}(i \circ h \circ f \circ h^{-1} \circ r|_{\bar{U}_1}) = h(\mathcal{F}(f))$. If $\Lambda(f, U) = 0$ then we may apply Proposition 4.7 to obtain a function $g : \bar{U}_1 \to M$ with $Gr(g) \subset \mathbf{B}(Gr(i \circ h \circ f \circ h^{-1} \circ r|_{\bar{U}_1}); \gamma/3)$. If $\Lambda(f, U) \neq 0$ then there is a slight technical problem insofar as it may be the case that $\mathcal{F}(f) = \{x^*\}$ is a singleton but $\Lambda(f, U) \notin \{-1, 0, 1\}$. In this case we can modify f without disturbing our hypotheses by fixing a coordinate system at x^* (so that M near x^* is an open subset of a half space) and replacing $f(x)$ with $\theta(x)x + (1 - \theta(x))f(x)$, where $\theta : M \to [0, 1]$ is a map that is identically 1 in a neighborhood of x^* and identically 0 outside a ball around x^* whose radius is small relative to γ and is also small enough that the domain of the coordinate system contains the images under f of points in the ball. Combining this trick with Proposition 4.7, when $\Lambda(f, U) \neq 0$ there is a map $g : \bar{U}_1 \to M$ with $Gr(g) \subset \mathbf{B}(Gr(i \circ h \circ f \circ h^{-1} \circ r|_{\bar{U}_1}); \gamma/3)$ all of whose fixed points are in the $\gamma/3$-ball around $h(x^*)$. Without going into details it is clear that when 3 is sufficiently large we will have $Gr(g) \subset \mathbf{B}(Gr(f); \gamma)$, and if $\Lambda(f, U) \neq 0$ then all fixed points of g lie in the γ-ball around x^*.//

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