The Algebra of Coherent Algebraic Sheaves

with an Explicated Translation of Serre's

Faisceaux Algébriques Cohérents

by

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For Shino, whose advice for those starting out in research is to understand one paper better than the author.

Contents

Preface

The preface of a book is traditionally devoted to remarks that have some personal character, and for most books these are mundane and reassuring. "Even in these turbulent times" the author's feelings gravitate, like a pendulum under the slow influence of friction, to appreciation of his or her parents, the delightful domestic environment he or she currently enjoys, the support and encouragement of colleagues, that nice person from the publisher who took care of all those pesky details, and so forth.

I could easily profess to such sentiments, but it would be evasive, because what everyone *really* wants to know is: Why in the world is an economist writing a book of algebra and algebraic geometry? What sort of hubris might inspire him to think he has any competence for such a task? What could he possibly hope to gain? And in the face of these questions, how do I conceive of my efforts, and what sort of "public face" am I trying to present to the world?

The actual answers are quite a bit less dramatic than this sounds. I am a mathematical economist, which means that if I am perhaps not exactly a mathematician, I am certainly not exactly not a mathematician. From the point of view of pure mathematics, mathematical economics is a fringy thing, perhaps mildly interesting, but suspiciously justified by appeals to values beyond mathematics, and inessential to the central thrust and foundations of the discipline. Be that as it may, it does present a rich menu of technical challenges, and is perhaps not more distant from the main currents of research than various other subfields within mathematics proper. Over the years it has attracted the interest of many mathematicians, including the Fields medalists Stephen Smale and Pierre-Louis Lions. Relative to other specializations, the technical foundations of mathematical economics are quite broad. Economic phenomena can be modelled in many ways, so if there's a tool out there that can be put to use, probably somebody will do so eventually. Real analysis, topology, functional analysis, and mathematical statistics underly fundamental economic models.

In the mid 1980's it occurred to me that algebraic geometry might have some relevance to game theory, because the notion of Nash equilibrium is a matter of polynomial equations and inequalities. (It turns out that the seemingly nearby but actually quite distant field of *semi*-algebraic geometry does indeed provide quite useful results and insights.) So, I walked across campus and started attending a course based on Hartshorne (1977). The first chapter was not impossible (if one accepted the cited results from algebra) but after that I quickly became hopelessly confused. At first I could understand the logic of the definitions related to scheme theory, if not their motivations, but before too long I was just lost in the jungle at night.

I did learn that, along with logic, topology, measure theory, and functional analysis, the reformulation of algebraic geometry in the 1960's was one of the profound transformations of mathematics during the 20$^{\text{th}}$ century, and that if I didn't find some way to learn more, large swaths of contemporary mathematics would be far beyond my comprehension and appreciation. But I think that my persistence really had more to do with not wanting to accept such a defeat. Now Hartshorne makes it pretty clear that he expects a strong background in algebra, so I obtained various books and read each one up to some point. It's a gorgeous subject, but it has its own motivations and internal agenda, as did each of the authors. Certainly I learned a lot, but each time I tried to return to Hartshorne I was rebuffed, and this was also the case after I read lower level books on algebraic geometry.

About a decade ago it occurred to me that I might try reading Serre's "Faisceaux Algébrique Cohérents" (henceforth FAC) which was obviously an important milestone in the history of the subject, presumably much closer to the original motivations and ways of thinking, and universally praised in the highest terms. My high school French is barely adequate for mathematics, and it is a journal article, not a textbook, so this also proved quite difficult. However, I had the thought that instead of reading it, it might work better to prepare a translation. This had the advantage of slowing me down, so that I could patiently work through each logical detail. Both for my own benefit, and because I could imagine it becoming accessible to readers at a much lower level than would otherwise be the case, I interpolated explanatory remarks when Serre elided some details, appealed to some not entirely elementary result, wrote in a way that later became obsolete, and so forth. This had the effect of creating a sense of dialogue, making it at least a quite original mathematical document. Everything seemed to be going nicely, and "working" on it was a delightfully relaxing activity.

Serre's style is very gentle throughout, and up to a certain point FAC is effectively self-contained, but then there are a flurry of citations to Cartan and Eilenberg's *Homological Algebra* (henceforth CE) which would appear in print the following year. I acquired this (still very useful) book, and set about figuring out what these results were. It quickly emerged that they were central to Serre's project, and that my translation couldn't succeed unless the reader could access them easily. At the same time CE was not an acceptable source, since what the reader of FAC needs is mixed in with a great many other things, and some of the cited results are exercises. No other source seemed suitable, so I set out to write a minimal treatment, working backward in CE in order to extract only what was required. The result was a "Supplement" consisting of several dozen pages.

Within FAC the work went forward again, until I reached the last few pages, where I learned, somewhat to my horror, that Serre appeals to various results of commutative algebra that are not in Atiyah and McDonald's *An Introduction to Commutative Algebra*. Again, I set out to extract a minimal treatment from various sources, and the supplemental material expanded accordingly. Eventually it became clear that I would need to take control of the subject from the beginning, so I wrote what is now Chapter A, even though this material is basic and very well treated elsewhere. Somehow the "Supplement" ballooned to over two hundred pages, dwarfing (at least in bulk) the original intent and spirit of the project.

By this point you have probably figured out that I enjoy writing mathematics. Early in my career I had great difficulties with writing (had it not been for the advent of personal computers my career might have been lost) so I tried to take that aspect of the work seriously, tracking down written advice about exposition and (more usefully) thinking about what it was that made the writings of John Milnor, Michael Spivak, J.S. Milne, Allen Hatcher, and others, work so well. I also put a lot of effort into writing, learning much from various mistakes and other experiences. Perhaps most important, I developed a taste for mathematical exposition as a medium of aesthetic expression. If you asked why I spent so much time on this project, I would say that no one would think it odd if I spent the occasional Saturday afternoon dabbling in watercolors, and this really isn't any different.

As you might expect, I have strong and well developed views concerning mathematical exposition, but for the most part I hope that they are better expressed implicitly in the text than I could state them here. I should say two things about the book. First, it is the sort of book I would like for myself, insofar as it is meant to be read, not "studied." (Readers who like to work exercises should have no difficulty finding them elsewhere.) I am a busy professional who doesn't like being told that he can't learn a subject without going back to the course work ghetto, or doing endless problem sets. As much as possible, I have tried to craft a book that can be absorbed and appreciated with minimal effort. The reader should be aware of two particular aspects of this. I have kept the coverage almost as minimal as possible, subject to the nature of the project. (At a certain point I thought that factoriality of regular local rings would be required. This turned out to be wrong, but it would be a shame to stop within spitting distance of this glorious theorem.) Sometimes I have added inessential results that illustrate or apply the ideas under discussion, but I have deliberately avoided trying to make the coverage of any topic "complete." Second, I have allowed the organization complete freedom to fall in line with the logic of the material. Possibly this book might serve as the main text of quite a nice course, but relative to any established curriculum or concept of what every young algebraist needs to know, and when she needs to know it, there are large and obvious gaps.

The second point is that this is the work of an amateur, in several senses. It was pursued for its own sake, outside of any strategy for "career development." (At best it might push my reputation in economics further sideways.) Consequently work on it proceeded in "slow cooking mode," and I could indulge a kind of perfectionism that the pressure to publish can easily quash. Also, I am no expert in commutative algebra, and perhaps was better able to appreciate the logic of the material as something fresh, and to convey some sense of that to the reader. In retrospect the surprising degree of coherence it attained is, I think, ultimately a reflection of Serre's long range vision.

Expository projects in this spirit will almost certainly continue to be not that well rewarded, because accomplishment in research will continue to be the only acceptable qualification for membership in the academy. (As an economist I could advance various points of view concerning whether that is a good or bad thing, but I see no likelihood that it will change soon.) Perhaps the example of this book may inspire others to think that such work can, in and of itself, be more than ample reward.

Ordinarily at this point in a preface there would be a long list of names of all the people who provided feedback, encouragement, and various forms of assistance. However, economists tend to be quite dubious when they learn that one of their colleagues is indulging a taste for pure math, so I have been completely secretive about this project while it was underway. Hopefully things won't be that bad ex post, when people see that during the decade or so that I have been noodling around with FAC, I have also done roughly the usual amount of the usual sort of research. But just to be on the safe side, if you happen to meet an economist, please don't tell them about this, OK?

Introduction

> Le travail [14] (cité (FAC)) de J.-P. Serre peut aussi être considéré comme un exposé intermédiare entre le point de vue classique et le point de vue des schémas en Géométrie algébrique, et à ce titre, sa lecture peut constituer une excellente préparation à celle de nos *Éléments*.
>
> –Alexander Grothendieck, *Éléments de Géométrie Algébrique*

For many, including your author, learning about algebraic geometry can be a daunting undertaking. There are introductory books that give some glimpse of the subject, mostly from a classical point of view, but to go beyond them, to the substance and consequences of the reformulation of the subject by Grothendieck and his colleagues in the 1960's, requires a very substantial technical background, in both commutative and homological algebra. It may seem "logical" to take courses in these subjects before attempting to tackle the theory of schemes, but these are large subjects that draw their motivations from many sources, and have active, independent research agendas. One can learn quite a bit about both of them and still be far from ready to tackle a text such as Hartshorne (1977).

Moreover, this "logical" approach flies in the face of the way people actually learn, which is by accumulating experiences that make the topic increasingly familiar. Now one might try to read some algebraic geometry while accepting some of the background on faith, to be studied seriously at some later time, but this never works for me. Unless I'm grasping each step in the logic, I'm not really learning mathematics, and then I just get confused.

This book embodies a different approach, which is to undertake a substantial, logically self contained project of some relevance. Specifically, we're going to study Serre's 1955 *Annals of Mathematics* paper "Faisceaux Algébriques Cohérents," (FAC) together with all of the algebra required to understand it, completely, by which I mean that you will see that everything that Serre says is justified. You won't see all the algebra that is relevant to algebraic geometry, by a long ways, but all the algebra herein *is* highly relevant. The material is very close to being a minimal rendering of all the results that Serre applies, and all the material that is logically prior to that, so the selection of topics is the product of happenstance, but somehow it comes together into quite a coherent package.

Although FAC is a journal article, it is, in its style and overall approach, quite close to being a textbook. Serre was laying out a new approach to algebraic geometry, and throughout he is systematic and patient with his readers. He does expect them to be mature mathematicians of that era, so he passes lightly over some tedious details, and expects a pretty solid algebraic background. In addition to translating to English, I've included a chapter titled "What It's All About" that gives a brief overview of the main concepts, and I've interpolated some explanatory remarks when I thought it might help contemporary students get past various bumps in the road. A reader at an early stage of her education should be aware that these comments are intended to provide some guidance and concrete sense of what is involved in reading journal articles. For many students going beyond textbooks to the primary literature is hard, and perhaps put off too long. The typical difficulties will be confronted here in one of the gentlest possible settings.

The algebraic text is self contained, and it would be "logical" to read it in its entirety before beginning FAC, but I can hardly imagine a less enjoyable way to approach this material, or one that is more contrary to the spirit of this project, which is to illuminate its various aspects by applying them in meaningful contexts. The reader should be familiar with the five lemma before starting FAC, but otherwise one can go quite a way into FAC before the results in the preceeding text become important. At a certain point, however, Serre cites results that are quite advanced. For almost everyone the best approach will be to go back and forth, taking prior background and the interests of the moment into account.

Classic papers have many advantages, in comparison with textbooks. In addition to the presentation of the material itself, they open a window into the mind of the author at that time, and through that into the surrounding mathematical landscape as leading researchers of the era saw it. In most areas of research there are a few papers whose deep study is the main source of inspiration for years of research activity, and there is something quite strained about trying to learn about this research without studying its wellspring directly. For algebraic geometry in particular, it is difficult to appreciate Mumford's or Hartshorne's way of thinking without going back to Grothendieck, and in turn Grothendieck's work was in large part a matter of seizing the opportunities opened up by FAC. Later authors of texts are always writing from the point of view of an expert, which tends to smooth things out in various ways, but it also creates a certain distance between the material and the research that brought it into existence. In contrast, the author of the original paper is writing from and for the mind of a beginner.

A real understanding of FAC involves much more than absorbing its logical content. In several ways it stands at the midpoint of 20^{th} century mathematics, a culmination of developments in topology and complex analysis whose echoes continue to resound.

From around the year 1900, largely beginning with the work of Poincaré, mathematics developed two methods of associating invariants with a well enough behaved topological space, namely homotopy groups and what started out as integers, such as the Euler characteristic and the Betti numbers, and developed (largely due to the influence of Emmy Noether) *Noether, Emmy* into homology-cohomology groups. In each case a continuous function induces a homomorphism of the associated groups, functorially. Each of these ways of associating a group with a space is, then, a collection of functors from the category of "well enough behaved" topological spaces to the category of groups. The most obvious application of this information is that it provides a method of proving that two spaces are *not* homeomorphic, but it is important in many other ways.

Naturally, an important goal is to compute the objects given by the theory. This concern leads naturally to the investigation of the groups associated with spaces constructed by combining given spaces in certain ways, and in various cases the groups associated with the constructed space stand in certain algebraic relationships with the groups associated with the given spaces. Thus the exploration of topological issues gave rise to a body of related, and seemingly subordinate, algebraic theory. Incidentally, in 1954 Serre had become the youngest ever recipient of the Fields Medal in recognition of contributions to the theory of spectral sequences, which provide sophisticated computations of this sort.

How homological algebra outgrew its topological origins is a curious story. Both homotopy groups and homology/cohomology are invariant under homotopy: if $f_0, f_1 : X \to Y$ are continuous and homotopic, then they induce the same homomorphisms of associated groups. A continuous function $f : X \to Y$ is a *homotopy equivalence* if there is a continuous function $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the respective identities. If this is the case, then f and g must induce inverse isomorphisms of the associated groups.

One key result of homotopy theory is that for any finitely presented group G, there is a compact topological space that has G as it fundamental group, and has all higher order homotopy groups vanishing. Moreover, any two such spaces are homotopy equivalent. This means that we can start with any given G, pass to a uniquely (up to homotopy equivalence) defined topological space, and from there pass to the associated homology and cohomology groups. The starting point and end result are algebraic structures, which suggests that this process should have a purely algebraic description, and indeed, eventually such a description was found. Similar developments occurred in the theory of Lie algebras *Lie algebra* and the theory of associative algebras.

CE organized the algebraic aspects of homological algebra as a large body of common methods, together with additional bodies of theory related to each of the specific applications mentioned above. It was and still is credited with transforming homological algebra from a somewhat scattered collection of results and computational methods into an independent subject, drawing motivation from several sources, but not subordinate to any one of them.

A few years before FAC Weil (1949) had formulated certain conjectures concerning the number of roots of certain systems of polynomial equations over finite fields, establishing them in certain cases, and pointing out that they would follow easily from a well behaved cohomology theory for algebraic varieties over finite fields. Insofar as the geometric phenomenon analyzed by cohomology are present also in the case of nonzero characteristic, such a cohomology theory was seemingly a reasonable aspiration, but all the methods for defining cohomology known at the time depended on the topology of the real numbers. Weil's work suggested the existence of geometric structures governing arithmetic phenomena that were as yet entirely invisible.

Serre's breakthrough was to show that a new approach to the definition of cohomology was possible in algebraic geometry. Various definitions of homology and cohomology had been proposed for topological spaces in the preceeding decades, and the situation had recently been clarified by Eilenberg and MacLane, who showed that they were (within some range of well behaved spaces) all the same because they all satisfied a system of axioms that determined the theory completely. The axiom system describes the relationships between the homology and cohomology groups of different spaces, and the homomorphisms induced by continuous functions. In contrast, sheaf cohomology works with a single fixed space and develops relationships between the cohomology groups of different sheafs on that space. It had recently become quite influential in the theory of several complex variables and complex analytic varieties, which was quite active at that time.

Shortly after the appearance of FAC it became clear that Serre's cohomology could not be used to prove the Weil conjectures, but during the next fifteen years the methods he pioneered were developed in great depth and generality, particularly by Grothendieck and his colleagues. This resulted in a radical reformulation of the foundations of algebraic geometry, which had languished after the work of the Italian school had gone beyond what the foundations of the subject at the time were able to support. Application of these methods to a different cohomology theory pioneered by Grothendieck led eventually to the complete verification of the Weil conjectures by Deligne in 1972. The structures developed by Grothendieck are still the basis of work in algebraic geometry, as well as arithmetic geometry.

Chapter A

Elements of Commutative Algebra

This chapter provides an introduction to commutative algebra. It presumes very little background, and is logically self contained for anyone who has absorbed the rudiments of linear algebra, groups, rings, and fields. It is just a bit more than the minimal treatment of the subject required by the subsequent chapters, which has the consequence that, although it is dense with important topics and results, it is really only the skeleton of the subject, leaving out numerous details and subsidiary results, as well as problems and other material that would be part of any initial course. Whether it is self contained in practice is a matter of the reader's mathematical maturity; most beginners will derive maximal benefit if they also study an introductory book such as Atiyah and MacDonald's *Introduction to Commutative Algebra* that provides a more comprehensive coverage of the beginnings of the field.

A1 Rings and Modules

We fix once and for all a commutative ring with unit R. We always assume that $1 \neq 0$, which is to say that $R \neq \{0\}$, unless the other possibility is explicitly mentioned. Homomorphisms for such rings are always assumed to map the multiplicative identity of the domain to the multiplicative identity of the range.

A nonzero ring element r is:

- *nilpotent*, or *a nilpotent*, if there is an integer m such that $r^m = 0$;
- a *zerodivisor* if there is a nonzero $s \in R$ such that $rs = 0$;
- a *unit* if there is an $s \in R$ such that $rs = 1$.

Note that a nilpotent is a zero divisor while a unit cannot be a zero divisor. The set of units is evidently an abelian group with multiplication as the group operation.

The ring R is:

- *reduced* if it has no nilpotents;
- an *integral domain* (or just a *domain*) if it has no zerodivisors;
- a *field* if its group of units is all of $R \setminus \{0\}$.

Evidently an integral domain is reduced, and of course a field is an integral domain.

An *R-module* is an abelian group M (whose group operation is written additively) that is endowed with a scalar multiplication by elements of R such that

$$1m = m, \quad r(sm) = (rs)m, \quad (r + s)m = rm + sm, \quad r(m + n) = rm + rn$$

for all $r, s \in R$ and $m, n \in M$. If M and N are R-modules, a function $\varphi : M \to N$ is an *R-module homomorphism* if it is a homomorphism of the underlying abelian groups and $\varphi(rm) = r\varphi(m)$ for all $r \in R$ and $m \in M$. It is easy to check that R-modules and their homomorphisms constitute a category.

If M is an R-module, a subset $M' \subset M$ is a *submodule* if it is a subgroup of the underlying abelian group and $rM' \subset M'$ for all $r \in R$. In this circumstance the *quotient module* is the quotient group M/M' endowed with the scalar multiplication $r(m + M') = rm + M'$; it is straightforward to check that M/M' is an R-module.

We note two elementary isomorphisms.

Lemma A1.1. *If L is an R-module and M and N are submodules, then*

$$(L/N)/((M + N)/N) \cong L/(M + N).$$

In particular, if $N \subset M$, then $(L/N)/(M/N) \cong L/M$.

Proof. The map $x + N \mapsto x + M + N$ is a surjective homomorphism from L/N to $L/(M + N)$, and its kernel is $(M + N)/N$. $\square$

Lemma A1.2. *If M is an R-module with submodules M_1 and M_2, then*

$$(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2).$$

Proof. The composition $M_2 \to M_1 + M_2 \to (M_1 + M_2)/M_1$ is surjective with kernel $M_1 \cap M_2$. $\square$

A2 Ideals

Evidently R is itself an R-module; its submodules are called *ideals*. For the most part R itself is not regarded as an ideal, but sometimes there arise situations in which whether a submodule is proper is in doubt, in which case we will use the phrase "proper ideal" to describe a submodule that is a proper subset of R.

The verification of the following general fact requires only a bit of thought.

Lemma A2.1. *If $\mathcal{C}$ is a nonempty set of ideals, then $\bigcap_{I \in \mathcal{C}} I$ is an ideal.*

If S is any subset of R, the *ideal generated by* S is the smallest (not necessarily proper) ideal containing S. In view of the result above, this is a well defined concept. Alternatively, the ideal generated by S is the collection of all finite sums $r_1 s_2 + \cdots + r_k s_k$ where $r_1, \dots, r_k \in R$ and $s_1, \dots, s_k \in S$. If $S = \{a_1, \dots, a_k\}$ is finite, the ideal it generates is denoted by $(a_1, \dots, a_k)$. An ideal is *principal* if it is generated by a singleton.

If $I_1, \dots, I_k$ are ideals, then, by definition, $I_1 + \cdots + I_k$ is the ideal

$$\{\, a_1 + \cdots + a_k : a_1 \in I_1, \dots, a_k \in I_k \,\}$$

and $I_1 \cdots I_k$ is the ideal generated by

$$\{\, a_1 \cdots a_k : a_1 \in I_1, \dots, a_k \in I_k \,\}.$$

Evidently $I_1 \cdots I_k \subset I_1 \cap \cdots \cap I_k$.

The *radical* of an ideal I is

$$\operatorname{rad} I = \{\, r \in R : r^m \in I \text{ for some integer } m \geq 1 \,\}.$$

If $r, s \in \operatorname{rad} I$, then $r + s \in \operatorname{rad} I$ because for sufficiently large m, every term in the binomial expansion of $(r+s)^m$ is in I. If $r \in \operatorname{rad} I$ and s is any element of R, then $sr \in \operatorname{rad} I$, obviously. Thus $\operatorname{rad} I$ is an ideal.

The ideal I is:

- *radical* if $\operatorname{rad} I = I$;
- *prime* if $ab \notin I$ whenever $a, b \notin I$;
- *maximal* if it is not a proper subset of another (proper) ideal.

Usually P will denote a prime ideal and $\mathfrak{m}$ will denote a maximal ideal.

The R-module R/I is endowed with a multiplication defined by

$$(r + I)(s + I) = rs + I.$$

It is easily checked that this definition does not depend on the choice of representatives of cosets, and that it makes R/I a commutative ring with unit. It is called the *quotient ring* of I.

Proposition A2.2. *An ideal I is radical if and only if R/I is reduced, it is prime if and only if R/I is an integral domain, and it is maximal if and only if R/I is a field.*

Proof. The assertions for radical and prime ideals are immediate consequences of the definitions. Suppose I is maximal. If $a + I$ was a nonzero element of R/I that was not a unit, $(a) + I$ would be an ideal that had I as a proper subset, and that was proper because it did not contain 1, which is impossible. Conversely, if every element of R/I is a unit, then $(a) + I = R$ for every $a \in R \setminus I$, so I is maximal. □

Corollary A2.3. *A prime ideal is radical, and a maximal ideal is prime.*

The next result, with $S = \{1\}$, implies that every ideal is contained in a maximal ideal.

Proposition A2.4. *If I is an ideal, $S \subset R$ and $I \cap S = \emptyset$, then the set $\mathcal{I}$ of ideals that contain I and have empty intersection with S has an element that is maximal, in the sense of not being properly contained in another element of $\mathcal{I}$.*

Proof. A *chain* in $\mathcal{I}$ is a subset of $\mathcal{I}$ that is completely ordered by inclusion. The union of the elements of a chain is easily seen to be an ideal, which of course contains I and has empty intersection with S, so any chain has an upper bound in $\mathcal{I}$. Therefore Zorn's lemma implies that $\mathcal{I}$ has a maximal element. □

If $S \subset R$ and P is a prime that contains S, we say that P is *minimal over* S if there is no prime that contains S and is properly contained in P.

Proposition A2.5. *If a set $S \subset R$ is contained in a prime ideal, then there is a prime ideal that is minimal over S.*

Proof. Let $\mathcal{P}$ be the set of prime ideals that contain S. By hypothesis $\mathcal{P}$ is nonempty. Let P be the intersection of the elements of a chain in $\mathcal{P}$. Of course $S \subset P$. If r and s are ring elements that are not in P, then each of them is outside of some element of the chain, and since the chain is completely ordered, there is an element of the chain that contains neither of them, so their product is also outside this element, and thus outside P. Therefore P is prime. We have shown that any chain in $\mathcal{P}$ has a lower bound in $\mathcal{P}$, so Zorn's lemma implies that $\mathcal{P}$ has a minimal element. □

Zorn's lemma is equivalent to the axiom of choice, so we see from these results that commutative algebra cannot get off the ground outside of a version of set theory that includes that axiom.

A *multiplicatively closed subset of* R is a set $S \subset R$ that contains 1 and all products st of elements $s, t \in S$, but does not contain 0. Important examples include $\{1\}$, more generally $\{1, r, r^2, \ldots\}$ for a nonnilpotent r, the group of units, and $R \setminus P$, where P is a prime.

Proposition A2.6. *If S is multiplicatively closed and I is an ideal that is maximal among those that do not meet S, then I is prime.*

Proof. Suppose, on the contrary, that $a, b \notin I$ and $ab \in I$. Then $I + (a)$ meets S, so there are $i \in I$ and $p \in R$ with $i + pa \in S$, and similarly there are $j \in I$ and $q \in R$ such that $j + qb \in S$. Then $(i + pa)(j + qb) \in S \cap I$, contrary to hypothesis. □

An R-*algebra* is a commutative ring with unit S that is an R-module. If this is the case, then there is an ring homomorphism $\phi : R \to S$ taking r to $r \cdot 1$ where $1 \in S$ is the multiplicative identity element. Conversely, if $\phi : R \to S$ is a ring homomorphism, then the scalar multiplication $(r, s) \mapsto \phi(r)s$ makes S into an R-module. If $I \subset R$ is an ideal, then the ideal generated by $\phi(I)$ is the *extension* of I, and if $J \subset S$ is an ideal, the ideal $\phi^{-1}(J)$ is the *contraction* of J. (This terminology easily becomes ambiguous in complex settings, and we will usually not use it, but the reader should know it.)

Suppose that I is an ideal of R and J is an ideal of S. The definitions immediately imply that I is a subset of the contraction of its extension, and that J contains the extension of its contraction. If I is the contraction of J, then the contraction of its extension is a subset of I because it is the contraction of the extension of the contraction of J. Similarly, if J is the extension of I, then it a superset of the extension of its contraction. Thus:

Lemma A2.7. *An ideal of R is the contraction of an ideal of S if and only if it is the contraction of its extension, and an ideal of S is the extension of an ideal of R if and only if it is the extension of its contraction.*

If J is prime, then its contraction is prime, obviously.

Proposition A2.8. *If a prime P of R is the contraction of an ideal of S, then the set of ideals of S that contract to P has a maximal element, and any such maximal element is prime.*

Proof. Let $U = R \setminus P$. The extension of P contracts to P, so it does not meet $\phi(U)$, and Proposition A2.4 gives a maximal ideal Q among those that contain $\phi(P)$ and do not meet $\phi(U)$. Since P is prime, U is a multiplicative subset of R, so $\phi(U)$ contains $\phi(1) = 1$ and all products of its elements. Also, $0 \notin \phi(U)$ because the complement of $\phi(U)$ contains an ideal. Thus $\phi(U)$ is a multiplicative subset of S, so Q is prime by Proposition A2.6. $\square$

The *nilradical* of R is $\operatorname{rad}(0)$. In view of the following result it is the intersection of the minimal prime ideals.

Corollary A2.9. *For any ideal I,* $\operatorname{rad} I$ *is the intersection of the primes that are minimal over I.*

Proof. It is enough to show that $\operatorname{rad} I$ is the intersection of the primes that contain I, because Proposition A2.5 implies that each such prime contains a minimal such prime. Obviously any prime that contains I must contain $\operatorname{rad} I$. On the other hand, if r is not in the radical of I, then Proposition A2.4 and the last result give a prime ideal P that contains I and is maximal among those that do not meet $\{1, r, r^2, \ldots\}$. $\square$

We can also say something now about the set of zerodivisors. There will be a more precise characterization later, in connection with the study of associated primes.

Lemma A2.10. *If R is reduced, then every zerodivisor is contained in a minimal prime.*

Proof. Suppose x is a zero divisor, say $xy = 0$ and $y \neq 0$. Since R is reduced, its nilradical is (0), so the last result implies that y is outside some minimal prime P. Since $xy \in P$, x must be an element of P. $\square$

The intersection of all of the maximal ideals of R is an important ideal called the *Jacobson radical* of R. It has the following concrete description.

Proposition A2.11. *The Jacobson radical of R is the set of $x \in R$ such that for all $y \in R$, $1 - xy$ is a unit.*

Proposition A2.12. *The Jacobson radical of R is the set of $x \in R$ such that for all $y \in R$, $1 - xy$ is a unit.*

Proof. Suppose that x is an element of the Jacobson radical. If, for some y, $1 - xy$ was not a unit, then $(1 - xy)$ would be a proper ideal, and would be contained in some maximal ideal $\mathfrak{m}$. Since x is in the Jacobson radical, $x \in \mathfrak{m}$ and thus $1 \in \mathfrak{m}$, which is impossible.

If x is not in the Jacobson radical, then it is outside some maximal ideal $\mathfrak{m}$. Maximality implies that $\mathfrak{m} + (x) = R$, so $r + xy = 1$ for some $r \in \mathfrak{m}$ and $y \in R$, and $1 - xy$ is not a unit because it is an element of $\mathfrak{m}$. $\square$

The ring R is *local* if it has a unique maximal ideal $\mathfrak{m}$. If this is the case, $k = R/\mathfrak{m}$ is called the *residue field* of R. Local rings are extremely important in algebraic geometry, and will be prominent in our work. It is annoying to have to reintroduce the maximal ideal and the residue field whenever one works with a local ring. Many authors deal with this by defining a local ring to be a triple $(R, \mathfrak{m}, k)$, but this is cumbersome in its own way, and would clash with our approach in which R is simply present at all times. Instead we adopt the convention that *whenever R is local, it is automatically the case (i.e., it goes with saying) that $\mathfrak{m}$ is its maximal ideal and k is its residue field.* The reader should be warned that $\mathfrak{m}$ will often denote a maximal ideal when R is or may not be local.

Since the Jacobson radical of a local ring is the unique maximal ideal, the following result can easily be applied, and in fact it is invoked quite frequently.

Theorem A2.13 (Nakayama's Lemma)**.** *Suppose M is a finitely generated R-module and I is an ideal contained in the Jacobson radical of R. If $IM = M$, then $M = 0$. If the images of $x_1, \ldots, x_n \in M$ in M/IM generate it as an R-module, then $x_1, \ldots, x_n$ generate M.*

Proof. Suppose that M is generated by $x_1, \dots, x_n$, but not by fewer generators. Then $Ix_1, \dots, Ix_n$ also generate M, so $x_n = a_1x_1 + \cdots + a_nx_n$ for some $a_1, \dots, a_n \in I$. Since I is contained in the Jacobson radical, $1 - a_n$ is a unit, and if $b(1-a_n) = 1$, then $x_n = ba_1x_1 + \cdots ba_{n-1}x_{n-1}$, which contradicts minimality unless $n = 1$, and even when $n = 1$ we have $(1 - a_1)x_1 = 0$ and thus $x_1 = 0$, i.e., $M = 0$.

For the second assertion let $M' = \sum_i Rx_i$ and $N = M/M'$. By hypothesis $M' + IM = M$, so Lemma A1.1 gives

$$N/IN = (M/M')/(IM + M'/M') = M/(IM + M') = M/M = 0.$$

Now the first assertion implies that $N = 0$, so $M' = M$. □

Corollary A2.14. *If I is an ideal contained in the Jacobson radical, M is a finitely generated R-module, and N is a submodule of M such that $IM+N = M$, then $N = M$.*

Proof. Since $I(M/N) = (IM+N)/N = M/N$, the claim follows from Nakayama's lemma applied to M/N. □

Suppose R is local and M is a finitely generated R-module. Then $M/\mathfrak{m}M$ is annihilated by $\mathfrak{m}$, so it may be regarded as a vector space over k. A particularly important example is $\mathfrak{m}/\mathfrak{m}^2$. Via the following result, one can sometimes use facts of linear algebra to prove that a collection of elements is a system of generators for a module.

Proposition A2.15. *Suppose R is local and M is an R-module.*

(a) If the images of $x_1, \dots, x_n \in M$ in $M/\mathfrak{m}M$ are a basis of this vector space, then $x_1, \dots, x_n$ is a system of generators for M.

(b) If $x_1, \dots, x_n$ is a minimal system of generators of M, then their images $\tilde{x}_1, \dots, \tilde{x}_n$ are a basis of $M/\mathfrak{m}M$.

Proof. (a) Let N be the submodule generated by $x_1, \dots, x_n$. The composition $N \to M \to M/\mathfrak{m}M$ maps N onto $M/\mathfrak{m}M$, so $N + \mathfrak{m}M = M$, and the last result gives $N = M$.

(b) Since $\tilde{x}_1, \dots, \tilde{x}_n$ span $M/\mathfrak{m}$, if they were not a basis there would be some proper subset that was a basis, and by (a) their preimages would be a system of generators of M, contrary to minimality. □

A3 The Cayley-Hamilton Theorem

Most people learn Cramer's rule, at a very early stage of their education, as a formula for the inverse of a nonsingular matrix. In a ring one cannot always divide by the determinant, but the determinant of a matrix is still defined, and the principle underlying Cramer's rule is valid and important.

Let M be an R-module, and let $\varphi : M \to M$ be a homomorphism. There is an R-algebra whose elements are the endomorphisms of M that are polynomial functions of $\mathbf{1}_M$ and φ, with coefficients in R. In this ring the product of two endomorphisms is defined to be their composition.

Theorem A3.1 (Cayley-Hamilton Theorem)**.** *Suppose M is generated by $m_1, \ldots, m_n$, I is a (possibly improper) ideal of R, and $\varphi(M) \subset IM$. Then there is a monic polynomial $p(x) = x^n + p_1 x^{n-1} + \cdots + p_n$ with $p_j \in I^j$ for all j such that $p(\varphi) = 0$.*

Proof. The hypotheses imply that for each i there are $a_{i1}, \ldots, a_{in} \in I$ such that $\varphi(m_i) = \sum_j a_{ij} m_j$. We will work in the space of $n \times n$ matrices whose entries are elements of the ring of endomorphisms generated by $\mathbf{1}_M$ and φ. Let $C = (c_{ij})$ be the matrix

$$C = \begin{pmatrix} a_{11}\mathbf{1}_M - \varphi & \cdots & a_{1n}\mathbf{1}_M \\ \vdots & \ddots & \vdots \\ a_{n1}\mathbf{1}_M & \cdots & a_{nn}\mathbf{1}_M - \varphi \end{pmatrix}.$$

Evidently $Cm = 0$ where $m = (m_1, \ldots, m_n) \in M^n$.

Let $p(\varphi) = \det(C)$. The formula for the determinant as a sum over permutations implies that the coefficients of p lie in the asserted ideals. The *adjugate* or *classical adjoint* of C is the $n \times n$ matrix D with entries

$$d_{ij} = \sum_{\sigma \in S_n, \sigma(i) = j} \operatorname{sgn}(\sigma) c_{\sigma(1)1} \cdots c_{\sigma(i-1)i-1} \cdot \mathbf{1}_M \cdot c_{\sigma(i+1)i+1} \cdots c_{\sigma(n)n}.$$

Cramer's rule boils down to the formula $DC = \det(C)I$ where I is the diagonal matrix whose diagonal entries are all $\mathbf{1}_M$. (The proof of Cramer's rule is left as an exercise because it is just a straightforward, bulky, and uninformative computation that is valid when the entries are from any commutative ring.) Therefore $0 = DCm = p(\varphi)m$. That is, $p(\varphi)m_j = 0$ for all j, so $p(\varphi) = 0$ because $m_1, \ldots, m_n$ generate M. □

Corollary A3.2. *If M is a finitely generated R-module, I is an ideal of R, and $IM = M$, then there is an $a \in I$ such that $am = m$ for all $m \in M$.*

Proof. If we take $\varphi = \mathbf{1}_M$ in the last result we arrive at the formula

$$(1 + p_1 + \cdots + p_n)\mathbf{1}_M = 0,$$

where $p_1, \ldots, p_n \in I$, so we can set $a = -(p_1 + \cdots + p_n)$. □

Corollary A3.3. *If M is a finitely generated R-module and $f : M \to M$ is a surjective homomorphism, then f is an isomorphism.*

Proof. In the obvious way we regard M as an $R[f]$-module. By assumption $(f)M = M$, so the last result implies that $\mathbf{1}_M \in (f)$, which is to say that there is a $g \in R[f]$ such that $gf = \mathbf{1}_M$. □

Corollary A3.4. *If $x_1, \ldots, x_n \in R^n$ generate R^n, then they generate R^n freely.*

Proof. The map $(r_1, \ldots, r_n) \mapsto r_1x_1 + \cdots + r_nx_n$ is surjective by assumption, so it is also injective. □

When R is local we can say a bit more.

Corollary A3.5. *If R is local, M is a finitely generated R-module, $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are minimal systems of generators of M, $x_i = \sum_j a_{ij}y_j$, and A is the matrix with entries a_{ij}, then the determinant of A is a unit in R, so A is invertible.*

Proof. Let $\tilde{x}_i$ and $\tilde{y}_j$ be the images of x_i and y_j in $M/\mathfrak{m}M$, let $\tilde{a}_{ij}$ be the image of a_{ij} in k, and let $\tilde{A}$ be the matrix with entries $\tilde{a}_{ij}$. Then $\tilde{x}_1, \ldots, \tilde{x}_n$ and $\tilde{y}_1, \ldots, \tilde{y}_n$ are bases of $M/\mathfrak{m}M$ (Proposition A2.15) so the determinant of $\tilde{A}$ is nonzero, and is the image in k of the determinant of A, which is consequently a unit because it is not an element of $\mathfrak{m}$. Now Cramer's rule computes the inverse of A. □

A4 Noetherian and Artinian Rings and Modules

An R-module M is *Noetherian* if it satisfies the *ascending chain condition*: every increasing sequence of submodules $M_0 \subset M_1 \subset M_2 \subset \cdots$ is eventually constant. The terminology honors Emmy Noether, who first demonstrated that this condition could be used to simplify and generalize material that had previously been treated using what is now known as "elimination theory."

In addition to the definition, there are two other formulations of the condition that are applied frequently.

Lemma A4.1. *For an R-module M the following are equivalent:*

(a) M is Noetherian.

(b) Every nonempty set of submodules of M has a maximal element.

(c) Every submodule of M is finitely generated.

Proof. If (b) fails one can easily construct an infinite ascending chain, so (a) implies (b). If (b) holds and N is a submodule, then the set of finitely generated submodules of N has a maximal element, which clearly must be N itself (otherwise we could expand it by adding one more generator) so (b) implies (c). If (c) holds and $M_1 \subset M_2 \subset \cdots$ is an increasing sequence of submodules, then $\bigcup_i M_i$ is a submodule, which is finitely generated, so it must be some M_n. Thus (c) implies (a). □

The module M is *Artinian* if it satisfies the descending chain condition: every descending sequence of ideals $M_0 \supset M_1 \supset M_2 \supset \cdots$ is eventually constant. This piece of terminology honors Emil Artin. For Artinian modules there is a slightly simpler result, whose proof should be obvious.

Lemma A4.2. *For an R-module M the following are equivalent:*

(a) M is Artinian.

(b) Every nonempty set of submodules of M has a minimal element.

We say that R itself is Noetherian or Artinian if it is a Noetherian or Artinian R-module, which means that its ideals satisfy the ascending or descending chain condition. Eventually we will see that Artinian rings and modules are much more special, and much less important, than Noetherian rings and modules, but for the time being it is logically efficient to treat them in parallel.

A composition $L \xrightarrow{f} M \xrightarrow{g} N$ of R-module homomorphisms is *exact at* M if the image of f is the kernel of g. A *short exact sequence* is a composition

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

that is exact at L, M, and N. That is, in addition to being exact at M, f is injective and g is surjective. Whenever $g : M \to N$ is surjective there is a short exact sequence $0 \to \operatorname{Ker}(g) \to M \to N \to 0$. Whenever $f : L \to M$ is injective (in particular, if L is a submodule of M) there is a short exact sequence $0 \to L \to M \to M/\operatorname{Im}(f) \to 0$. When $0 \to L \to M \to N \to 0$ is exact we will often identify L with its image in M.

Lemma A4.3. *If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence of R-modules, M' is a submodule of M, $f(L) \subset M'$, and $g(M') = N$, then $M' = M$.*

Proof. We regard L as a submodule of M and identify N with M/L. Then $M/L = M'/L$, so any $m \in M$ is $m' + \ell$ for some $m' \in M'$ and $\ell \in L$, and consequently $m \in M' + L = M'$. Thus $M = M'$. □

Proposition A4.4. *If $0 \to L \to M \to N \to 0$ is a short exact sequence of R-modules, then M is Noetherian (Artinian) if and only if L and N are both Noetherian (Artinian).*

Proof. An ascending sequence in L maps injectively into an ascending sequence in M. The preimages of the modules in a strictly ascending sequence in N are a strictly ascending sequence in M. Thus if M is Noetherian, then so are L and N.

Suppose that L and N are Noetherian. Let $M_0 \subset M_1 \subset M_2 \subset \cdots$ be an ascending sequence of submodules of M. For each j let L_j be the preimage of M_j in L, and let N_j be the image of M_j in N; clearly $0 \to L_j \to M_j \to N_j \to 0$ is a short exact sequence. Since the sequences $L_0 \subset L_1 \subset L_2 \subset \cdots$ and $N_0 \subset N_1 \subset N_2 \subset \cdots$ are eventually constant, Lemma A4.3 implies that $M_0 \subset M_1 \subset M_2 \subset \cdots$ is eventually constant.

The proof for Artinian modules is the same except that there are descending sequences of modules. □

Corollary A4.5. *If $M_1, \ldots, M_n$ are Noetherian (Artinian) R-modules, then $M_1 \oplus \cdots \oplus M_n$ is Noetherian (Artinian).*

Proof. This follows by induction on n, with the induction step being the application of the last result to $0 \to M_n \to \oplus_{i=1}^n M_i \to \oplus_{i=1}^{n-1} M_i \to 0$. □

Proposition A4.6. *If R is Noetherian (Artinian) and M is a finitely generated R-module, then M is Noetherian (Artinian).*

Proof. There is an exact sequence $0 \to K \to R^n \to M \to 0$. By the last result R^n is Noetherian (Artinian) so the preceeding result implies that M is Noetherian (Artinian). □

An R-module M is *finitely presented* if there is an exact sequence

$$R^q \to R^p \to M \to 0$$

for some integers p and q. In view of the importance of this condition in sheaf theory, one might expect it to be quite prominent. However, its explicit appearances are infrequent because in the most important settings it is a consequence of finite generation:

Proposition A4.7. *If R is Noetherian and M is an R-module, then the following are equivalent:*

(a) M is finitely generated;

(b) M is finitely presented;

(c) M is Noetherian.

Proof. The last result and Lemma A4.1 imply that (a) and (c) and equivalent, and of course (b) implies (a). If M is finitely generated there is an exact sequence $0 \to K \to R^p \to M \to 0$, and R^p and K are Noetherian by virtue of the lemmas above. In particular, K is finitely generated, so there is a surjection $R^q \to K$ whose composition with $K \to R^p$ achieves finite presentation. □

A submodule M' of an R-module M is *irreducible* if it is not the intersection of two submodules which each contain it strictly.

Lemma A4.8. *If M is a Noetherian R-module, then each of its submodules is a finite intersection of irreducible submodules.*

Proof. If the set of submodules that are not finite intersections of irreducible submodules is nonempty, then it has a maximal element M'. Since M' is not itself irreducible, it is the intersection of two submodules that each contain it strictly. Each of these is a finite intersection of irreducible submodules, so M' is also such an intersection, but of course this is a contradiction. □

The representation of M' as an intersection of irreducible modules need not be unique. For example, a linear subspace of a finite dimensional vector space can be the intersection of codimension one subspaces in many ways. For ideals one can say somewhat more.

Lemma A4.9. *A prime ideal P is irreducible.*

Proof. If $P = I \cap J$, where I and J are distinct ideals that contain P strictly, then for any $a \in I \setminus P$ and $b \in J \setminus P$ we have $ab \in P$, contradicting primality. □

The following basic fact was first pointed out by Emmy Noether.

Proposition A4.10. *If R is Noetherian, any ideal of R has finitely many primes that are minimal over it.*

Proof. If the set of ideals with infinitely many minimal primes is nonempty, there is a maximal element I. Of course I cannot be prime, so there are $a, b \in R \setminus I$ with $ab \in I$. Any prime that contains I contains ab, so it must contain either a or b because it is prime. Therefore a minimal prime over I is minimal over either $I + (a)$ or $I + (b)$, and by construction there are finitely many such primes. □

We now have the following refinement of Corollary A2.9.

Proposition A4.11. *If R is Noetherian, then a radical ideal I is the intersection of the primes that are minimal over it, which are finite in number, and this is the unique representation of I as a minimal (in the sense that no ideal can be omitted) intersection of prime ideals.*

Proof. Let $P_1, \ldots, P_k$ be the primes that are minimal over I. Corollary A2.9 implies that $\bigcap_i P_i = I$. Let $I = Q_1 \cap \cdots \cap Q_\ell$ be a representation of I as an intersection of prime ideals. If some P_i is not among the Q_j, then, because P_i is minimal over I, for each j there is some $r_j \in Q_j \setminus P_i$, and $r_1 \cdots r_\ell \in I \subset P_i$, contradicting the primality of P_i. □

Clearly a field is both Noetherian and Artinian. The ring $K[X_1,\dots,X_n]$ of polynomials in the variables $X_1,\dots,X_n$ with coefficients in a field K is Noetherian by virtue of repeated applications of the following famous result. In general, if $f = a_dX^d + \cdots + a_1X + a_0 \in R[X]$ is a univariate polynomial with $a_d \neq 0$, the *degree* indexdegree of a polynomial of f is d and the *leading coefficient* of f is a_d. (The leading coefficient of $0 \in R[X]$ is zero, and a common convention is that its degree is -1, but we will not rely on that.)

Theorem A4.12 (Hilbert Basis Theorem)**.** *If R is Noetherian, then the polynomial ring $R[X]$ is Noetherian.*

Proof. Let I be an ideal in $R[X]$, and let J be the set of leading coefficients of elements of I. It is easy to see that J is an ideal, and that for each $d = 0, 1, 2, \dots$ the set J_d of leading coefficients of elements of I of degree $\leq d$ is also an ideal. Since R is Noetherian, each J_d is generated by the leading coefficients of finitely many elements of I of degree $\leq d$, say $f_{d1},\dots,f_{dn_d}$. Multiplying by a power of X, we may arrange for the degree of each f_{di} to be exactly d. Since $J_0 \subset J_1 \subset J_2 \subset \cdots$ and $\bigcup_d J_d = J$, there is some $\overline{d}$ such that $J_{\overline{d}} = J$.

We claim that I is the ideal I' generated by $\{f_{di}\}_{0\leq d\leq\overline{d},\,1\leq i\leq n_d}$. Aiming at a contradiction, let g be an element of $I\setminus I'$ of least degree d_0. For $d = \min\{d_0, \overline{d}\}$ there are $c_1,\dots,c_{n_d} \in R$ such that the leading coefficient of $c_1f_{d1}+\cdots+c_{n_d}f_{dn_d}$ is the same as the leading coefficient of g. Therefore

$$g - X^{d_0-d}(c_1f_{d1} + \cdots + c_{n_d}f_{dn_d})$$

is an element of $I \setminus I'$ of degree less than d_0, contradicting the choice of g. ☐

A5 Localization

Localization is a method of defining rings of fractions. In the most common applications in algebraic geometry these are rings of germs of rational functions, each of which is defined in some neighborhood of a given set.

Let S be a multiplicatively closed subset of R. (Recall that this means that $1 \in S \subset R$ and $st \in S$ for all $s, t, \in S$.) If M is an R-module, $S^{-1}M$ is the set of equivalence class of symbols of the form m/s where $m \in M$, $s \in S$, and m/s and n/t are equivalent if there is a $u \in S$ such that $u(tm - sn) = 0$. Addition and multiplication by elements in R are defined by the formulas

$$\frac{m}{s} + \frac{n}{t} = \frac{tm + sn}{st} \quad \text{and} \quad a \cdot \frac{m}{s} = \frac{am}{s}.$$

These definitions do not depend on the choice of representatives: if $\frac{m'}{s'} = \frac{m}{s}$, so that $u(m's - ms') = 0$ for some $u \in S$, then $\frac{tm'+s'n}{s't} = \frac{tm+sn}{st}$ and $\frac{am'}{s'} = \frac{am}{s}$ because

$$u\big(st(tm' + s'n) - s't(tm - sn)\big) = t^2u(sm' - s'm) = 0$$

and

$$u(sam' - s'am) = au(sm' - s'm) = 0.$$

Thus $S^{-1}M$ is an R-module.

Multiplication in $S^{-1}R$ is defined by the formula

$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

Again, if $u(sa' - s'a) = 0$, then $\frac{a'b}{s't} = \frac{ab}{st}$ because

$$u(st(a'b) - s't(ab)) = tbu(sa' - s'a) = 0.$$

Evidently $S^{-1}R$ is a commutative ring with unit. With the scalar multiplication

$$\frac{a}{s} \cdot \frac{m}{t} = \frac{am}{st},$$

$S^{-1}M$ is also an $S^{-1}R$-module. (The proof that this multiplication is independent of the choice of representatives is left as an exercise.) In fact the ring homomorphism $r \mapsto \frac{r}{1}$ makes $S^{-1}R$ into an R-algebra.

If $f : M \to N$ is an R-module homomorphism, then there is a $S^{-1}R$-module homomorphism $S^{-1}f : S^{-1}M \to S^{-1}N$ given by

$$S^{-1}f(m/s) = f(m)/s.$$

If $g : N \to P$ is a second homomorphism, then $S^{-1}(g \circ f) = S^{-1}g \circ S^{-1}f$, obviously, so the formation of fractions with denominators in S is a functor.

Proposition A5.1. *If* $M \xrightarrow{f} N \xrightarrow{g} P$ *is exact, then*

$$S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N \xrightarrow{S^{-1}g} S^{-1}P$$

is exact.

Proof. This has two parts: (a) $S^{-1}g \circ S^{-1}f = S^{-1}(g \circ f) = 0$; (b) if $S^{-1}g(n/t) = g(n)/t = 0$, then there is $u \in S$ such that $0 = ug(n) = g(un)$, so $un = f(m)$ for some $m \in M$, and $S^{-1}f(m/tu) = un/tu = n/t$. □

If Q is a second ring, a functor F from the category of R-modules to the category of Q-modules is *exact* if

$$F(L) \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(N)$$

is exact whenever $L \xrightarrow{f} M \xrightarrow{g} N$ is exact. Dressed up as abstract nonsense, Proposition A5.1 asserts that $M \mapsto S^{-1}M$ is an exact functor from R-modules to $S^{-1}R$-modules.

Corollary A5.2. *If M is a submodule of N, then $S^{-1}N/S^{-1}M$ and $S^{-1}(N/M)$ are isomorphic.*

Proof. Apply the last result to $0 \to M \to N \to N/M \to 0$. $\square$

A similar situation arises in algebraic geometry. One may pass from a ring of germs of continuous functions at a point to the restrictions of the germs to a subdomain containing that point, then form the quotient ring with the restricted germs that do not vanish at that point as the denominators. Alternatively, one may first form the ring of quotients of germs, then pass to the restrictions of the quotients to the subdomain. For actual functions the two procedures obviously give the same result, but when we are dealing with ring elements that are thought of as "representing" functions this becomes an isomorphism. Suppose that I is an ideal that does not meet S. Let $S/I = \{ s+I : s \in S \} \subset R/I$. This is a multiplicative subset of R/I.

Proposition A5.3. $(S/I)^{-1}(R/I) \cong S^{-1}R/S^{-1}I$.

Proof. We first show that the map $\frac{r+I}{s+I} \mapsto \frac{r}{s} + S^{-1}I$ is well defined. We have $\frac{r+I}{s+I} = \frac{r'+I}{s'+I}$ if and only if there is a $t+I \in S/I$ such that

$$(t+I)\big((r+I)(s'+I) - (r'+I)(s+I)\big) = 0,$$

which boils down to there being a $t \in S$ such that $t(rs' - r's) \in I$. On the other hand $\frac{r}{s} + S^{-1}I = \frac{r'}{s'} + S^{-1}I$ if and only if there are $i \in I$ and $t \in S$ such that $\frac{r}{s} - \frac{r'}{s'} = \frac{i}{t}$. If this is the case, then $t(rs' - r's) \in I$, and if there is a $t \in S$ such that $t(rs' - r's) \in I$, then $\frac{r}{s} - \frac{r'}{s'} = \frac{t(rs'-r's)}{tss'} \in S^{-1}I$.

Specializing this reasoning, we see that $\frac{r+I}{s+I} = \frac{0+I}{1+I}$ if and only if there is a $t \in S$ such that $tr \in I$, which is the case if and only if $\frac{r}{s} + S^{-1}I = \frac{0}{1} + S^{-1}I$. Thus the map is injective, and it is obviously surjective. $\square$

The functor S^{-1} preserves images and kernels. Proposition A5.5 below may be understood as asserting that "localization commutes with homology."

Lemma A5.4. *If $\varphi : M \to N$ is an R-module homomorphism, then* $\text{Im } S^{-1}\varphi = S^{-1}(\text{Im } \varphi)$ *and* $\text{Ker } S^{-1}\varphi = S^{-1}(\text{Ker } \varphi)$.

Proof. That $\text{Im } S^{-1}\varphi = S^{-1}(\text{Im } \varphi)$ follows immediately from the definition of $S^{-1}\varphi$, and the containment $S^{-1}(\text{Ker } \varphi) \subset \text{Ker } S^{-1}\varphi$ is also immediate. If $m/r \in \text{Ker } S^{-1}\varphi$, then $\varphi(m)/r = 0 \in S^{-1}N$, so there is some $s \in S$ such that $s\varphi(m) = 0$, but this implies that $\varphi(sm) = 0$, so that $m/r = sm/sr \in S^{-1}(\text{Ker } \varphi)$. Therefore $\text{Ker } S^{-1}\varphi \subset S^{-1}(\text{Ker } \varphi)$. $\square$

Proposition A5.5. *If $M \xrightarrow{f} N \xrightarrow{g} P$ is a composition of homomorphisms with $g \circ f = 0$, then* $\text{Ker } S^{-1}g/\text{Im } S^{-1}f = S^{-1}(\text{Ker } g/\text{Im } f)$.

Proof. The exactness of localization implies that

$$0 \to S^{-1}\mathrm{Im}\ f \to S^{-1}\mathrm{Ker}\ g \to S^{-1}(\mathrm{Ker}\ g/\mathrm{Im}\ f) \to 0$$

is exact. Combining this with the last result gives

$$S^{-1}(\mathrm{Ker}\ g/\mathrm{Im}\ f) \cong S^{-1}\mathrm{Ker}\ g/S^{-1}\mathrm{Im}\ f \cong \mathrm{Ker}\ S^{-1}g/\mathrm{Im}\ S^{-1}f.$$

□

We now study the ideals of $S^{-1}R$. If I is an ideal of R and $r \in R$, let

$$(I : r) = \{ a \in R : ar \in I \}.$$

Evidently $(I : r)$ is an ideal that contains I. If $a \in (I : s)$ and $b \in (I : t)$, then $a + b \in (I : st)$, so $\bigcup_{s \in S}(I : s)$ is an ideal. Note that I is prime precisely when $\bigcup_{r \in R}(I : r) = I$.

The next result gives a bijection between the ideals of $S^{-1}R$ and the ideals $I \subset R$ that do not meet S and satisfy $I = \bigcup_{s \in S}(I : s)$, i.e., $(I : s) = I$ for all $s \in S$. This bijection restricts to a bijection between the respective prime ideals.

Proposition A5.6. *Let $\varphi : r \mapsto r/1$ be the natural map from R to $S^{-1}R$.*

(a) If I is an ideal of R that does not intersect S, then $S^{-1}I$ is an ideal of $S^{-1}R$.

(b) If I is an ideal of R, then $\varphi^{-1}(S^{-1}I) = \bigcup_{s \in S}(I : s)$.

(c) If J is an ideal of $S^{-1}R$, then $\varphi^{-1}(J)$ is an ideal of R that does not intersect S, and $S^{-1}\varphi^{-1}(J) = J$.

(d) If P is a prime of R that does not intersect S, then $S^{-1}P$ is prime, and $\varphi^{-1}(S^{-1}P) = P$.

(e) If Q is a prime of $S^{-1}R$, then $\varphi^{-1}(Q)$ is prime.

Proof. (a) Using the definitions of addition and multiplication in $S^{-1}R$, it is simple to check that $S^{-1}I$ is an ideal. It is a proper ideal because it cannot contain $1/1$: if $ir/s = 1/1$, then $t(ir - s) = 0$ for some $t \in S$, so that $tir = st \in I \cap S$.

(b) If $a \in \varphi^{-1}(S^{-1}I)$, then $\varphi(a) = i/s$ for some $i \in I$ and $s \in S$, which means that $t(as - i) = 0$ for some $t \in S$, so $a \in (I : st)$. On the other hand, if $a \in (I : s)$, then $\varphi(a) = as/s \in S^{-1}I$.

(c) Of course $\varphi^{-1}(J)$ is an ideal. There cannot be an $s \in \varphi^{-1}(J) \cap S$, because then J would contain $\varphi(s) \cdot 1/s = 1/1$. Clearly $S^{-1}\varphi^{-1}(J) \subset J$. For the reverse inclusion observe that if $j/s \in J$, then $j/1 \in J$ so $j \in \varphi^{-1}(J)$ and thus $j/s \in S^{-1}\varphi^{-1}(J)$.

(d) If $a, b \notin P$ and $a/s \cdot b/t = i/u$ for some $s, t, u \in S$ and $i \in P$, then there is $v \in S$ such that $vuab = vsti \in P$, which is impossible. Therefore $S^{-1}P$ is prime. Since P is prime, $(P : s) = P$ for all $s \in S$, so (b) gives $\varphi^{-1}(S^{-1}P) = P$.

(e) If Q is prime, then so is $\varphi^{-1}(Q)$ because the preimage of a prime ideal under a ring homomorphism is always prime. □

In particular, (c) gives an inclusion preserving bijection between the ideals of $S^{-1}R$ and a subset of the set of ideals of R, so:

Corollary A5.7. *If R is Noetherian, then so is $S^{-1}R$.*

One common application of localization is the formation of the ring

$$R[x^{-1}] = \{1, x, x^2, \ldots\}^{-1}R$$

where x is not nilpotent. But by far the most important application of localization occurs when P is a prime ideal of R and $S = R \setminus P$. We write R_P and M_P in place of $(R \setminus P)^{-1}R$ and $(R \setminus P)^{-1}M$. When $\varphi : M \to N$ is an R-module homomorphism we write φ_P in place of $(R \setminus P)^{-1}\varphi$. If R is an integral domain, $R_{(0)}$ is called the *field of fractions* of R, and will sometimes be denoted by $K(R)$. The map $r \mapsto r/1$ allows us to regard R as a subring of $K(R)$.

In view of Proposition A5.6 the ideals of R_P are precisely the ideals $I_P = \{\, i/s : i \in I, s \in R \setminus P \,\}$ where I is an ideal contained in P, and the prime ideals are precisely the Q_P where Q is a prime contained in P. In particular, P_P is the unique maximal ideal, so R_P is a local ring.

From topology we are familiar with a number of important properties that are "local," insofar as they hold in a space if and only if they hold in a neighborhood of each point. There will be a number of such properties of R-modules, of which the following is perhaps the most basic.

Lemma A5.8. *If M is an R-module, $m \in M$, and m goes to zero in each localization $M_{\mathfrak{m}}$ at a maximal ideal, then $m = 0$. Consequently $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$.*

Proof. Suppose $m \neq 0$, and let I be the annihilator of m. Since I does not contain a unit, it is a proper ideal, and is contained in some maximal ideal $\mathfrak{m}$. Since $\mathfrak{m}$ contains I, there is no $a \in R \setminus \mathfrak{m}$ such that $am = 0$, so m does not go to zero in $M_{\mathfrak{m}}$. □

Since localization preserves images and kernels, this implies that:

Lemma A5.9. *An R-module homomorphism $\varphi : M \to N$ is injective (surjective, bijective) if and only if, for each maximal ideal $\mathfrak{m}$, $\varphi_{\mathfrak{m}}$ is injective (surjective, bijective).*

A6 Tensor Products

Let M and N be R-modules. A function $\psi : M \times N \to P$, where P is a third R-module, is *bilinear* if, for all $m \in M$ and $n \in N$, the functions $\psi(m, \cdot) : N \to P$ and $\psi(\cdot, n) : M \to P$ are R-module homomorphisms. We can let $\psi_M = \psi(\cdot, 0)$ and $\psi_N = \psi(0, \cdot)$, or equivalently, we may be given R-module homomorphisms $\psi_M : M \to P$ and $\psi_N : N \to P$, after which we can set $\psi = \psi_M \times \psi_N$. The tensor product is a construction that handles bilinear functions systematically.

Let F be the free R-module with $M \times N$ as its set of generators. As a set, the tensor product $M \otimes_R N$ is the set of equivalence classes of elements of F induced by the transitive closure of the three relations

$$(m_1 + m_2, n) = (m_1, n) + (m_2, n), \quad (m, n_1 + n_2) = (m, n_1) + (m, n_2),$$

$$r(m, n) = (rm, n) = (m, rn).$$

What this means precisely is that $M \otimes_R N = F/E$ where E is the submodule of F generated by all elements of the forms

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n), \quad (m, n_1 + n_2) - (m, n_1) - (m, n_2),$$

$$r(m, n) - (rm, n), \quad r(m, n) - (m, rn).$$

The equivalence class or coset of (m, n) is denoted by $m \otimes n$, or perhaps $m \otimes_R n$ if more than one ring is under consideration.

This is a quite cumbersome and tedious construction, and it turns out that a categorical perspective provides a much simpler method of handling tensor products. Since the concepts are quite important, and they illustrate the nature and use of categorical notions, we are going to address this in a rather leisurely fashion.

To begin with we consider products of sets. Let X and Y be sets, and let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the standard projections. If S is a set and $f_X : S \to X$ and $f_Y : S \to Y$ are functions, then there is a unique function $f : S \to X \times Y$ such that the following diagram commutes.

$$\begin{tikzcd}
S \arrow[dr, "f"] \arrow[drr, "f_Y"] \arrow[ddr, "f_X"'] & & \\
& X \times Y \arrow[r, "p_Y"'] \arrow[d, "p_Y"] & Y \\
& X &
\end{tikzcd}$$

In fact the cartesian product $X \times Y$ is uniquely defined, up to unique isomorphism, by the fact that for any S, f_X, and f_Y there is a unique f making this diagram commute. For any category $\mathcal{C}$, a *product* of objects X and Y is an object $X \times Y$ such that for any object S and morphisms $f_X : S \to X$ and $f_Y : S \to Y$, there is a unique morphism $f : S \to X \times Y$ such that this diagram commutes. For example, the product of R-modules M and N is, obviously, their cartesian product, endowed with its natural R-module structure.

The phrase 'up to unique isomorphism' may seem disturbing, but blurring the distinction between identity of R-modules and canonical isomorphism is actually liberating. In fact such blurring pervades commutative algebra, to such an extent that it very frequently goes unmentioned, and the reader should start to get used to it.

In category theory, whenever the construction of a "whatever" is defined by the ability to complete a particular diagram, the construction of a "cowhatever" is defined by the ability to complete the same diagram with all arrows reversed. Thus an object Z, with morphisms $q_X : X \to Z$ and $q_Y : Y \to Z$, is the *coproduct* of X and Y if, for any object S and morphisms $g_X : X \to S$ and $q_Y : Y \to S$, there is a unique morphism $g : Z \to S$ such that the following diagram commutes.

$$\begin{tikzcd}
S & & \\
 & X \times Y \arrow[ul, "g"'] & Y \arrow[ull, "g_Y"'] \arrow[l, "q_Y"] \\
 & X \arrow[uul, "g_X"] \arrow[u, "q_Y"'] &
\end{tikzcd}$$

For sets, and also for topological spaces, the coproduct of two objects is their disjoint union, which seems pretty uninteresting.

But the coproduct of R-modules M and N is $M \otimes_R N$! There is a map

$$\phi : M \times N \to M \otimes_R N, \quad \phi(m, n) = m \otimes n,$$

which is obviously bilinear. (To make things a bit more compact, we work with a bilinear ϕ rather than homomorphisms $\phi_M : M \to M \otimes_R N$ and $\phi_N : N \to M \otimes_R N$, and similarly for ψ below.) If $\kappa : M \otimes_R N \to P$ is a homomorphism, where P is a third R-module, then $\kappa \circ \phi$ is bilinear. For every R-bilinear $\psi : M \times N \to P$ there is a unique homomorphism $\kappa : M \otimes_R N \to P$ such that $\psi = \kappa \circ \phi$, namely the one satisfying $\kappa(m \otimes n) = \psi(m, n)$. (To show that κ is a well defined R-homomorphism observe that E is in the kernel of the obvious homomorphism $F \to P$.) These properties characterize the tensor product up to unique isomorphism:

Proposition A6.1. *If B is an R-module and $\phi' : M \times N \to B$ is a bilinear function such that, whenever $\psi' : M \times N \to P'$ is bilinear, there is a unique R-module homomorphism $\kappa' : B \to P'$ such that $\kappa' \circ \phi' = \psi'$, then B is canonically isomorphic to $M \otimes_R N$.*

Proof. Setting $P = B$ in the preceeding discussion and $B' = M \otimes_R N$, κ and κ' are inverse isomorphisms. □

Instead of working with the explicit construction of the tensor product, it is usually much easier to work with the characterization given by this result. For example, this approach does much to simplify the presentation of the basic properties of the tensor product:

Lemma A6.2. *The tensor product is commutative, associative, and distributive with respect to direct sums: $(\bigoplus_{i \in I} M_i) \otimes_R N = \bigoplus_{i \in I} M_i \otimes_R N$. In addition R acts as an identity element: $R \otimes_R M = M$.*

Proof. To prove commutativity, consider the map $\phi : (m, n) \mapsto n \otimes m \in N \otimes_R M$. This is evidently bilinear, and if $\psi : M \times N \to P$ is bilinear, then there is a unique homomorphism $\kappa : N \otimes_R M \to P$ such that $\psi = \kappa \circ \phi$, namely the one satisfying $\kappa(n \otimes m) = \psi(m, n)$.

Now let $\phi : (\bigoplus_i M_i) \times N \to \bigoplus_i M_i \otimes N$ be $(\sum_i m_i, n) \mapsto \sum_i m_i \otimes n$. If $\psi : (\bigoplus_I M_i) \times N \to P$ is bilinear, then $\psi = \kappa \circ \phi$ where $\kappa : m_i \otimes n \mapsto \psi(m_i, n)$.

Similarly, the map $\phi : (r, m) \mapsto rm$ is easily seen to have the properties required to verify that $R \otimes_R M = M$.

The proof of associativity follows the same general pattern, but is rather bulky, so it is left to the reader. (It can also be found on pp. 26–27 of Atiyah and McDonald (1969).) □

The method provided by Proposition A6.1 is a bit indirect, so identities that eventually become second nature can be difficult for the beginner to reconfirm mentally when they arise. We now provide a number of such results, and we will cite them rather systematically when they occur.

Lemma A6.3. *If I is an ideal of R, then $M \otimes_R R/I = M/IM$.*

Proof. Let $\phi : M \times R/I \to M/IM$ be the map $\phi(m, r + I) = rm + IM$. For a bilinear $\psi : M \times R/I \to P$ we can let $\kappa : m + IM \mapsto \psi(m, 1 + I)$. □

Let $S \subset R$ be a multiplicatively closed set.

Lemma A6.4. *$S^{-1}M = S^{-1}R \otimes_R M$.*

Proof. Let $\phi : S^{-1}R \times M \to S^{-1}M$ be the map $\phi(\frac{r}{s}, m) = \frac{rm}{s}$. It is easy to check that if $\frac{r'}{s'} = \frac{r}{s}$, then $\frac{r'm}{s'} = \frac{rm}{s}$, so this definition does not depend on the choice of representatives. Clearly ϕ is bilinear. If $\psi : S^{-1}R \times M \to P$ is R-bilinear, then the function $\kappa : S^{-1}M \to P$ given by $\kappa(\frac{m}{s}) = \psi(\frac{1}{s}, m)$ is easily shown to be a well defined R-module homomorphism such that $\psi = \kappa \circ \phi$. □

Localization commutes with tensor products:

Proposition A6.5. $S^{-1}(M \otimes_R N) = S^{-1}M \otimes_{S^{-1}R} S^{-1}N$.

Proof. Let $\phi : S^{-1}M \times S^{-1}N \to S^{-1}(M \otimes_R N)$ be the map $\phi(\frac{m}{s}, \frac{n}{t}) = \frac{m \otimes n}{st}$. To check that this definition does not depend on the choice of representatives, suppose that $\frac{m'}{s'} = \frac{m}{s}$, so that $u(sm' - s'm) = 0$ for some $u \in S$. Then $\frac{m' \otimes n}{s't} = \frac{m \otimes n}{st}$ because

$$u(st(m' \otimes n) - s't(m \otimes n)) = u(sm' - s'm) \otimes tn = 0.$$

Clearly ϕ is bilinear. If $\psi : S^{-1}M \times S^{-1}N \to P$ is bilinear, then the map $\kappa : \frac{m \otimes n}{s} \mapsto \psi(\frac{m}{s}, \frac{n}{1})$ is the unique $S^{-1}R$-module homomorphism such that $\psi = \kappa \circ \phi$. □

Similarly:

Lemma A6.6. *If M is an R-module and N is an $S^{-1}R$-module, then*

$$S^{-1}M \otimes_{S^{-1}R} N = M \otimes_R N.$$

Proof. First, observe that $M \otimes_R N$ is an $S^{-1}R$-module with scalar multiplication $(r/s)(m \otimes n) = m \otimes rn/s$. Let $\phi : S^{-1}M \times N \to M \otimes_R N$ be the map $\phi(m/s, n) = m \otimes n/s$. To check that ϕ is well defined, suppose that $m'/s' = m/s$, so that $u(sm' - s'm) = 0$ for some $u \in S$. Then $m' \otimes n/s' = m \otimes n/s$ because

$$u(s(m' \otimes n) - s'(m \otimes n)) = u(sm' - s'm) \otimes n = 0.$$

Clearly ϕ is $S^{-1}R$-bilinear. If $\psi : S^{-1}M \times N \to P$ is $S^{-1}R$-bilinear, then the map $\kappa : m \otimes n \mapsto \psi(m, n)$ is the unique $S^{-1}R$-module homomorphism such that $\psi = \kappa \circ \phi$. □

If $f : M \to M'$ is a homomorphism, we define

$$f \otimes_R N : M \otimes_R N \to M' \otimes_R N$$

to be the homomorphism taking each $m \otimes n$ to $f(m) \otimes n$. This is well defined because (as is easy to see but tedious to write out) if F' is the free R-module on the set of generators $M' \times N$, then the map $\sum_{i=1}^k r_i(m_i, n_i) \mapsto \sum_{i=1}^k r_i(f(m_i), n_i)$ maps E into the corresponding submodule E' of F'. Evidently

$$(f' \circ f) \otimes_R N = f' \otimes_R N \circ f \otimes_R N$$

whenever f and f' are composable R-module homomorphisms, and $\mathbf{1}_M \otimes_R N = \mathbf{1}_{M \otimes_R N}$, so $- \otimes N$ is a covariant functor from the category of R-modules to itself. Similarly, for a homomorphism $g : N \to N'$ let $M \otimes_R g : M \otimes_R N \to M \otimes_R N'$ be the homomorphism taking each $m \otimes n$ to $m \otimes g(n)$. Again, for any given M, $M \otimes_R -$ is a covariant functor.

Lemma A6.7. *If $f : M \to M'$ is a R-module homomorphism, then*

$$S^{-1}f = \mathbf{1}_{S^{-1}R} \otimes_R f.$$

Proof. The meaning of this assertion is that if $\phi' : S^{-1}R \times M' \to S^{-1}M'$ is the map $\phi'(\frac{r}{s}, m') = \frac{rm'}{s}$, then $\phi' \circ (\mathbf{1}_{S^{-1}R} \times f) = S^{-1}f \circ \phi$, which is easily verified. □

We say that M is *flat* if $M \otimes_R -$ and $- \otimes_R M$ are exact functors. In view of the last result, the diagram considered in Proposition A5.1 can be rewritten as

$$S^{-1}R \otimes_R M \xrightarrow{\mathbf{1}_{S^{-1}R} \otimes f} S^{-1}R \otimes_R N \xrightarrow{\mathbf{1}_{S^{-1}R} \otimes g} S^{-1}R \otimes_R P$$

and the result can be reexpressed as follows:

Proposition A6.8. *The R-module $S^{-1}R$ is flat.*

We will have much more to say about flatness in Chapter B.

There is a tensor product of R-algebras that is closely related to the tensor product of R-modules, but with additional features. Suppose that $\pi_S : R \to S$ and $\pi_T : R \to T$ are ring homomorphisms. The *tensor product* of these R-algebras is $S \otimes_R T$ endowed with the multiplication

$$(s_1 \otimes t_1)(s_2 \otimes t_2) = s_1 s_2 \otimes t_1 t_2.$$

To show that this product is well defined we should verify that if F is the free R-module generated by the elements of $S \times T$ and E is the submodule defined at the beginning of the section, then $\sum_{i,j} r_i r_j (s_i s_j, t_i t_j) \in E$ whenever $\sum_{i=1}^k r_i(s_i, t_i) \in E$ or $\sum_{j=1}^\ell r_j(s_j, t_j) \in E$. This is obvious, even if writing out the details would be quite tedious. Evidently this product is associative, commutative, and distributive, and $1_S \otimes 1_T$ is a multiplicative identity, so $S \otimes_R T$ is a commutative ring with unit.

There are ring homomorphisms $\chi_S : S \to S \otimes_R T$ and $\chi_T : T \to S \otimes_R T$ given by $\chi_S(s) = s \otimes 1_T$ and $\chi_T(t) = 1_S \otimes t$, so $S \otimes_R T$ is both an S-algebra and a T-algebra, and the ring homomorphism $\chi_S \circ \pi_S = \chi_T \circ \pi_T$ makes $S \otimes_R T$ into an R-algebra. If $\psi_S : S \to Z$ and $\psi_T : T \to Z$ are ring homomorphisms, then there is a ring homomorphism $\kappa : S \otimes_R T \to Z$ given by $\kappa(s \otimes t) = \psi_S(s)\psi_T(t)$. (Again, we need to verify that $\sum_i r_i \psi_S(s_i)\psi_T(t_i) = 0$ whenever $\sum_{i=1}^k r_i(s_i, t_i) \in E$, and again this is both obvious on inspection and bulky to write out.) Of course κ is uniquely determined by the requirement that $\psi_S = \kappa \circ \chi_S$ and $\psi_T = \kappa \circ \chi_T$.

Evidently we have defined a coproduct in the category of R-algebras. However, in the categorical perspective that is most useful in algebraic geometry, R is allowed to vary. This leads to the notion of a cofibered product.

We first define fibered products. If W, X, and Y are objects in a category $\mathcal{C}$, and $e_X : X \to W$ and $e_Y : Y \to W$ are morphisms, then an object Z, together with morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$, is a *fibered product* of the data (W, X, Y, e_X, e_Y) if $e_X \circ p_X = e_Y \circ p_Y$ and, for any object S and morphisms $f_X : S \to X$ and $f_Y : S \to Y$ such that $e_X \circ f_X = e_Y \circ f_Y$, there is a unique morphism $f : S \to Z$ such that $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$. For the category of sets the fibered product is given by setting

$$Z = \{\, (x, y) \in X \times Y : e_X(x) = e_Y(y) \,\}$$

and letting p_X and p_Y be the restrictions of the usual projections from $X \times Y$ to X and Y. This construction also gives the fibered product for the category of topological spaces if we endow Z with the relative topology inherited from the product topology of $X \times Y$.

$$\begin{tikzcd}
S & & \\
& Z \arrow[ul, "f"] & Y \arrow[l, "p_Y"'] \arrow[ull, "f_Y"'] \\
& X \arrow[u, "p_X"'] \arrow[uul, "f_X"] & W \arrow[l, "e_X"'] \arrow[u, "e_Y"]
\end{tikzcd}$$

The *cofibered product* is obtained by reversing all arrows. That is, if W, X, and Y are objects in a category $\mathcal{C}$, and $e_X : W \to X$ and $e_Y : W \to Y$ are morphisms, then an object Z, together with morphisms $p_X : X \to Z$ and $p_Y : Y \to Z$, is a *cofibered product* of (W, X, Y, e_X, e_Y) if, for any object S and morphisms $f_X : X \to S$ and $f_Y : Y \to S$ such that $f_X \circ e_X = f_Y \circ e_Y$, there is a unique morphism $f : Z \to S$ such that the diagram above commutes. We now see that the tensor product is a cofibered product for the category of commutative rings with unit.

A7 Principal Ideals, Factorization, and Normality

We now take up a set of concepts related to factorization and the appropriate generalization of the notion of an integer. We will see that if all of a ring's ideals are principal, then the rings elements can be factored uniquely, up to units. We begin with a very specific way in which this can happen that pertains to the integers, and also to the ring $\mathbb{Z}[i]$ of Gaussian integers.

The ring R is *Euclidean* if there is function $|\cdot|$ from R to the nonnegative integers, called the *norm*, such that $|a| = 0$ if and only if $a = 0$ and, for all nonzero $a, b \in R$ there are $q, r \in R$ such that $a = qb + r$ and $|r| < |b|$. The ring R is a *principal ideal domain* if it is an integral domain and every ideal is principal.

Proposition A7.1. *If R is Euclidean, then it is a principal ideal domain.*

Proof. Let I be an ideal. We may assume that $I \neq (0)$, so let b be a nonzero element of I of minimal norm. Any nonzero $a \in I$ is $qb + r$ for some q and r with $|r| < |b|$, and since $r \in I$ it follows that $r = 0$, so that $a \in (b)$. □

The integers $\mathbb{Z}$ are Euclidean; the usual absolute value is the norm. The ring of polyomials in a single variable with coefficients in a field is Euclidean; the norm is the degree. The *Gaussian integers* $\mathbb{Z}[i]$ are Euclidean; the norm is $c + id$ is $c^2 + d^2$. (Proving that this is a norm is not entirely trivial.) Of course these examples are quite prominent. Nevertheless, Euclidean rings are quite special, and in fact principle ideal domains turn out to be rather uncommon.

An element of R is *irreducible* if it cannot be written as a product of two nonunits. A ring element p is *prime* if (p) is a prime ideal. Concretely, p is prime if, whenever p divides a product ab, either p divides a or p divides b.

Lemma A7.2. *If R is an integral domain, then a prime element is irreducible.*

Proof. Suppose that r is prime but not irreducible, so that $r = ab$ where a and b are nonunits. Since (r) is prime, it contains either a or b. If $b \in (r)$, say $b = cr$, then $r = acr$, so $1 = ac$ because R is a domain. But a is not a unit, so this is impossible. □

The ring R is *factorial*, or a *unique factorization domain* (UFD), if it is an integral domain and every nonzero element is uniquely (up to multiplication by units) a product of irreducible elements.

Lemma A7.3. *An irreducible element of a UFD is prime.*

Proof. Let r be irreducible. If there were $a, b \notin (r)$ such that $ab \in (r)$, say $ab = cr$, then representations of a, b, and c as products of irreducibles would give two representations of $ab = cr$ as a product of irreducibles that are distinct because r a factor in one but not the other. □

Lemma A7.4. *If R is an integral domain and every ascending chain of principal ideals stabilizes, then every nonzero element of R is a product of irreducible elements. If, in addition, irreducibles are prime, then R is factorial.*

Proof. Suppose that $0 \neq a \in R$ cannot be written as a product of irreducibles. It must not be irreducible itself, so it is a product $a = bc$ of nonunits. Either b or c must not have a representation as a product of irreducibles, and we may suppose it is b. We cannot have $b \in (a)$, say $b = ad$, because $a = adc$ implies $1 = dc$, so (a) is proper subset of (b). Again we can represent b as a product of two nonunits, one of which is not a product of irreducibles, and continuing in this fashion gives an infinite ascending chain of principal ideals $(a) \subset (b) \subset \cdots$, contrary to hypothesis.

Now assume that irreducibles are prime, and suppose that two products of irreducibles $r_1 \cdots r_k$ and $s_1 \cdots s_\ell$ are equal. Since $s_1 \cdots s_\ell \in (r_1)$, this ideal must contain some s_j, which is necessarily the product of r_1 and a unit, because it is irreducible. We may divide by r_1, because R is a domain. Repeated reductions of this sort lead eventually at the conclusion that the two products are the same up to units. $\square$

Proposition A7.5. *If R is a Noetherian integral domain and every prime minimal over a principal ideal is itself principal, then R is factorial.*

Proof. In view of the last result we only need to show that any irreducible $a \in R$ is prime. Of course (a) is contained in a maximal ideal, which is prime, so Proposition A2.5 implies that there is a prime P that is minimal over (a). By hypothesis $P = (p)$ for some p, so $a = rp$ for some r. Since a is irreducible and p is not a unit, r must be a unit, so that $(a) = P$. $\square$

Theorem A7.6. *A principle ideal domain is factorial.*

Proof. After the last result it suffices to show that R is Noetherian. If $(a_1) \subset (a_2) \subset \cdots$ is an increasing chain of principal ideals, then its union is an ideal, which is necessarily principal, say (b). There is some n such that $b \in (a_n)$ which implies that $(a_n) = (a_{n+1}) = \cdots$. $\square$

For geometric applications it is important to understand the effect of localization.

Proposition A7.7. *If R is a UFD and S is a multiplicatively closed subset of R, then $S^{-1}R$ is a UFD.*

Proof. We first study when an element $\frac{x}{s}$ of $S^{-1}R$ is a unit. If this is the case, say $\frac{x}{s} \cdot \frac{y}{t} = \frac{1}{1}$, then $xy = st$, so all the prime factors of x and y divide elements of S. Conversely, if all the prime factors of x divide elements of S, then x divides an element of S, say $ax = t$, and $\frac{x}{s} \cdot \frac{as}{t} = \frac{1}{1}$. In particular, if $p \in R$ is irreducible, then $\frac{p}{1}$ is a unit if and only if p divides an element of S.

Now suppose that p is a prime that does not divide any element of S. We claim that $\frac{p}{1}$ is prime. Suppose that $\frac{p}{1} = \frac{x}{s} \cdot \frac{y}{t}$. Then $pst = xy$. Uniqueness of prime factorization implies that p divides x or y but not both, and that all other prime factors of xy are factors of st. In particular, if p divides x, then $\frac{y}{t}$ is a unit in $S^{-1}R$, so $\frac{p}{1}$ divides $\frac{x}{s}$.

Next suppose that $\frac{a}{s}$ is irreducible in $S^{-1}R$, and that $a = p_1 \cdots p_k$ is the prime factorization of a. We have $\frac{a}{s} = \frac{1}{s} \cdot \frac{p_1}{1} \cdots \frac{p_k}{1}$, so precisely one of the $\frac{p_i}{1}$ is a nonunit. Thus every irreducible of $S^{-1}R$ is (up to units) of the form $\frac{p}{1}$ where p is a prime of R that is not a factor of an element of S.

Combining all this with unique factorization in R, any $\frac{a}{s}$ has a representation of the form $\frac{a}{s} = \frac{x}{t} \cdot \frac{p_1}{1} \cdots \frac{p_k}{1}$ where $p_1, \dots, p_k$ are (not necessarily distinct) primes that do not divide elements of S and x divides an element of S. Furthermore, any representation of $\frac{a}{s}$ as a product of irreducibles can be brought to this form by multiplying by units. Finally, if $\frac{a}{s} = \frac{x'}{t'} \cdot \frac{p_1'}{1} \cdots \frac{p_k'}{1}$ is a second such representation, then unique factorization applied to $t'xp_1 \cdots p_k = tx'p_1' \cdots p_k'$ reveals that $p_1, \ldots, p_k$ and $p_1', \ldots, p_k'$ are the prime factors that do not divide elements of S, so the second list is a reordering of the first. Thus prime factorization in $S^{-1}R$ is unique. $\square$

Just which algebraic numbers deserve to be regarded as integers is a basic conceptual issue in algebraic number theory. For a variety of reasons the most satisfying definition, by far, is that an algebraic number is an integer if it satisfies a monic equation with integral coefficients. Somewhat surprisingly, this concept is also quite important in algebraic geometry.

Suppose that S is a ring that contains R as a subring. An element $s \in S$ is *integral over* R if it is a root of a monic polynomial with coefficients in R. That is, there is an integer n and $r_0, r_1, \ldots, r_{n-1} \in R$ such that

$$s^n + r_{n-1}s^{n-1} + \cdots + r_1 s + r_0 = 0.$$

The Cayley-Hamilton theorem gives a surprisingly general test for integrality.

Proposition A7.8. *For $s \in S$ the following are equivalent:*

(a) s is integral over R;

(b) there is an S-module N and a finitely generated R-submodule M that is not annihilated by any nonzero element of S, such that $sM \subset M$.

Proof. Supposing that (a) holds, take $N = S$ and $M = R[s]$. Then $sM \subset M$, and M is finitely generated because for some n it is generated by $1, s, \ldots, s^{n-1}$. Since $1 \in M$, M is not annihilated by any nonzero element of S.

Suppose that (b) holds. We may regard multiplication by s as an endomorphism of M. Applying the Cayley-Hamilton theorem (with improper ideal $I = R$) gives a monic polynomial p with coefficients in R such that $p(s)M = 0$, and $p(s) = 0$ because M is not annihilated by any other element of S. $\square$

We say that S is *finitely generated as an R-algebra*, or simply *finitely generated*, if there are $x_1, \ldots, x_k$ such that $S = R[x_1, \ldots, x_k]$. This is a much weaker condition than being finitely generated as an R-module.

Corollary A7.9. *For $s \in S$, s is integral over R if and only if $R[s]$ is finitely generated as an R-module.*

Proof. If s is integral over R, then for some n every element of $R[s]$ is an R-linear combination of $1, s, \ldots, s^{n-1}$. Conversely, if $R[s]$ is finitely generated as an R-module, we may take $N = S$ and $M = R[s]$ in the last result. (Since $1 \in R[s]$, $R[s]$ is not annihilated by any nonzero element of S.) $\square$

Lemma A7.10. *If $R \subset S \subset T$ with S finitely generated as an R-module and S finitely generated as an S-module, then T is finitely generated as an R-module.*

Proof. We may assume that there is an s such that S is generated over R by $1, s, \dots, s^{m-1}$ and there is a t such that T is generated over S by $1, t, \dots, t^{n-1}$, since the general case follows from this special case by induction. Evidently T is generated over R by $\{ s^i t^j : 0 \le s \le m-1, 0 \le j \le n-1 \}$. □

We say that S is *integral over R* if each of its elements is integral over R.

Proposition A7.11. *If S is finitely generated as an R-algebra, then S is integral over R if and only if is finitely generated as an R-module.*

Proof. Suppose that S is finitely generated as an R-module and $s \in S$. Then (b) of Proposition A7.8 holds with $N = S$ and $M = R[s]$, because $1 \in R[s]$ is not annihilated by any $s \in S$, so s is integral over R.

Suppose that $S = R[s_1, \dots, s_n]$ is integral over R. Then $R[s_1]$ is finitely generated as an R-module. Since s_2 is integral over R, it is integral over $R[s_1]$, and consequently $R[s_1, s_2]$ is finitely generated as an $R[s_1]$-module, and so forth. The last result implies that S is finitely generated as an R-module. □

Proposition A7.12. *If $R \subset S \subset T$ are rings with T integral over S and S integral over R, then T is integral over R.*

Proof. Any $t \in T$ satisfies a monic equation $t^n + s_{n-1}t^{n-1} + \cdots + s_0 = 0$ where $s_0, \dots, s_{n-1} \in S$, in which case $R[s_0, \dots, s_{n-1}, t]$ is a finitely generated $R[s_0, \dots, s_{n-1}]$-module. Since S is integral over R, $R[s_0, \dots, s_{n-1}]$ is a finitely generated R-module, and consequently Lemma A7.10 implies that $R[s_0, \dots, s_{n-1}, t]$ is finitely generated as an R-module, hence integral over R, and in particular t is integral over R. □

The *integral closure* of R in S is the set of elements of S that are integral over R.

Proposition A7.13. *The integral closure of R in S is a subring of S that contains R.*

Proof. Each $r \in R$ is a root of the monic polynomial $X - r$, so the integral closure contains R. We need to show that the integral closure contains all sums and products of its elements, so suppose that s and t are integral over R. For some integers m and n, $R[s,t]$ is generated by $\{ s^i t^j : 0 \le s \le m-1, 0 \le j \le n-1 \}$. Now (b) of Proposition A7.8 is satisfied by $N = S$ and $M = R[s,t]$, so $s + t$ and st are integral over R. □

We say that R is *integrally closed* in S if it is itself its integral closure in S. This terminology makes sense:

Proposition A7.14. *The integral closure of R in S is integrally closed in S.*

Proof. Let R' is the integral closure of R in S, and let R'' be the integral closure of R' in S. Proposition A7.12 implies that R'' is integral over R, hence contained in R'. □

We now study integrality in relation to integral domains.

Proposition A7.15. *If S is an integral domain, and integral over R, then:*

(a) If I is a nonzero ideal of S, then $R \cap I \neq \emptyset$.

(b) An element $r \in R$ if and only if it is a unit of S.

(c) R is a field if and only if S is a field.

Proof. (a) If $0 \neq s \in I$, then s satisfies an equation $s^n + r_{n-1}s^{n-1} + \cdots + r_0 = 0$, which implies that $r_0 \in I$. By taking n minimal we can obtain $r_0 \neq 0$.

(b) If r is a unit of R, it is automatically a unit of S. Suppose that $r \in R$ is a unit of S, so $rs = 1$ for some s. If $s^n + r_{n-1}s^{n-1} + \cdots + r_0 = 0$, then multiplying by r^{n-1} gives $s = -(r_{n-1} + r_{n-2}r + \cdots + r_0 r^{n-1})$, so $s \in R$.

(c) If S is a field, then (b) implies that R is a field. Suppose that R is a field and $0 \neq s \in S$. If $s^n + r_{n-1}s^{n-1} + \cdots + r_0 = 0$ and $r_0 \neq 0$, then $s(s^{n-1} + r_{n-1}s^{n-2} + \cdots + r_1)(-r_0)^{-1} = 1$, so s is a unit. □

If R is an integral domain, the *normalization* of R is its integral closure in its field of fractions $K(R)$. We say that R is *normal*, or a *normal domain*, or *integrally closed*, if it is integrally closed in $K(R)$, which is to say that it is its own normalization. Like the earlier steps in this section (from Euclidean rings to PIDs, then to UFDs) this is an increase in generality.

Theorem A7.16. *If R is factorial, it is normal.*

Proof. We need to show that if r/s is a fraction that is integral over R, then it is an element of R. Without loss of generality we may assume that r and s have no common prime factors. There is an n and $a_{n-1}, \ldots, a_0 \in R$ such that

$$(r/s)^n + a_{n-1}(r/s)^{n-1} + \cdots + a_0 = 0.$$

Multiplying by s^n and rearranging gives

$$r^n = -a_{n-1}r^{n-1}s - \cdots - a_0 s^n,$$

so any prime factor of s is a prime factor of r^n, and thus a prime factor of r, which is impossible. We conclude that s is a unit, and that $r/s \in R$. □

We conclude this section with some results relating integral closure to the main operations on rings.

Proposition A7.17. *Let $\varphi : S \to S'$ be a ring homomorphism. If S is integral over R, then $\varphi(S)$ is integral over $\varphi(R)$. In particular, if I is an ideal of S, then S/I is integral over $R/(R \cap I)$.*

Proof. If $s^n + r_{n-1}s^{n-1} + \cdots + r_0 = 0$, then

$$\varphi(s)^n + \varphi(r_{n-1})\varphi(s)^{n-1} + \cdots + \varphi(r_0) = 0.$$

□

Proposition A7.18. *If $R \subset T \subset S$ are rings, T is the integral closure of R in S, and $U \subset R$ is multiplicatively closed, then $U^{-1}T$ is the integral closure of $U^{-1}R$ in $U^{-1}S$.*

Proof. If $t \in T$ satisfies the monic equation $x^n + r_{n-1}x^{n-1} + \cdots + r_0$, and $u \in U$, then t/u satisfies the monic equation $x^n + (r_{n-1}/u)x^{n-1} + \cdots + r_0/u^n = 0$. Thus the integral closure of $U^{-1}R$ contains $U^{-1}T$.

Suppose that s/u satisfies a monic equation

$$(s/u)^n + (r_{n-1}/u_{n-1})(s/u)^{n-1} + \cdots + r_0/u_0 = 0.$$

Multiplying by $(uu_{n-1} \cdots u_0)^n$ gives an equation showing that $su_{n-1} \cdots u_0$ is integral over R, hence an element of T, so that $s/u \in U^{-1}T$. Thus the integral closure of $U^{-1}R$ is contained in $U^{-1}T$. □

Corollary A7.19. *If $U \subset R$ is multiplicatively closed and S is integral over R, then $U^{-1}S$ is integral over $U^{-1}R$.*

Corollary A7.20. *If R is an integrally closed integral domain and $U \subset R$ is a multiplicatively closed set, then $U^{-1}R$ is integrally closed.*

Normality is a local condition:

Proposition A7.21. *If R is an integral domain, then the following are equivalent:*

(a) R is integrally closed.

(b) R_P is integrally closed for all prime ideals P.

(c) $R_\mathfrak{m}$ is integrally closed for all maximal ideals $\mathfrak{m}$.

Proof. In view of the result above, (a) implies (b), and of course (c) follows automatically from (b). We will show that if R is not integrally closed, then some $R_\mathfrak{m}$ is not integrally closed. Let x be an element of $K(R) \backslash R$ that satisfies a monic equation $x^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0 = 0$ with coefficients in R. Let $I = \{ r \in R : rx \in R \}$. (This is called the *ideal of denominators* of x.) Since $x \notin R$, this is a proper ideal of R, and is contained in a maximal ideal $\mathfrak{m}$. Now $x \notin R_\mathfrak{m}$ (its ideal of divisors as an element of $R_\mathfrak{m}$ is $\{ i/s : i \in I, s \in R \setminus \mathfrak{m} \}$) but it satisfies the equation above, which may be understood as monic with coefficients in $R_\mathfrak{m}$. (Of course $K(R_\mathfrak{m}) = K(R)$.) Thus $R_\mathfrak{m}$ is not integrally closed, so (c) implies (a). □

A8 Noether Normalization and Hilbert's Nullstellensatz

This section presents two quite famous results. As we will see in the next section, the second of these has a crucial role in algebraic geometry from the very beginning, and the first also has an important geometric interpretation.

In preparation we discuss a useful piece of notation for polynomials in $R[X_1, \dots, X_n]$. An *exponent vector* is an n-tuple $e = (e_1, \dots, e_n)$ of non-negative integers. For such an e let $X^e = X_1^{e_1} \cdots X_n^{e_n}$. Now any element of $R[X_1, \dots, X_n]$ has the form $\sum r_e X^e$ where the sum is over all exponent vectors and only finitely many of the r_e are nonzero.

Theorem A8.1 (Noether Normatization)**.** *Let K be a field, and suppose that $A = K[x_1, \dots, x_n]$ is a finitely generated K-algebra. Then there are algebraicly independent $z_1, \dots, z_m \in A$ such that A is integral over $K[z_1, \dots, z_m]$.*

Here the relevant Noether is Max, Emmy's father.

Proof. We argue by induction on n. Of course the case $n = 0$ is trivial, so suppose the claim has been established with $n - 1$ in place of n. If $x_1, \dots, x_n$ are algebraicly independent we are done, so suppose that $f(x_1, \dots, x_n) = 0$ for some nonzero $f \in K[X_1, \dots, X_n]$. Let $\beta = (\beta_1, \dots, \beta_{n-1}, 1)$, where $\beta_1, \dots, \beta_{n-1}$ are positive integers to be specified later. For $i = 1, \dots, n-1$ let $y_i = x_i - x_n^{\beta_i}$, and let $B = K[y_1, \dots, y_{n-1}]$. Below we show that x_n is integral over B, after which Proposition A7.11 implies that A is integral over B. The induction hypothesis gives algebraicly independent $z_1, \dots, z_m \in B$ such that B is integral over $K[z_1, \dots, z_m]$. In view of Proposition A7.14, the integral closure of $K[z_1, \dots, z_m]$ in A is all of A.

For each exponent vector e we substitute the various $x_i = x_n^{\beta_i} + y_i$, obtaining

$$x^e = x_n^{\beta \cdot e} + g_e(y_1, \dots, y_{n-1}, x_n)$$

where g_e is a polynomial in which the maximal power of x_n is less than $\beta \cdot e$. (Here $\beta \cdot e$ is the usual inner product, an unexpected fringe benefit of using exponent vectors!) If $f = \sum a_e X^e$, then substituting these expressions gives

$$0 = \sum a_e x_n^{\beta \cdot e} + h(y_1, \dots, y_{n-1}, x_n)$$

where h is a polynomial whose maximal power of x_n is less than the maximum of $\beta \cdot e$ over all exponent vectors with $a_e \neq 0$.

In order to interpret this as a monic equation satisfied by x_n, we need to insure that the sum of the a_e, over those e with $a_e \neq 0$ for which $\beta \cdot e$ is maximal, is not zero. This is certainly the case if the integers $\beta \cdot e$ are distinct. An effective (if somewhat brutal) method is to let $\beta_i = d^i$ where d is an integer greater than the maximum degree of f in any variable. With this choice we can divide the equation above by a_e, where e is the exponent vector that is maximal for $\beta \cdot e$, thereby displaying x_n as a root of a monic polynomial, as desired. $\square$

If K and L are fields with $K \subset L$, then L is a *field extension* of K. We write L/K to indicate this situation. The extension is *finite* if L is finite dimensional as a vector space over K. An element of L is *algebraic over* K if it is the root of a polynomial with coefficients in K. Since we can divide any polynomial with coefficients in K by its leading coefficient, being algebraic over K is the same as being integral over K, and the theory we developed in the last section is entirely applicable. In particular, if L is finite, then each of its elements is algebraic over K. Conversely, L/K is a finite extension if L is generated by finitely many elements that are algebraic over K. In fact a weaker hypothesis suffices to imply that L/K is finite:

Lemma A8.2 (Zariski's Lemma)**.** *If L/K is a field extension, and L is finitely generated as a K-algebra, then L is a finite extension of K.*

Proof. Noether normalization gives $z_1, \dots, z_m \in L$ that are algebraicly independent over K such that L is a finitely generated $K[z_1, \dots, z_m]$-module. If $m \geq 1$, then $1/z_1$ satisfies some condition of the form

$$z_1^{-k} + p_{k-1} z_1^{1-k} + \cdots + p_0 = 0$$

where $p_0, \dots, p_{k-1} \in K[z_1, \dots, z_m]$. After rearranging terms, and possibly multiplying by a negative power of z_1, we obtain such an equation with $p_0, \dots, p_{k-1} \in K[z_2, \dots, z_m]$. Multiplying by z_1^k gives a violation of algebraic independence, so $m = 0$, which is the desired conclusion. □

Corollary A8.3 (Weak Nullstellensatz)**.** *If K is a field, R is a finitely generated K-algebra, and $\mathfrak{m}$ is a maximal ideal of R, then $R/\mathfrak{m}$ is a finite extension of K. Consequently $R/\mathfrak{m} \cong K$ if K is algebraicly closed.*

Proof. Since $R/\mathfrak{m}$ is a finitely generated K-algebra, and also a field, the last result implies that it is a finite extension of K. □

A9 Geometric Motivation

The concepts developed up to this point are enough to support a description of the initial motivations and definitions of algebraic geometry. This section has a different spirit from the rest of our work, which aims to do as little as possible consistent with achieving our main objectives. Here we aim to give as rich a sense of the interplay of algebra and geometry as we can without going too far afield.

There will be two main geometric settings. First, let K be a field, fix an integer n, and let *n-dimensional affine space* $\mathbf{A}^n$ be the n-fold cartesian product of K, regarded as a geometric setting. Any set $S \subset K[X_1, \dots, X_n]$ determines a set

$$V(S) = \{\, x \in \mathbf{A}^n : f(x) = 0 \text{ for all } f \in S \,\}.$$

Evidently $V(S) = V(I)$ where I is the radical of the ideal generated by S. A set of the form $V(I)$ is called is called an *affine algebraic set.* These sets, and the corresponding subsets of projective spaces, are the traditional focus of algebraic geometry, at least at the outset.

Finite unions of affine algebraic sets are affine algebraic sets, because for any ideals I_1, I_2 we have $V(I_1) \cup V(I_2) = V(I_1 \cdot I_2)$. Arbitrary intersections of affine algebraic sets are affine algebraic sets: if $\{I_j\}_{j \in J}$ is any collection of ideals, $\bigcap_j V(I_j) = V(J)$ where J is the ideal generated by $\bigcup_j I_j$. In addition, $\emptyset = V(R)$ and $\mathbf{A}^n = V(\emptyset)$ are affine algebraic. Therefore the affine algebraic sets are the closed sets of a topology, called the *Zariski topology.* For each $x \in \mathbf{A}^n$, $\{x\} = V((X_1 - x_1, \ldots, X_n - x_n))$, so singletons are closed. (For topologists, a space in which singletons are closed is a T_1*-space.*) But the Zariski topology is not Hausdorff except in trivial cases, and in other respects as well it is highly peculiar, at least for those who rarely venture beyond metric spaces, as we will see below.

A subset $A \subset X$ of a topological space X is *reducible* if it can be written as the union of two proper subsets that are relatively closed; otherwise it is *irreducible.* There is an algebraic characterization of irreducibility for affine algebraic sets. Any set $S \subset \mathbf{A}^n$ determines a set

$$I(S) = \{\, f \in K[X_1, \ldots, X_n] : f(x) = 0 \text{ for all } x \in S\,\}.$$

It is easy to see that $I(S)$ is a radical ideal.

Lemma A9.1. *An affine algebraic set Y is irreducible if and only if $I(Y)$ is prime.*

Proof. Suppose that $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are affine algebraic set that are both proper subsets of Y. There is a polynomial f that vanishes on Y_1 but not on Y_2, and there is a polynomial g that vanishes on Y_2 but not on Y_1. Then $f, g \notin Y$ and $fg \in I(Y)$, so $I(Y)$ is not prime.

Now suppose that Y is irreducible. If $fg \in I(Y)$, then $Y \subset V(\{fg\}) = V(\{f\}) \cup V(\{g\})$, so

$$Y = (Y \cap V(\{f\})) \cup (Y \cap V(\{g\})).$$

Since Y is irreducible, either $Y = Y \cap V(\{f\})$ or $Y = Y \cap V(\{g\})$, so that either $f \in I(Y)$ or $g \in I(Y)$. Thus $I(Y)$ is prime. □

When V is an affine algebraic set, $\Gamma(V) = R/I(V)$ is a finitely generated K-algebra, called the *coordinate ring* of V. We think of this as the set of polynomial functions on V taking values in K. Note that $\Gamma(V)$ is reduced, because $I(V)$ is radical.

Conversely, let $\tilde{R}$ be a reduced finitely generated K-algebra, let $x_1, \cdots, x_n$ be a system of generators, and let I be the kernel of the map $f \mapsto f(x_1, \ldots, x_n)$ from R to $\tilde{R}$. Then $\tilde{R} \cong R/I$ is the coordinate ring of $V(I)$. *Thus the class of* *affine algebraic sets is exactly mirrored in the class of reduced finitely generated K-algebras.*

We can also say something about the geometric significance of integrality. Let R be a reduced finitely generated K-algebra, that we think of as the coordinate ring of some affine algebraic variety. Let $x_1, \ldots, x_n$ be a system of generators. After reordering, we may assume that $x_1, \ldots, x_r$ are algebraicly independent, and that $x_{r+1}, \ldots, x_n$ are algebraic over $K[x_1, \ldots, x_r]$. Suppose that each such x_j is actually integral over $K[x_1, \ldots, x_r]$, so that it is a root of a monic polynomial:

$$p_j(x_j, y) = x_j^{m_j} + a_{j,m_j-1}(y)x_j^{m_j-1} + \cdots + a_{j,1}(y)x_j + a_{j,0}(y) = 0$$

where $a_{j,0}(y), \ldots, a_{j,m-1}(y) \in K[x_1, \ldots, x_r]$. For each $y \in K^r$ the fiber

$$F(y) = \{\, (x_{r+1}, \ldots, x_n) : p_j(x_j, y) = 0 \text{ for all } j = r+1, \ldots, n \,\}$$

is a cartesian product $F(y) = \prod_j F_j(y)$ where $F_j(y) = \{\, x_j : p_j(x_j, y) = 0 \,\}$. Because p_j is monic, the coefficient of $x_j^{m_j}$ in $p_j(\cdot, y)$ never vanishes, so each $F_j(y)$ contains m_j roots when these are counted according to multiplicity. In connection with $K = \mathbb{R}$ or $K = \mathbb{C}$, the geometric picture is that a branch of the set of solutions cannot go to infinity as one approaches some $y \in K^r$. The point of the Noether normalization theorem is that it is possible to perturb the system of generators so as to bring this situation about.

What we have seen so far suggests that the correspondence between radical ideals and affine algebraic sets may allow a quite fruitful algebraic analysis of geometric issues. This raises the question of whether the algebraic and geometric perspective are exact mirrors of each other. For any affine algebraic set $V(S)$ we have $V(I(V(S))) \subset V(S)$ because $S \subset I(V(S))$. On the other hand $V(S) \subset V(I(V(S)))$ because for any $x \in V(S)$, every element of $I(V(S))$ vanishes at x, so $x \in V(I(V(S)))$. Thus $V \circ I$ takes each affine algebraic set $V(S)$ to itself. In particular, the map $V(S) \mapsto I(V(S))$ from affine algebraic sets to radical ideals is injective.

But the map $I \mapsto V(I)$ from radical ideals to affine algebraic sets is not injective in general. For example the ideal $(X_1^2 - 2)$ and the improper ideal $K[X_1]$ are both mapped to the null set if $K = \mathbb{Q}$.

We will consider two responses to this conundrum. The first is to require that K be algebraically closed, in which case the correspondence is bijective by virtue of the following result.

Theorem A9.2 (Hilbert's Nullstellensatz)**.** *If K is algebraicly closed and $I \subset R$ is a radical ideal, then $I(V(I)) = I$.*

Proof. Obviously $I \subset I(V(I))$. Aiming at a contradiction, suppose that $f \in I(V(I)) \setminus I$. Since I is the intersection of the minimal primes that contain it (Proposition A4.11) there is a prime ideal $P \subset R$ that contains I but not f.

Let $\tilde{R} = R/P$, let $\tilde{f}$ be the image of f in $\tilde{R}$, and let $\mathfrak{m}$ be a maximal ideal of $\tilde{R}[1/\tilde{f}]$. Since $\tilde{R}[1/\tilde{f}]$ is generated by $1/\tilde{f}$ and the images of $X_1, \ldots, X_n$, it is finitely generated, so Corollary A8.3 implies that $\tilde{R}[1/\tilde{f}]/\mathfrak{m} \cong K$.

Let β be the composition of ring homomorphisms

$$R \to \tilde{R} \to \tilde{R}[1/\tilde{f}] \to \tilde{R}[1/\tilde{f}]/\mathfrak{m} \cong K.$$

Since $\beta(1) = \beta(1 \cdot 1) = \beta(1)^2$, either $\beta(1) = 0$ or $\beta(1) = 1$. The image of $1 \in R$ in $\tilde{R}$ is nonzero because P is a proper ideal, so it is $1 \in \tilde{R}$. In turn, its image in $\tilde{R}[1/\tilde{f}]$ is $1 \in \tilde{R}[1/\tilde{f}]$, and this is not in $\mathfrak{m}$, so $\beta(1) = 1$.

For each $i = 1, \ldots, n$ let $x_i = \beta(X_i)$, and let $x = (x_1, \ldots, x_n) \in K^n$. The set of $g \in R$ such that $\beta(g) = g(x)$ includes all elements of K and $X_1, \ldots, X_n$, and it is closed under addition and multiplication, so it is all of R. If $g \in P$, then $\beta(g) = 0$, so $x \in V(P) \subset V(I)$. Since the map $\tilde{R}[1/\tilde{f}] \to K$ takes $1 = \tilde{f} \cdot 1/\tilde{f}$ to 1, it must be the case that $f(x) = \beta(f) \neq 0$. This contradicts our supposition that $f \in I(V(I))$, which completes the proof. $\square$

Any field K can be embedded in an algebraic closure $\overline{K}$, leading to an embedding of $\mathbf{A}^n$ in the corresponding affine space $\overline{\mathbf{A}}^n$. Let $\overline{R} = \overline{K}[X_1, \ldots, X_n]$. For any subset $S \subset \overline{R}$ one may study the relationship between the subsets of $\mathbf{A}^n$ and $\overline{\mathbf{A}}^n$ that it defines, and one may study the special properties of these sets that result from S being a subset of R. The algebraic geometry of $\overline{R}$ and $\overline{\mathbf{A}}^n$ sets the stage, and is implicitly present in the background, of these more specific studies, so the study of algebraic geometry over algebraically complete fields should be simpler than, and prior to, the sorts of issues that might arise in arithmetic geometry. Hilbert's nullstellensatz is perhaps the most important basic result endowing these general considerations with concrete substance.

The second response begins with the idea that we may consider all the maximal ideals of $K[X_1, \ldots, X_n]$, not just those that correspond to points in $\mathbf{A}^n$. Note that there is a maximal ideal corresponding to each point in $\overline{\mathbf{A}}^n$. This method can be applied to any ring, and doing so should be interesting and useful because a variety of rings arise naturally during any sort of analysis. In this general context it makes sense to consider all prime ideals, including those that are not maximal. (For example, a local ring has a single maximal ideal, by definition, but the structure of its set of prime ideals can be quite rich.) This perspective is the starting point of the theory of schemes.

Now let R be an arbitrary commutative ring with unit. The *spectrum* of R, denoted by $\operatorname{Spec} R$, is the set of prime ideals of R. For any $S \subset R$, let

$$\tilde{V}(S) = \{ P \in \operatorname{Spec} R : S \subset P \}.$$

Clearly $\tilde{V}(S) = \tilde{V}(I)$ where I is the radical of the (possibly improper) ideal generated by S. For a nonempty $S \subset \operatorname{Spec} R$ let

$$\tilde{I}(S) = \bigcap_{P \in S} P,$$

and let $\tilde{I}(\emptyset) = R$. Evidently $\tilde{I}(S)$ is a (possibly improper) radical ideal.

As before, for any $S \subset R$ we have $\tilde{V}(\tilde{I}(\tilde{V}(S))) = \tilde{V}(S)$: $\tilde{V}(\tilde{I}(\tilde{V}(S))) \subset \tilde{V}(S)$ because $S \subset \tilde{I}(\tilde{V}(S))$, and $\tilde{V}(S) \subset \tilde{V}(\tilde{I}(\tilde{V}(S)))$ because $\tilde{I}(\tilde{V}(S))$ is a subset of each $P \in \tilde{V}(S)$, so that $P \in \tilde{V}(\tilde{I}(\tilde{V}(S)))$. Any radical ideal I is the intersection of the primes that contain it (Corollary A2.9) so $\tilde{I}(\tilde{V}(I)) = I$ and $\tilde{V}(\tilde{I}(\tilde{V}(I))) = \tilde{V}(I)$. Thus $\tilde{V}$ and $\tilde{I}$ authomatically restrict to inverse bijections between the set of (possibly improper) radical ideals and the set of subsets of $\operatorname{Spec} R$ of the form $\tilde{V}(S)$.

For ideals I_1, I_2, $\tilde{V}(I_1) \cup \tilde{V}(I_2) = \tilde{V}(I_1 \cap I_2)$. (Any prime P that contains either I_1 or I_2 contains $I_1 \cap I_2$, and if $I_1 \cap I_2 \subset P$, then P contains either I_1 or I_2 because otherwise the product of an element of $I_1 \setminus P$ and an element of $I_2 \setminus P$ would be an element of $(I_1 \cap I_2) \setminus P$.) More trivially, for any set of ideals $\mathcal{I}$, $\bigcap_{I \in \mathcal{I}} \tilde{V}(I) = \tilde{V}(J)$ where J is the ideal generated by $\bigcup_{I \in \mathcal{I}} I$. In addition $\tilde{V}((0)) = \operatorname{Spec} R$ and $\tilde{V}(R) = \emptyset$. Therefore the sets $\tilde{V}(I)$ are the closed sets of a topology on $\operatorname{Spec} R$ that is also called the *Zariski topology*. We now study $\operatorname{Spec} R$ in relation to basic concepts from topology.

In this version of the Zariski topology singletons need not be closed. Concretely, Q is an element of the closure of $\{P\}$ whenever P and Q are primes with $P \subset Q$. For this reason the maximal ideals of R are called *closed points*. For the most part, when K is algebraicly closed there are corresponding properties of the Zariski topology of affine algebraic sets that can be derived by considering the relative topology of the set of closed points of $\operatorname{Spec} K[X_1, \ldots, X_n]$.

We have seen that passing from $\operatorname{Spec} R$ to its set of closed points, endowed with the relative topology, throws away pertinent geometric information except in the circumstance described by Hilbert's nullstellensatz. More generally, no information is lost if R is a *Jacobson ring*, which is to say that every prime ideal (and thus every radical ideal) is the intersection of the maximal ideals that contain it. Of course a field is a Jacobson ring, and it turns out that R is Jacobson if and only if $R[X]$ is Jacobson. This result provides the proper (in the sense of maximal generality) understanding of the nullstellensatz; it is treated in Sections 1-3 of Kaplansky (1974) and also in Section 4.5 of Eisenbud (1995).

Working with $\operatorname{Spec} R$ retains the information lost in passing to its space of closed points. One may also ask what is lost in the passage from R to $\operatorname{Spec} R$. Let N be the nilradical of R. The prime and radical ideals of R/N are precisely the P/N and I/N where P and I are prime and radical ideals of R, so $\operatorname{Spec} R/N$ and $\operatorname{Spec} R$ are homeomorphic. (With respect to certain issues studied below, the picture is clarified if we assume that R is reduced.) In a sense, the passage to $\operatorname{Spec} R$ throws away information about R that is not retained in the passage from R to R/N. The theory of schemes retains this information by embedding $\operatorname{Spec} R$ in a larger structure called an *affine scheme* which, in effect, has R as part of its data. Nilpotents were systematically exploited by some of the important advances in algebraic geometry following the introduction of schemes.

Recall that a topological space X is *disconnected* if it is the union of two disjoint nonempty open (or closed) sets. It is *connected* if it is not disconnected.

Proposition A9.3. *If R is reduced, then* $\operatorname{Spec} R$ *is disconnected if and only if $R = R_1 \times R_2$ where R_1 and R_2 are commutative rings with unit.*

Proof. First suppose that $R = R_1 \times R_2$. Let I be an ideal, and let

$$I_1 = \{\, r_1 : (r_1, r_2) \in I \,\} \quad \text{and} \quad I_2 = \{\, r_2 : (r_1, r_2) \in I \,\}.$$

Clearly I_1 and I_2 are ideals of R_1 and R_2 respectively, and $I_1 \times I_2$ is an ideal of R that contains I. If $(r_1, r_2) \in I$, then $(r_1, 0) = (r_1, r_2) \cdot (1, 0) \in I$, so $I_1 \times \{0\} \subset I$, and symmetrically $\{0\} \times I_2 \subset I$. Taking sums gives $I_1 \times I_2 \subset I$ and thus $I = I_1 \times I_2$. Evidently I is radical if and only if I_1 and I_2 are radical.

Now suppose that I is prime. Any $(r_1, r_2) \in I$ is the product of $(r_1, 1)$ and $(1, r_2)$, one of which must be in I, so either $I_1 = R_1$ or $I_2 = R_2$. If $I_1 = R_1$, then I_2 must be prime, and I_1 is prime if $I_2 = R_2$. Conversely, if P_1 and P_2 are prime ideals of R_1 and R_2, then $P_1 \times R_2$ and $R_1 \times P_2$ are easily seen to be prime ideals of I.

Since R is reduced, R_1 and R_2 are reduced, so for each its respective (0) is radical. Writing

$$\operatorname{Spec} R = \tilde{V}((0) \times R_2) \cup \tilde{V}(R_1 \times (0))$$

displays $\operatorname{Spec} R$ as the union of two disjoint nonempty closed sets.

Now suppose that $\operatorname{Spec} R$ is disconnected. Then there are radical ideals R_1 and R_2 such that every prime ideal contains either R_1 and R_2 and no prime ideal contains both. Therefore $R_1 \cap R_2$ is the nilradical of R, which is (0), and $R_1 + R_2$ cannot be proper ideal, so it is R. Therefore there are unique $i_1 \in R_1$ and $i_2 \in R_2$ such that $1 = i_1 + i_2$. For each $i = 1, 2$, if $r_i \in R_i$, then $i_1 r_i = (i_1 + i_2) r_i = r_i$. Thus i_1 and i_2 are identity elements for R_1 and R_2 respectively, which means that R_1 and R_2 are commutative rings with unit such that $R = R_1 \times R_2$. ☐

Let X be a topological space. For any $x \in X$, the *connected component* of x, denoted by C_x is the union of all the subsets of X that contain x and are connected; it is connected (supposing otherwise quickly leads to a contradiction) and is the largest connected subset of X containing x. The closure of a connected set is connected (supposing otherwise quickly yields a contradiction) so C_x is closed. The collection $\{\, C_x : x \in X \,\}$ is a partition of X because if $x_0 \in C_x \cap C_{x'}$, then $C_x \cup C_{x'}$ is a connected set contained in C_{x_0} that contains both x and x', so $C_x = C_{x'}$.

A set $A \subset X$ is irreducible if and only if it does not have two disjoint nonempty relatively open subsets, so a nonempty open subset of an irreducible space is dense, and an irreducible set is connected. Thus the collection of irreducible sets refines the collection of all connected components. For any $x \in X$, the closure of $\{x\}$ is irreducible, and the union of a collection of irreducible sets that is completely ordered by inclusion is irreducible, so Zorn's lemma implies that x is contained in a maximal irreducible subset. Such a set is called an *irreducible component*. The closure of an irreducible set is irreducible (supposing otherwise quickly leads to absurdity) so irreducible components are closed. However, unlike connected components, two different irreducible components can have a nonempty intersection.

The bijection between radical ideals and closed subsets of $\operatorname{Spec} R$ specializes to a bijection between prime ideals and irreducible subsets.

Proposition A9.4. *For any radical ideal I, $\tilde{V}(I)$ is irreducible if and only if I is prime.*

Proof. If I is not prime, then there are $f, g \notin I$ with $fg \in I$. Let I_f and I_g be the smallest ideals containing I and f and g respectively. Then $I \subset I_f \cap I_g$, so $\tilde{V}(I_f) \cup \tilde{V}(I_g) \subset \tilde{V}(I)$. If $P \notin \tilde{V}(I_f) \cup \tilde{V}(I_g)$, then there are $r + af \in I_F \setminus P$ and $s + bg \in I_g \setminus P$ where $r, s \in I$, and $(r + af)(s + bg) \in I \setminus P$, so $P \notin \tilde{V}(I)$. Thus $\tilde{V}(I) \subset \tilde{V}(I_f) \cup \tilde{V}(I_g)$. Finally, because I is the intersection of the prime ideals that contain it, $\tilde{V}(I_f)$ and $\tilde{V}(I_g)$ are proper subsets of $\tilde{V}(I)$, so $\tilde{V}(I)$ is reducible.

Now suppose that $\tilde{V}(I)$ is the union of two proper subsets $\tilde{V}(I_1)$ and $\tilde{V}(I_2)$. We may assume that I_1 and I_2 are radical, which implies that $I_1 \cap I_2$ is radical. Then $I = I_1 \cap I_2$ because I is the unique radical ideal such that $\tilde{V}(I) = \tilde{V}(I_1) \cup \tilde{V}(I_2)$. Thus I is a reducible ideal and consequently (Lemma A4.9) not prime. □

A topological space X is *Noetherian* if any descending (ascending) sequence of closed (open) sets stabilizes. Obviously this terminology is derived from the fact that if R is a Noetherian ring, then $\operatorname{Spec} R$ is a Noetherian topological space. An argument similar to those for Noetherian rings and modules shows that X is Noetherian if and only if any nonempty collection of closed (open) sets has a minimal (maximal) element. In a Noetherian topological space, any closed set is a finite union of irreducible sets. (If the set of closed sets that are not finite unions of irreducibles was nonempty, it would contain a minimal element C, which could not itself be irreducible, but could also not be a finite union of closed proper subsets, because each of these would be a finite union of irreducibles.) Any subspace Y of a Noetherian X, with the induced topology, is Noetherian: if $\{C_j\}$ is a collection of relatively closed subsets of Y, then the collection $\{\overline{C}_j\}$ of closures in X has a minimal element $\overline{C}_j^*$, and C_j^* is minimal in $\{C_j\}$.

Following Bourbaki, algebraic geometers say that a topological space is *quasi-compact* if every open cover has a finite subcover. (It is *compact* if it is quasi-compact and Hausdorff.) A *base* of a topology is a collection of open sets such that every open set is the union of sets in the base. Since any open cover can be refined to a cover by base elements, a space is quasicompact if every cover by base elements has a finite subcover.

For $f \in R$ let

$$D(f) = \{ P \in \operatorname{Spec} R : f \notin P \}.$$

For any $S \subset R$ we have

$$\operatorname{Spec} R \setminus \tilde{V}(S) = \operatorname{Spec} R \setminus \bigcap_{f \in S} \tilde{V}((f)) = \bigcup_{f \in S} D(f),$$

so the sets $D(f)$ are a base for $\operatorname{Spec} R$.

Proposition A9.5. $\operatorname{Spec} R$ *is quasicompact.*

Proof. If $\{D(f_i)\}_{i \in I}$ is an open cover of $\operatorname{Spec} R$, then the ideal generated by $\{f_i\}$ cannot be proper because if it was, it would be contained in a maximal ideal $\mathfrak{m}$, and $\mathfrak{m}$ would not be a member of any $D(f_i)$. Therefore there are $i_1, \ldots, i_k$ and $r_1, \ldots, r_k$ such that $r_1 f_{i_1} + \cdots + r_k f_{i_k} = 1$, and $\operatorname{Spec} R = D(f_{i_1}) \cup \cdots \cup D(f_{i_k})$. $\square$

A Noetherian space (and thus also any subspace) is quasi-compact, because we can choose elements $U_1, U_2, \ldots$ of an open cover with $U_{n+1} \not\subset U_1 \cup \ldots \cup U_n$ whenever $U_1 \cup \ldots \cup U_n$ is not the entire space, and the sequence $U_1, U_1 \cup U_2, U_1 \cup U_2 \cup U_3, \ldots$ must stabilize. Since all subsets of $\operatorname{Spec} R$ are quasi-compact when R is Noetherian, in one sense quasicompactness plays a very small role in algebraic geometry, but the existence of finite subcovers is invoked frequently.

If $X \subset \mathbf{A}^m$ and $Y \subset \mathbf{A}^n$ are affine algebraic sets, perhaps the most natural functions $p : X \to Y$ are those given by polynomials. That is, we have a polynomial function $p : K^m \to K^n$ such that $p(X) \subset Y$. Such a function induces a ring homomorphism $\varphi : g \mapsto g \circ p$ from the coordinate ring of Y to the coordinate ring of X, and in effect we study p by analyzing φ. Roughly, φ is injective if p is surjective. (More precisely, φ is injective if an element of the coordinate ring of Y is determined by its values on $p(X)$.) In this sense it seems simplifying to replace Y with $p(X)$, so that φ is injective.

The corresponding setup in the realm of spectra has rings $R \subset S$, thought of as the coordinate rings of Y and X respectively. There is a map

$$\pi : \operatorname{Spec} S \to \operatorname{Spec} R$$

given by $\pi(Q) = R \cap Q$. When $P = R \cap Q$ we say that Q *lies over* P. If I is a radical ideal of R and J is the radical of the ideal generated by I as a subset of S, then $\pi^{-1}(\tilde{V}(I)) = \tilde{V}(J)$, so π is continuous.

Corresponding to our earlier discussion of the geometric significance of Noether normalization, we should expect that π is well behaved when S is integral over R. There are several important results in this direction. First of all, we should hope that the fibers of π are zero dimensional:

Proposition A9.6. *If S is integral over R and Q is a prime of S, then Q is maximal if and only if $\pi(Q)$ is maximal.*

Proof. Let $P = R \cap Q$. The induced map $R/P \hookrightarrow S/Q$ is an inclusion of integral domains, and (by Proposition A7.17) S/Q is integral over R/P. Of course, P is maximal if and only if R/P is a field, and similarly for Q, and Proposition A7.15(c) implies that R/P is a field if and only if S/Q is a field. □

The geometric picture of the next result is that if two irreducible varieties in the domain have the same image, and one is contained in the other, then they are the same.

Proposition A9.7. *If S is integral over R, $\pi(Q) = \pi(Q') = P$, and $Q \subset Q'$, then $Q = Q'$.*

Proof. The set $U = R \setminus P$ is multiplicative, both as a subset of R and as a subset of S. There is an induced inclusion $U^{-1}R \hookrightarrow U^{-1}S$, and (Corollary A7.20) $U^{-1}S$ is integral over $U^{-1}R$. Since $Q \cap U = \emptyset$, $U^{-1}Q$ is a prime ideal of $U^{-1}S$ (Proposition A5.6(c)) and it lies over $U^{-1}P$, which is a maximal ideal of $U^{-1}R$. (In most other contexts we would of course write R_P and P_P instead of $U^{-1}R$ and $U^{-1}P$.) Therefore the last result implies that $U^{-1}Q$ is maximal. This argument applies equally to Q', so $U^{-1}Q = U^{-1}Q'$ and consequently (Proposition A5.6(d)) $Q = Q'$. □

Theorem A9.8 (Lying Over)**.** *If S is integral over R, then π is surjective.*

Proof. Let P be a prime ideal of R, and let $U = R \setminus P$. Then $U^{-1}S$ is integral over $U^{-1}R$ (Corollary A7.20) so the hypotheses are satisfied with $U^{-1}R$ and $U^{-1}S$ in place of R and S. If Z is a prime ideal of $U^{-1}S$ that lies over $U^{-1}P$, then (Proposition A5.6) $\{\, s \in S : s/1 \in Z \,\}$ is a prime ideal that lies over P. Therefore we may assume that R is local, and it suffices to find a prime of S lying over $\mathfrak{m}$.

We now show that $\mathfrak{m}S$ is a proper ideal of S. If not there are $r_1, \ldots, r_k \in \mathfrak{m}$ and $s_1, \ldots, s_k \in S$ such that $r_1 s_1 + \cdots + r_k s_k = 1$, so $S = R[s_1, \ldots, s_k]$ is a finitely generated R-algebra, and consequently (Proposition A7.11) finitely generated as an R-module. But now $\mathfrak{m}S = S$ implies $S = 0$ by Nakayama, which is of course impossible.

Since $\mathfrak{m}S$ is a proper ideal of S, it is contained in a maximal ideal Q. Now $\pi(Q)$ is an ideal containing $\mathfrak{m}$, which is maximal, so $\pi(Q) = \mathfrak{m}$. □

The next two results are quite famous. The first is often stated as a matter of being able to extend a chain of primes of S lying over a chain of ideals of R. That is, for given primes $P_0 \subset \cdots \subset P_n$ of R and $Q_0 \subset \cdots Q_m$ of S with $\pi(Q_i) = P_i$, there are primes $Q_{m+1} \subset \cdots \subset Q_n$ with $Q_m \subset Q_{m+1}$ and $\pi(Q_j) = P_j$. Of course this more elaborate version is proved by applying the following special case inductively.

Theorem A9.9 (Going Up, Cohen-Seidenberg). *If S is an integral extension of R, $P \subset P'$ are primes of R, and $\pi(Q) = P$, then there is a prime Q' of S such that $Q \subset Q'$ and $\pi(Q') = P'$.*

Proof. By Proposition A7.17, S/Q is integral over R/P. Every prime ideal of S/Q is Q'/Q for some prime Q' of S, and the last result gives such a Q' with $Q'/Q \cap R/P = P'/P$, so that $Q' \cap R = P'$. □

The *Krull dimension* of R is the maximum length d of a chain of distinct prime ideals $P_0 \subset \cdots \subset P_d$. Proposition A9.7 implies that if S is integral over R, then the Krull dimension of S cannot be larger than the Krull dimension of R. Given a chain $P_0 \subset \cdots \subset P_d$ in R, lying over implies that there is a Q_0 with $\pi(Q_0) = P_0$, and going up then gives a preimage chain $Q_0 \subset \cdots \subset Q_d$ in S, so together they imply the opposite inequality.

The next result is also often stated as a matter of extending chains of prime ideals, and again the general form is proved by applying the special case below inductively. It has been included here because of its geometric content, and because it completes the picture in a sense, but (like everything else in this section) it will not be applied later, so the reader may choose to bypass the proof, which is relatively elaborate.

Theorem A9.10 (Going Down, Cohen-Seidenberg). *Suppose R is a normal domain and S is an integral extension of R that is an integral domain. If $P' \subset P$ are primes of R and $\pi(Q) = P$, then there is a prime Q' of S such that $Q' \subset Q$ and $\pi(Q') = P'$.*

Three preliminary results prepare the proof. If $R \subset S$ is an inclusion of rings and I is an ideal of R, we say that $s \in S$ is *integral over I* if it satisfies a monic equation $s^n + a_{n-1}s^{n-1} + \cdots + a_0$ with $a_0, \ldots, a_{n-1} \in I$. The set of such s is the *integral closure of I in S*.

Lemma A9.11. *If S is an integral extension of R and I is an ideal of R, then the the integral closure of I in S is the radical of IS.*

Proof. If s is integral over I, then the monic equation of the definition puts s in the radical of IS. On the other hand suppose that $s^n = a_1s_1 + \cdots + a_ks_k$ for some n, $a_1, \ldots, a_k \in I$, and $s_1, \ldots, s_k \in S$. Since S is integral over R, $M = R[s_1, \ldots, s_k]$ is a finitely generated R-module (Proposition A7.11). If $\varphi : M \to M$ is multiplication by s^n, then $\varphi(M) \subset IM$, so the Cayley-Hamilton theorem implies that φ satisfies some monic equation

$$p(\varphi) = \varphi^m + \alpha_{m-1}\varphi^{m-1} + \cdots + \alpha_0 = 0$$

with $\alpha_0, \ldots, \alpha_{m-1} \in I$. Multiplication by $p(s)$ annihilates M, and $1 \in M$, so $0 = p(s) = s^{mn} + \alpha_{m-1}s^{(m-1)n} + \cdots + \alpha_0$. □

Lemma A9.12. *Let R be an integrally closed domain, and let K be its field of fractions. If $f, g \in K[x]$ are monic and $fg \in R[x]$, then $f, g \in R[x]$.*

Proof. Let L/K be a field extension in which f and g split as products of monic linear factors, and let S be the integral closure of R in L. Since f and g are monic, their roots are elements of S. The coefficients of f and g are polynomials in these roots, hence elements of S, and they are also elements of K. But $K \cap S = R$ because R is integrally closed. □

Lemma A9.13. *Suppose that S is an integral extension of the integrally closed domain R, and R does not contain any zero divisors of S. For a nonzero $s \in S$, let $\varepsilon_s : R[x] \to S$ be the evaluation map $\varepsilon_s(f) = f(s)$. Then the kernel I of ε_s is a principal ideal generated by a monic polynomial.*

Proof. Since s is integral over R, it satisfies some monic polynomial $h \in R[x]$, so $I \neq (0)$. Let K be the field of fractions of R. Then $IK[x]$ is an ideal of $K[x]$, which is a PID, and $IK[x] \neq (0)$, so $IK[x]$ is generated by a polynomial f, which may be taken to be monic. We have $h = fg$ for some $g \in K[x]$, which must be monic, so from the last result $f, g \in R[x]$.

For any nonzero $j \in IK[x]$ there is a nonzero $r \in R$ that clears denominators, so that $rj \in IR[x] = I$, whence $rj(s) = 0$. Therefore (since r is not a zero divisor) $j(s) = 0$. In particular, $f(s) = 0$, which is to say that $f \in I$.

To complete the proof that $I = fR[x]$ we must show that f divides an arbitrary $p \in I$. Since $fK[x] = IK[x]$, there is a $q \in R[x]$ and a nonzero $r \in R$ such that $p = fq/r$. In view of our goal, we may assume that r is not a unit. Passing to residue classes in $(R/(r))[x]$, the equation $rp = fq$ becomes $0 = \tilde{f}\tilde{q}$. Since f is monic, $\tilde{f} \neq 0$, so $\tilde{q} = 0$, and thus $q/r \in R[x]$, as desired. □

Proof of Theorem A9.10. Let T be the set of all products rs of elements $r \in R \setminus P'$ and $s \in S \setminus Q$. Since S is an integral domain, T does not contain 0 and is consequently a multiplicative subset of S. Also, $R \setminus P'$ and $S \setminus Q$ are subsets of T because $1 \in (R \setminus P') \cap (S \setminus Q)$.

The bulk of the proof is concerned with showing that $P'S \cap T = \emptyset$, but first we explain why this implies the result. Basic facts concerning localization (Proposition A5.6) imply that $P'S_T$ is a proper ideal of S_T and is consequently contained in a maximal ideal $\mathfrak{m}$, which is $Q'S_T$ for some prime Q' of S that does not meet T, so that $Q' \cap R \subset P'$ and $Q' \subset Q$. Since $P' \subset \{\, s \in S : s/1 \in \mathfrak{m} \,\} = Q'$ we actually have $P' = Q' \cap R = \pi(Q')$, as desired.

Aiming at a contradiction, suppose that $rs \in P'S \cap T$ where $r \in R \setminus P'$ and $s \in S \setminus Q$. Because rs is an element of (the radical of) $P'S$, Lemma A9.11 implies that there is a monic polynomial $f(x) = x^m + a_{n-1}x^{m-1} + \cdots + a_0$ with $a_0, \ldots, a_{m-1} \in P'$ and $f(rs) = 0$. Let

$$g(x) = r^m x^m + r^{m-1} a_{m-1} x^{m-1} + \cdots + a_0.$$

Then $g(s) = f(rs) = 0$. By Lemma A9.13 there is a monic polynomial $h \in R[x]$ that generates the kernel of the evaluation map $\varepsilon_s : R[x] \to S$. Therefore $g = hj$ for some $j \in R[x]$. Passing to residue classes in the polynomial ring $(R/P')[x]$, we have $\tilde{g} = \tilde{h}\tilde{j}$. Since R/P' is an integral domain, $\tilde{g} = \tilde{r}^m x^m \neq 0$, where $\tilde{r}$ is the residue of r, and $\tilde{h}$ and $\tilde{j}$ are monomials. Since h is monic, $\tilde{h} = x^k$ for some $k \leq m$, so

$$h(x) = x^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_0$$

for some $\alpha_0, \ldots, \alpha_{k-1} \in P'$. Since $h(s) = 0$, Lemma A9.11 implies that s belongs to the radical of $P'S$. But $P'S \subset PS \subset Q$ and Q is prime, so $s \in Q$, contrary to assumption. □

A10 Associated Primes

Fix an R-module M. A prime P is *associated* to M if P is the annihilator of some element of M. Put another way, for $m \in M$ the ideal $\mathrm{Ann}(m)$ is an associated prime of M if and only if it is proper (i.e., $m \neq 0$) and prime. For a geometric interpretation one may imagine that R is the coordinate ring of an affine variety and M is an R-module of functions defined on that variety. The prime ideal consisting of those functions that vanish on a certain irreducible component of the variety is associated to M if some $m \in M$ is nonzero on a dense subset of that component and vanishes outside it.

The set of associated primes of M is denoted by $\mathrm{Ass}(M)$, or $\mathrm{Ass}_R(M)$ if more than one ring is under discussion. An automatic consequence of the definition is that $\mathrm{Ass}(N) \subset \mathrm{Ass}(M)$ whenever N is a submodule of M, and by extension whenever there is an injection mapping N into M.

Even though an ideal I is an R-module, by convention the set of *associated primes of* I is $\mathrm{Ass}(R/I)$. (One of the reasons this convention works rather well is that when $M \cong R$, the associated primes of M are the associated primes of the ideal (0).) Note that if P is a prime of R, then $\mathrm{Ass}(R/P) = \{P\}$ because P is the annihilator of every nonzero element of R/P. This simple example is implicitly the key building block of much of the following analysis.

Lemma A10.1. *A prime P is associated to M if and only if there is an injective homomorphism $R/P \to M$.*

Proof. When $P = \mathrm{Ann}(m)$ there is the injection $r + P \mapsto rm$. Conversely, if $R/P \to M$ is injective, then P is the annihilator of the image of $1 + P$. □

The next result is frequently applied to show that $\mathrm{Ass}(M) \neq \emptyset$ when R is Noetherian.

Lemma A10.2. *The maximal elements of $\{\, \mathrm{Ann}(m) : 0 \neq m \in M \,\}$ are associated primes.*

Proof. We show that if $\mathrm{Ann}(m)$ isn't prime, then it isn't maximal. If $ab \in \mathrm{Ann}(m)$ and $a, b \notin \mathrm{Ann}(m)$, then $bm \neq 0$ and $\mathrm{Ann}(bm)$ is a proper superset of $\mathrm{Ann}(m)$ because it contains a. $\square$

The set of zerodivisors of R is $\bigcup_{r \in R} \mathrm{Ann}(r)$, so this has the following simple consequence.

Corollary A10.3. *If R is Noetherian, the set of zerodivisors of M is the union of the maximal associated primes of* (0)*.*

When R is reduced, every zerodivisor is contained in a prime that is minimal over (0) (Lemma A2.10) so each associated prime is contained in the union of the minimal primes. For Noetherian reduced rings there will be a more precise result below.

The *support* indexmodule!support of of a M, denoted by $\mathrm{Supp}(M)$, is the set of prime ideals P such that $M_P \neq 0$.

Proposition A10.4. $\mathrm{Ass}(M) \subset \mathrm{Supp}(M)$.

Proof. Suppose that $P \in \mathrm{Ass}(M)$. The result above gives an exact sequence $0 \to R/P \to M$. Since R_P is flat (Proposition A6.8) applying the exact functor $- \otimes_R R_P$ to this and $0 \to P \to R \to R/P \to 0$ gives exact sequences $0 \to R/P \otimes_R R_P \to M_P$ and $0 \to P_P \to R_P \to R/P \otimes_R R_P \to 0$. Since P_P is a proper subset of R_P, $R/P \otimes_R R_P \neq 0$, and it follows that $M_P \neq 0$. $\square$

When M is finitely generated $\mathrm{Supp}(M)$ has a simple characterization.

Lemma A10.5. *If M is finitely generated, then* $\mathrm{Supp}(M)$ *is the set of primes that contain* $\mathrm{Ann}(M)$*.*

Proof. Fix a prime P. An element $m \in M$ goes to zero in M_P if and only if there is some $r \notin P$ with $rm = 0$. Since M is finitely generated, $M_P = 0$ if and only if there is such an element for each generator. Since the product of these elements is not in P, $M_P = 0$ if and only if there is some $r \notin P$ that takes all generators to 0, which is to say that there is some $r \in \mathrm{Ann}(M) \setminus P$. Equivalently, $P \in \mathrm{Supp}(M)$ if and only if $\mathrm{Ann}(M) \subset P$. $\square$

Let S be a multiplicatively closed subset of R. As we explained in Proposition A5.6, the prime ideals of $S^{-1}R$ are the $S^{-1}P$ where P is a prime of R that does not intersect S.

Proposition A10.6. *Suppose that R is Noetherian and P is a prime.*

(a) If $P = \mathrm{Ann}(m)$ and $P \cap S = \emptyset$, then $S^{-1}P = \mathrm{Ann}(m/1)$.

(b) If $m/s \neq 0$, $S^{-1}P = \mathrm{Ann}_{S^{-1}R}(m/s)$ then $P \cap S = \emptyset$ and $P = \mathrm{Ann}(tm)$ for some $t \in S$.

Consequently $\mathrm{Ass}_{S^{-1}R}(S^{-1}M) = \{\, S^{-1}P : P \in \mathrm{Ass}(M)$ *and* $P \cap S = \emptyset \,\}$.

Proof. (a) Clearly $S^{-1}P \subset \mathrm{Ann}_{S^{-1}R}(m/1)$. If $(a/s)(m/1) = 0$, then $atm = 0$ for some $t \in S$. We have $at \in P$ and thus $a \in P$ because $t \notin P$. Therefore $\mathrm{Ann}_{S^{-1}R}(m/1) \subset S^{-1}P$.

(b) If $s \in P \cap S$, then $S^{-1}P = (s/s) = S^{-1}R$, which is impossible (only $0/1$ is annihilated by all of $S^{-1}R$) so $P \cap S = \emptyset$. Since R is Noetherian, P has a finite set of generators $a_1, \dots, a_k$. For each i there is some $t_i \in S$ with $t_i a_i m = 0$. Let $t = t_1 \cdots t_k$. Now $P \subset \mathrm{Ann}(tm)$. If $atm = 0$, then $(sat/1) \cdot (m/s) = 0$, so that $sat \in P$ and consequently $a \in P$. Therefore $\mathrm{Ann}(tm) = P$. □

Theorem A10.7. *If R is Noetherian, then* $\mathrm{Supp}(M)$ *is the set of primes that contain elements of* $\mathrm{Ass}(M)$. *Consequently the minimal elements of* $\mathrm{Ass}(M)$ *and* $\mathrm{Supp}(M)$ *coincide.*

Proof. Suppose that $P \in \mathrm{Supp}(M)$. By definition $M_P \neq 0$. Since R is Noetherian, Lemma A10.2 implies that $\mathrm{Ass}(M_P) \neq \emptyset$. In view of Proposition A10.6 this means precisely that there is some $P' \in \mathrm{Ass}(M)$ with $P' \subset P$.

Conversely, if P is a prime that contains some associated prime $P' = \mathrm{Ann}(m)$, then $M_P \neq 0$ because $\mathrm{Ann}(m/1) = P'_P$ is a proper subset of R_P. □

An associated prime that is not minimal is said to be *embedded*. We summarize the consequences of Lemma A10.5 and the last result when their hypotheses hold.

Theorem A10.8. *If R is Noetherian and M is finitely generated, then* $\mathrm{Supp}(M)$ *is the set of primes that contain* $\mathrm{Ann}(M)$, *and* $\mathrm{Ass}(M)$ *is a subset that includes the primes that are minimal over* $\mathrm{Ann}(M)$.

The specific consequences of this for ideals are worth emphasizing.

Theorem A10.9. *If R is Noetherian and I is an ideal, then* $\mathrm{Supp}(R/I)$ *is the set of primes ideals containing I, and the set of primes associated to I is a subset that includes all the primes that are minimal over I.*

From Lemma A2.10 we know that if R is reduced, then the zerodivisors are contained in the union of the primes that are minimal over (0), and from Corollary A10.3 it follows that each prime associated to (0) is contained in this union. When R is Noetherian there is a more precise result whose proof uses the following very useful fact, which is known as *prime avoidance*.

Lemma A10.10 (Prime Avoidance)**.** *If $I, J_1, \dots, J_n$ are ideals, $I \subset \bigcup_j J_j$, and at most two of the J_j are not prime, then I is contained in some J_j.*

Proof. We use induction on n, with the case $n = 1$ being trivial. The induction hypothesis implies the claim if I is contained in any union of $n-1$ of the ideals $J_1, \dots, J_n$, so we may suppose that for each j there is an $x_j \in I \setminus \bigcup_{i \neq j} J_i$.

When $n = 2$ this gives a contradiction because $x_1 + x_2$ is in I but not in J_1 or J_2. If $n > 2$ we may suppose that J_1 is prime, so that $x_2 \cdots x_n \notin J_1$ and $x_1 + x_2 \cdots x_n$ is in I, but not in any J_j, which is again a contradiction. □

We rephrase prime avoidance to put it in the form in which it is most commonly applied later, and to explain its name.

Corollary A10.11. *If $J_1, \dots, J_n$ are ideals, at most two of which are not prime, and I is an ideal that is not contained in any J_j, then there is an $x \in I \setminus \bigcup J_j$.*

Proposition A10.12. *If R is Noetherian and reduced, then the primes associated to (0) are the minimal primes.*

Proof. Above we saw that each prime associated to (0) is contained in the union of the minimal primes, which are finite in number (Proposition A4.10). Prime avoidance implies that each prime associated to (0) must be contained in (and therefore coincide with) one of the minimal primes. But above we saw that each minimal prime is associated to (0). □

Suppose R is the coordinate ring of an affine algebraic set V and I is an ideal. Each prime that is minimal over I is the set of functions that vanish on one of the irreducible components of the set $W \subset V$ where all elements of I vanish. If I is radical, so that R/I is reduced, there are no other primes associated to I. If I is not radical and P is an embedded prime associated to I, then we think of the set $Z \subset W$ where all elements of P vanish as a set where the elements of I satisfy some additional condition, for instance that some partial derivative or Jacobean vanishes.

The remainder of this section studies the finiteness of $\mathrm{Ass}(M)$.

Lemma A10.13. *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, then*

$$\mathrm{Ass}(M') \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'').$$

Proof. The first inclusion is immediate. For the second, suppose $P = \mathrm{Ann}(m)$ for some $m \in M$. For any $a \in R \setminus P$ we have $am \neq 0$ and $\mathrm{Ann}(am) = P$ because $P \subset \mathrm{Ann}(am)$ automatically, and if $b \in \mathrm{Ann}(am)$, then $abm = 0$, which implies that $ab \in P$ and therefore $b \in P$ because P is prime. Since this is the case for every nonzero $am \in Rm$, either $Rm \cap M' = 0$ or $P \in \mathrm{Ass}(M')$. In the former case the restriction of $M \to M''$ to Rm is injective, and consequently P is the annihilator of the image of m in M''. □

Corollary A10.14. *For any R-modules M and N,* $\mathrm{Ass}(M \oplus N) = \mathrm{Ass}(M) \cup \mathrm{Ass}(N)$.

Proof. Apply the last result to the short exact sequences

$$0 \to M \to M \oplus N \to N \to 0 \quad \text{and} \quad 0 \to N \to M \oplus N \to M \to 0.$$

□

Proposition A10.15. *If R is Noetherian and M is finitely generated, then there is a chain of submodules $0 = M_0 \subset \cdots \subset M_n = M$ with each M_i/M_{i-1} isomorphic to R/P_i for some prime P_i.*

Proof. Lemma A10.2 implies that M has an associated prime, say $P_1 = \mathrm{Ann}(m_1)$. Then $M_1 = Rm_1$ is a submodule of M that is isomorphic to R/P_1. Similarly, if $P_2 = \mathrm{Ann}(m_2 + M_1)$ is an associated prime of M/M_1, then $M_2 = Rm_1 + Rm_2$ is a submodule with M_1/M_2 isomorphic to R/P_2, and so forth. Since M is Noetherian (Proposition A4.7) this construction eventually arrives at $M_n = M$. □

Theorem A10.16. *If R is Noetherian and M is finitely generated, then* $\mathrm{Ass}(M)$ *is finite.*

Proof. Let $0 = M_0 \subset \cdots \subset M_n = M$ be as in the last result. Applying Lemma A10.13 to the short exact sequence $0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$ gives $\mathrm{Ass}(M_i) \subset \mathrm{Ass}(M_{i-1}) \cup \{P_i\}$ for all i, so $\mathrm{Ass}(M) \subset \{P_1, \ldots, P_n\}$. □

A11 Primes Associated to Principal Ideals

This section develops a specific result cited by Serre. In this way we are led to briefly touch upon discrete valuation rings, and to apply related methods. Of the many topics not considered in this book, for the sake of minimality, the theory of discrete valuation rings and Dedekind domains is certainly one of the most important.

Let K be a field, and let $K^* = K \setminus \{0\}$. A *discrete valuation* for K is a function $v : K^* \to \mathbb{Z}$ such that for all $x, y \in K$:

(a) $v(xy) = v(x) + v(y)$;

(b) $v(x + y) \geq \min\{v(x), v(y)\}$.

For example, given a prime p, any nonzero $r \in \mathbb{Q}$ can be written as $p^{v(r)}s/t$ where s and t are integers that are not divisible by p. Similarly, if $P \in K[X]$ is irreducible, then any nonzero $f \in K(X)$ is $P^{v(f)}Q/R$ where Q and R are polynomials that are not divisible by P.

The *valuation ring* of v is

$$R_v = \{\, x \in K : v(x) \geq 0 \,\} \cup \{0\}.$$

Evidently (a) and (b) imply that R_v is closed under addition and multiplication of nonzero elements, and it is trivially closed under addition of 0 and multiplication by 0, so it is in fact a ring. Let

$$\mathfrak{m}_v = \{\, x \in R_v : v(x) > 0 \,\} \cup \{0\}.$$

It is equally obvious that $\mathfrak{m}_v$ is an ideal. By (a), $v(1) = v(1) + v(1)$, so $v(1) = 0$, and $v(x^{-1}) = -v(x)$, so every x such that $v(x) = 0$ is a unit of R_v. Consequently $\mathfrak{m}_v$ is the unique maximal ideal, so R_v is local. Furthermore, $\mathfrak{m}_v = (y)$ for any y such that $v(y) = 1$, because the ratio of any two such elements is a unit. In fact if I is any ideal and y is an element of smallest valuation, then $I = (y)$, so the ideals of R_v are the powers of $\mathfrak{m}_v$.

Turning the definitions around, R is a *discrete valuation ring* (DVR) if it is an integral domain and there is a valuation on its field of fractions for which it is the valuation ring.

Proposition A11.1. *If R is an integral domain, then it is a DVR if and only if it is a Noetherian local ring and $\mathfrak{m}$ is principal.*

Proof. That R is local and $\mathfrak{m}$ is principal when R is a DVR was argued above. Since the ideals are the powers of $\mathfrak{m}$, it is Noetherian.

Suppose that R is Noetherian and local, and $\mathfrak{m}$ is principal, say $\mathfrak{m} = (t)$. Nakayama's lemma implies that $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$, so for any nonzero $a \in R$ there is a maximal n such that $a \in \mathfrak{m}^n$. Then $a = ut^n$ for some $u \in R$, which must be a unit because $a \notin \mathfrak{m}^{n+1}$. Thus R is the disjoint union $\{0\} \cup \bigcup_{n \geq 0} t^n U$ where U is the group of units, and the quotient field is the disjoint union $\{0\} \cup \bigcup_{n \in \mathbb{Z}} t^n U$. Set $v(r) = n$ for $r \in t^n U$. It is easy to check that this function satisfies (a) and (b). $\square$

Theorem A11.2. *If R is a Noetherian normal domain and $I = (x)$ is a principal ideal, then any prime P associated to I is minimal over I.*

Proof. There is a $y \in R$ such that $P = (I : y)$. Let K be the quotient field of R, and let $a = y/x \in K$. For any nonzero $r, s \in R$, if $a = r/s$, then $ys = rx \in I$ and consequently $s \in P$. Therefore $a \notin R_P$. We also have $aP = \{\, py/x : py \in I \,\} \subset R$, so $aP_P \subset R_P$. There are now two possibilities for aP_P.

First suppose that aP_P is a proper ideal of R_P, and consequently $aP_P \subset P_P$. The Cayley-Hamilton theorem (applied to the endormorphism $p/s \mapsto ap/s$ of P_P) implies that $a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0$ for some n and $c_{n-1} = p_{n-1}/s_{n-1}, \ldots, c_0 = p_0/s_0 \in P_P$. Let $s = s_0 \cdots s_{n-1}$. Multiplying by s^n shows that sa satisfies a monic polynomial with coefficients in R, so $sa \in R$, because R is integrally closed, and consequently $a \in R_P$, but this is false.

Therefore $aP_P = R_P$. Let

$$P_P^{-1} = \{\, c \in K : cP_P \subset R_P \,\},$$

and let $P_P^{-1} \cdot P_P$ be the set of sums of products cb where $b \in P_P$ and $c \in P_P^{-1}$. Then the definition of P_P^{-1} gives $P_P^{-1} \cdot P_P \subset R_P$, and since $a \in P_P^{-1}$ we have $P_P^{-1} \cdot P_P = R_P$. Nakayama's lemma implies that $(P_P)^2 \neq P_P$. Choose a $t \in P_P \setminus (P_P)^2$. If $tP_P^{-1} \subset P_P$, then $tR_P = tP_P^{-1} \cdot P_P \subset (P_P)^2$, which is impossible because $t \notin (P_P)^2$. Therefore tP_P^{-1} is not contained in P_P, but it is an R_P-module contained in R_P, so $tP_P^{-1} = R_P$. Now $P_P = tP_P^{-1} \cdot P_P = tR_P$, so P_P is principal. Of course R_P is local with maximal ideal P_P, and it is Noetherian because R is. Therefore R_P is a DVR, and P_P and (0) are its only prime ideals. Correspondingly, (0) is the only prime properly contained in P. □

A12 Primary Decomposition

Primary decomposition is a weaker concept than prime factorization, and it is possible in a much wider range of circumstances. It was introduced for polynomial rings by Lasker (World Chess Champion and student of Hilbert) in 1905. The theory was subsequently generalized and simplified by Emmy Noether, using the ascending chain condition. Accordingly, throughout this section we assume that R is Noetherian.

Fix an R-module M. In preparation for the main definition we establish that various conditions are equivalent.

Proposition A12.1. *If M is finitely generated, then the following are equivalent:*

(a) $\mathrm{Ann}(m) \subset \mathrm{rad}(\mathrm{Ann}(M))$ *for all nonzero $m \in M$;*

(b) $\mathrm{Ass}(M) = \{\mathrm{rad}(\mathrm{Ann}(M))\}$*;*

(c) $\mathrm{Ass}(M)$ *is a singleton.*

Proof. Let $I = \mathrm{rad}(\mathrm{Ann}(M))$. First suppose that (a) holds. Each associated prime contains $\mathrm{Ann}(M)$, so it contains I because primes are radical, but (a) implies that it is contained in I. Thus I is the only possible associated prime, and Lemma A10.2 guarantees that there is at least one associated prime, so (a) implies (b). Obviously (b) implies (c).

Now suppose that (c) holds, so $\mathrm{Ass}(M) = \{P\}$ for some P. Theorem A10.8 implies that $\mathrm{Ass}(M)$ contains every prime that is minimal over $\mathrm{Ann}(M)$, and Proposition A2.5 implies that there is at least one such prime, so P is the unique such prime. Since I is the intersection of the primes that are minimal over $\mathrm{Ann}(M)$ (Corollary A2.9) we have $P = I$. If, for some m, $\mathrm{Ann}(m)$ was not contained in P, among such m there would be one for which $\mathrm{Ann}(m)$ was maximal and consequently (Lemma A10.2) a second element of $\mathrm{Ass}(M)$. Thus (c) implies (a). □

We say that a submodule $M' \subset M$ is *P-primary* if $\operatorname{Ass}(M/M') = \{P\}$, and that M' is *primary* if it is P-primary for some P. In view of this result, M' is primary if and only if each zero divisor of M/M' is in $\operatorname{rad}(\operatorname{Ann}(M/M'))$.

Lemma A12.2. *If M' is an irreducible submodule of M, it is primary.*

Proof. Since R is Noetherian, Lemma A10.2 implies that M/M' has at least one associated prime, so if M' is not primary there are distinct $P_1, P_2 \in \operatorname{Ass}(M/M')$. Lemma A10.1 implies that $\operatorname{Ass}(M/M')$ has one submodule M_1' that is isomorphic to R/P_1 and another M_2' that is isomorphic to R/P_2. For $i = 1, 2$ the annihilator of every nonzero element of M_i' is P_i, so $M_1' \cap M_2' = \{0\}$. The preimages M_1 and M_2 of these modules in M contain M' strictly, and $M_1 \cap M_2$ is the preimage of $M_1' \cap M_2'$, so $M_1 \cap M_2 = M'$, contradicting irreducibility. □

Fix a submodule $N \subset M$. A *primary decomposition* of N is a representation of N as a finite intersection $N = \bigcap_{i=1}^k M_i$ where each M_i is primary.

Theorem A12.3. *If M is finitely generated, then N has a primary decomposition.*

Proof. Lemma A4.8 states that N is a finite intersection $\bigcap_{i=1}^k M_i$ of irreducible submodules. □

As we mentioned earlier, a linear subspace of a vector space can be an intersection of codimension 1 subspaces in many different ways, so primary decompositions are far from unique. Nevertheless, the primes are uniquely determined if the decomposition is irredundant.

Theorem A12.4. *Suppose that M is finitely generated and $N = \bigcap_{i=1}^k M_i$ is a primary decomposition. If each M_i is P_i-primary, then $\operatorname{Ass}(M/N) \subset \{P_1, \ldots, P_k\}$. If the decomposition is irredundant, in the sense that $\bigcap_{i \neq j} M_i \neq 0$ for all j, then $\operatorname{Ass}(M/N) = \{P_1, \ldots, P_k\}$.*

Proof. The assertions concern the primary decomposition $0 = \bigcup_{i=1}^k M_i/N$ in M/N, so we may assume that $N = 0$. The natural map $M \to \oplus_i M/M_i$ is injective, so (Corollary A10.14) $\operatorname{Ass}(M) \subset \{P_1, \ldots, P_k\}$.

Now suppose that the decomposition is irredundant. For a given j we will show that $P_j \in \operatorname{Ass}(M)$. We have $\bigcap_{i \neq j} M_i \neq 0$ and $M_j \cap \bigcap_{i \neq j} M_i = 0$, so Lemma A1.2 gives

$$\bigcap_{i \neq j} M_i = \Big(\bigcap_{i \neq j} M_i\Big) \Big/ \Big(M_j \cap \bigcap_{i \neq j} M_i\Big) \cong \Big(\bigcap_{i \neq j} M_i + M_j\Big) / M_j \subset M/M_j.$$

Therefore $\operatorname{Ass}(\bigcap_{i \neq j} M_i) \subset \operatorname{Ass}(M/M_j) = \{P_j\}$, and since R is Noetherian, Lemma A10.2 implies that $\operatorname{Ass}(\bigcap_{i \neq j} M_i) \neq \emptyset$. □

An ideal Q is *primary* if it is a primary submodule of R. We have $\mathrm{Ann}(R/Q) = Q$ (note that $\mathrm{Ann}(1+Q) = Q$) so if Q is primary, then $P = \mathrm{rad}(Q)$ is a prime and $\mathrm{Ass}(R/Q) = \{P\}$. In this circumstance we say that Q is *P-primary.* Condition (a) of Proposition A12.1 gives the concrete condition that is most commonly given as the definition of a primary ideal: $r, s \in R \setminus Q$ and $rs \in Q$ (so $r \in \mathrm{Ann}(s+Q)$) then $r \in \mathrm{rad}(Q)$.

If R is a UFD and $a = p_1^{j_1} \cdots p_n^{j_k}$ is a prime factorization of a ring element a, then $(a) = \bigcap_{i=1}^k (p_i^{j_i})$, so primary decomposition is a generalization of prime factorization. However, an ideal Q need not be primary even if $P = \mathrm{rad}(Q)$ is prime and Q is a power of P. For example, let $R = K[X,Y,Z]/(XY - Z^2)$, let x, y, and z be the images of X, Y, and Z in R, and let $P = (x,z)$ and $Q = P^2$. Then $R/P \cong K[Y]$ is an integral domain, so P is prime. Of course $\mathrm{rad}(Q) \subset P$, and for any $f, g \in R$ we have $(fx + gz)^3 \in Q$, so $\mathrm{rad}(Q) = P$. An element of P has the form $f + zg$ where $f, g \in K[x,y]$ have no constant terms, and in each of their monomials the exponent of x is at least as large as the exponent of y. In addition all the monomials of an element of Q have total degree at least two. Therefore $xy = z^2 \in Q$, $x \notin Q$, and Q does not contain any power of y, so Q is not primary.

Nevertheless there is one positive result in this direction.

Proposition A12.5. *If Q is an ideal and $\mathrm{rad}(Q) = \mathfrak{m}$ is maximal, then Q is $\mathfrak{m}$-primary.*

Proof. The image of $\mathfrak{m}$ in R/Q is the nilradical of this quotient, by hypothesis, and a maximal ideal. Therefore every element of R/Q is either a unit or a nilpotent. In particular any zerodivisor is nilpotent, which is condition (a) above. □

It can easily happen that an ideal has a maximal ideal as its radical without being a power of the maximal ideal (e.g., $(X,Y) = \mathrm{rad}((X^2,Y))$ in $K[X,Y]$) so a P-primary ideal need not be a power of P.

We need one more result concerning primary decompositions of an ideal I. Recall that if $x \in R$, then $(I : x) = \{\, a \in R : ax \in I \,\}$.

Proposition A12.6. *Let $I = \bigcap_{i=1}^k Q_i$ be a minimal primary decomposition of the ideal I, and for each i let $P_i = \mathrm{rad}(Q_i)$. Then*

$$\bigcup_{i=1}^k P_i = \{\, x \in R : (I : x) \neq I \,\}.$$

Proof. In R/I each Q_i/I is P_i/I-primary, and the claim follows if we can show that $\bigcup_{i=1}^k P_i/I$ is the set of zerodivisors of R/I. Therefore we may assume that $I = (0)$. Let D be the set of zerodivisors.

Since the decomposition is minimal, Theorem A12.3 implies that each P_i is a prime associated to (0), so there is an x_i such $P_i = \mathrm{Ann}(x_i)$. Thus

$\bigcup_i P_i \subset D$. To prove the reverse inclusion observe that

$$D = \mathrm{rad}(D) = \mathrm{rad}\big(\bigcup_{x \neq 0}(0:x)\big) = \bigcup_{x \neq 0} \mathrm{rad}(0:x).$$

It suffices to show that $(0:x) \subset \bigcup_i P_i$ for a given $x \neq 0$. We have

$$(0:x) = \big(\bigcap Q_i : x\big) = \bigcap (Q_i : x) = \bigcap_{i : x \notin Q_i} (Q_i : x)$$

because $(Q_i : x) = R$ when $x \in Q_i$. There is at least one i such that $x \notin Q_i$ because $x \neq 0$. If $y \in (Q_i : x)$, then $xy \in Q_i$, so $y \in \mathrm{rad}(Q_i) = P_i$ because Q_i is P_i-primary. Therefore

$$\mathrm{rad}(0:x) = \bigcap_{i : x \notin Q_i} \mathrm{rad}(Q_i : x) \subset \bigcap_{i : x \notin Q_i} P_i \subset \bigcup_i P_i.$$

□

A13 Chains of Submodules

In this section we study finiteness conditions on submodules of a given R-module M. Our aim is to apply these results to R itself, and then in the next section to achieve a good understanding of Artinian rings.

An R-module N is *simple* if N and 0 are its only submodules. If this is the case, then $Rx = N$ for any nonzero $x \in N$, and $r + \mathrm{Ann}(x) \mapsto rx$ is an isomorphism between $R/\mathrm{Ann}(x)$ and N. Moreover, $\mathrm{Ann}(x)$ must be a maximal ideal, because otherwise N would have a submodule that was isomorphic to an ideal that contained $\mathrm{Ann}(x)$ properly. Thus a simple module N is a module that is isomorphic to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Conversely, each such $R/\mathfrak{m}$ is simple, obviously.

A *chain* of submodules of M is a finite sequence of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_n$$

with all containments strict; the *length* of this chain is n. The chain is a *composition series* for M if each M_j/M_{j+1} is simple and $M_n = 0$. The *length* of M is the least length of any composition series for M, or ∞ if M has no composition series.

Lemma A13.1. *If M has finite length and M' is a proper submodule, then* $\mathrm{length}(M') < \mathrm{length}(M)$.

Proof. Let $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ be a composition series for M, and for each $j = 0, \ldots, n$ let $N_j = M' \cap M_j$. Of course $N_n = 0$. For $j < n$ Lemma A1.2 gives an isomorphism

$$N_j/N_{j+1} = N_j/(N_j \cap M_{j+1}) \cong (N_j + M_{j+1})/M_{j+1} \subset M_j/M_{j+1}.$$

Consequently each N_j/N_{j+1} is either simple or zero, and we can obtain a composition series for M' by removing redundant modules, so the only way the result can fail is if each N_j/N_{j+1} is nonzero. But $N_0 = 0$, and if $N_{j+1} = M_{j+1}$ and $N_j \neq N_{j+1}$, then $N_j = M_j$ because N_j/M_{j+1} is contained in the simple module M_j/M_{j+1}. Therefore, if all N_j/N_{j+1} were nonzero, we would have $M' = M$, contrary to hypothesis. □

Via a simple application of the last result, we find that the length can be studied using any chain.

Proposition A13.2. *If M has finite length, then the length of any chain for M is not greater than the length of M. Consequently all composition series have the same length, and any chain refines to a composition series.*

Proof. We argue by induction on the length of M. When M has length zero the claim is trivial. Let $M = M_0 \supset M_1 \supset \cdots \supset M_n$ be a chain for M. The last result implies that the length of M_1 is less than the length of M, and the induction hypotheses implies that the length of M_1 is at least $n-1$.

If a chain is not a composition series, it is possible, by definition, to refine it by inserting an intermediate submodule between two of its terms. □

Proposition A13.3. *An R-module M has finite length if and only if it is both Noetherian and Artinian.*

Proof. Suppose that M is Noetherian and Artinian. Since it is Noetherian we may choose a maximal proper submodule M_1, a maximal proper submodule M_2 of M_1, and so forth. Since M is Artinian, this sequence terminates, necessarily at 0. Then $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ is a composition series for M.

Conversely if M has finite length, then Proposition A13.2 implies that it is both Noetherian and Artinian. □

There are now three results, which are mostly quite obvious, providing information about the simplest situations in which different lengths might be compared.

Lemma A13.4. *If I is an ideal contained in $\operatorname{Ann}(M)$, then the length of M as an R-module is finite if and only the length as an R/I-module is finite, and the two lengths are the same when both are finite.*

Proof. The composition series for M as an R-module are the composition series for M as an R/I-module. □

Lemma A13.5. *If $M_0 \supset \cdots \supset M_n$ is a chain of R-modules and M_0/M_n has finite length, then*

$$\operatorname{length}(M_0/M_n) = \sum_{i=1}^{n} \operatorname{length}(M_{i-1}/M_i).$$

Proof. The chain $M_0/M_n \supset \cdots M_{n-1}/M_n \supset 0$ refines to a composition series for M_0/M_n, and taking the quotient by M_i of the resulting chain from M_{i-1}/M_n to M_i/M_n gives a composition series for M_{i-1}/M_i. □

Proposition A13.6. *If $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is a short exact sequence of R-modules and L and N have finite length, then*

$$\text{length}(M) = \text{length}(L) + \text{length}(N).$$

Proof. If $L = L_0 \supset L_1 \supset \cdots \supset L_m$ and $N = N_0 \supset N_1 \supset \cdots \supset N_p$ are composition series for L and N, then

$$M = \beta^{-1}(N_0) \supset \cdots \supset \beta^{-1}(N_p) = \alpha(L_0) \supset \cdots \supset \alpha(L_m) = 0$$

is a composition series for M. □

A14 Artinian Rings

As we demonstrate below, Artinian rings are a special type of Noetherian ring, and consequently they arise rather infrequently. Nevertheless, we will need to have a good understanding of them.

Proposition A14.1. *If R is Artinian, then each of its primes is maximal.*

Proof. Let P be a prime ideal. The ideals of R/P are derived from the ideals of R that contain P, so R/P is Artinian. Let a be a nonzero element of R/P. Since R/P is Artinian, $(a^{n+1}) = (a^n)$ for some n, whence $a^n = ba^{n+1}$ for some b, and $ab = 1$ because R/P is an integral domain. Thus every nonzero element of R/P has an inverse, i.e., R/P is a field. □

Let N be the nilradical of R.

Proposition A14.2. *If R is Artinian, then N is nilpotent.*

Proof. The sequence $N \supset N^2 \supset \cdots$ is eventually constant, say with value I. Seeking a contradiction, suppose that $I \neq (0)$. The set of ideals I' such that $II' \neq 0$ is nonempty because I is an element, so (Lemma A4.2) it has a minimal element J. Let a be an element of J with $aI \neq 0$. Then $J = (a)$ because J is minimal. In addition, $(aI)I = aI^2 = aI \neq 0$, and $aI \subset (a)$, so minimality implies that $aI = (a)$. Therefore there is a $b \in I$ such that $ab = a$, and we have $a = ab = ab^2 = \cdots$, but $b \in I \subset N$, so $b^k = 0$ for large k, and consequently $a = 0$, which contradicts $aI \neq 0$. □

Noetherian rings also have this property.

Lemma A14.3. *If the nilradical N is finitely generated, it is nilpotent.*

Proof. Suppose that $N = (x_1, \dots, x_n)$. For each i there is some k_i such that $x_i^{k_1} = 0$. For each K, N^K is generated by those $x_1^{\ell_1} \cdots x_n^{\ell_n}$ with $\sum_i \ell_i = K$, and if $K = 1 - n + \sum_i k_i$, then $N^K = 0$. $\square$

Our next result concerning Artinian rings requires a basic fact.

Proposition A14.4. *If $I_1, \dots, I_n$ are ideals and P is a prime ideal that contains the product $I_1 \cdots I_n$, then P contains some I_j. Consequently P contains some I_j whenever $\bigcap_{j=1}^n I_j \subset P$.*

Proof. If there is an $i_j \in I_j \setminus P$ for each j, then $i_1 \cdots i_n \in I_1 \cdots I_n \setminus P$. $\square$

Proposition A14.5. *If R is Artinian, then it has only finitely many maximal ideals.*

Proof. The set of finite intersections of maximal ideals has a minimal element (Lemma A4.2) say $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$. Each maximal ideal $\mathfrak{m}$ contains this intersection, and Proposition A14.4 implies that $\mathfrak{m}$ contains some $\mathfrak{m}_j$. $\square$

Proposition A14.6. *If (0) is a product $\mathfrak{m}_1 \cdots \mathfrak{m}_n$ of (not necessarily distinct) maximal ideals, then R is Noetherian if and only if it is Artinian.*

Proof. We consider the descending sequence

$$R \supset \mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n = 0.$$

For each $j = 0, \dots, n-1$ an additive subgroup of $\mathfrak{m}_1 \cdots \mathfrak{m}_{j-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_j$ is closed under multiplication by scalars in R if and only if it is closed under multiplication by scalars in the field $R/\mathfrak{m}_j$, where the scalar multiplication is

$$(a + \mathfrak{m}_j)(m + \mathfrak{m}_1 \cdots \mathfrak{m}_j) = am + \mathfrak{m}_1 \cdots \mathfrak{m}_j.$$

Thus the submodules of the R-module $\mathfrak{m}_1 \cdots \mathfrak{m}_{j-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_j$ are the linear subspaces of $\mathfrak{m}_1 \cdots \mathfrak{m}_{j-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_j$ regarded as a vector space over $R/\mathfrak{m}_j$. A finite dimensional vector space is both Noetherian and Artinian, and an infinite dimensional vector space is neither, so $\mathfrak{m}_1 \cdots \mathfrak{m}_{j-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_j$ is a Noetherian module if and only if it is an Artinian module.

Evidently R is Noetherian if and only if each $\mathfrak{m}_1 \cdots \mathfrak{m}_j$ (including R, corresponding to $j = 0$) is Noetherian. Multiple applications of Proposition A4.4 to the short exact sequences

$$0 \to \mathfrak{m}_1 \cdots \mathfrak{m}_j \to \mathfrak{m}_1 \cdots \mathfrak{m}_{j-1} \to \mathfrak{m}_1 \cdots \mathfrak{m}_{j-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_j \to 0$$

show that each $\mathfrak{m}_1 \cdots \mathfrak{m}_j$ is Noetherian if and only if each $\mathfrak{m}_1 \cdots \mathfrak{m}_{j-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_j$ is Noetherian. This is so if and only if each $\mathfrak{m}_1 \cdots \mathfrak{m}_{j-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_j$ is Artinian, and multiple applications of Proposition A4.4 show that the latter condition holds if and only if each $\mathfrak{m}_1 \cdots \mathfrak{m}_j$ is Artinian, which is the case if and only if R is Artinian. $\square$

Recall that the Krull dimension of R is the maximum length r of a chain

$$P_r \supset P_{r-1} \supset \cdots \supset P_1 \supset P_0$$

of distinct prime ideals. In particular, a 0-dimensional ring is one whose prime ideals are all maximal.

Theorem A14.7. *The following are equivalent:*

(a) R is Noetherian and 0-dimensional;

(b) R is Artinian.

Proof. Suppose that R is Noetherian and zero dimensional. Then (Corollary A2.9) N is the intersection of the prime ideals that are minimal over it, which are finite in number (Proposition A4.11), and since R is 0-dimensional, each of these is maximal. Thus $\mathfrak{m}_1 \cdots \mathfrak{m}_k \subset \bigcap_j \mathfrak{m}_j = N$ for some maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_k$. Lemma A14.3 implies that N is nilpotent, so (0) is a finite product of maximal ideals, and Proposition A14.6 implies that R is Artinian.

Now suppose that R is Artinian. Above we saw that R is 0-dimensional. so every prime ideal is maximal. In addition we saw that there are finitely many of these, say $\mathfrak{m}_1, \dots, \mathfrak{m}_k$. Corollary A2.9 gives $N = \bigcap_i \mathfrak{m}_i$, and of course $\mathfrak{m}_1 \cdots \mathfrak{m}_k \subset \bigcap_i \mathfrak{m}_i$. Since N is nilpotent, (0) is a finite product of maximal ideals, so (Proposition A14.6) R is Noetherian. ☐

This result has several easy and important consequences.

Corollary A14.8. *A ring is Artinian if and only if it has finite length.*

Proof. A ring has finite length if and only if it is both Noetherian and Artinian (Proposition A13.3) which (by the last result) is the same as being Artinian. ☐

Corollary A14.9. *If R is Artinian, then an R-module M has finite length if and only if it is finitely generated.*

Proof. If M has finite length, then it is Noetherian, hence finitely generated, because any ascending chain of submodules refines to a composition series.

If M is generated by $x_1, \dots, x_r$, then there is a surjective homomorphism $\varphi : R^r \to M$. In the obvious way we may use a composition series for R to create a composition series for R^r (incidently demonstrating that the length of R^r is r times the length of R) and the image of this is (after removing redundant terms) a composition series for M. ☐

Lemma A14.10. *If R is Noetherian, I is an ideal, and R/I is Artinian, then R/I^n is Artinian for all n.*

Proof. Every prime ideal of R/I^n is of the form P/I^n where P is a prime of R that contains I^n. For any $r \in I$, $r^n \in P$, so $r \in P$. Therefore P contains I, and P/I is a prime of R/I. Since R/I is Artinian, P/I is a maximal ideal of R/I, so P is a maximal ideal of R and P/I^n is a maximal ideal of R/I^n. Thus every ideal of R/I^n is maximal, and of course R/I^n is Noetherian. $\square$

Lemma A14.11. *If R is Noetherian, $\mathfrak{m}$ is a maximal ideal, and I is $\mathfrak{m}$-primary, then R/I is Artinian.*

Proof. Since $\mathfrak{m} = \mathrm{rad}(I)$ is finitely generated, $\mathfrak{m}^n \subset I$ for some large n. Since $R/\mathfrak{m}$ is a field, it is Artinian, so the last result implies that $R/\mathfrak{m}^n$ is Artinian, and consequently R/I is Artinian. $\square$

Lemma A14.12. *If R is Noetherian, M is a finitely generated R-module, and $S = R/\mathrm{Ann}(M)$ is Artinian, then M has finite length.*

Proof. Since M is a finitely generated, it is a Noetherian R-module and an Artinian S-module (Proposition A4.6). The R-submodules and S-submodules of M are the same, so M is also Artinian as an R-module, and Proposition A13.3 implies the claim. $\square$

Chapter B

Elements of Homological Algebra

Chapters B, C, and D provide an introduction to homological algebra of the sort that figures prominently in algebraic geometry. The main features are the theories of projective and injective modules and resolutions, derived functors, and Tor and Ext. Because it is terse and restricted to a simplified setting, while still encompassing many of the key arguments, I hope this material will be useful for readers at an early stage of their study of homological algebra. In particular, it is a suitable preparation for, or companion to, an introductory course in homological algebra.

Initially this topic may be rather difficult to appreciate for a reader who is unacquainted with homology and cohomology as they arise in algebraic topology, or perhaps some other setting. On the other hand any first exposure to homology is likely to be a trial by fire to a greater or lesser extent, precisely because a certain amount of initially unmotivated technique must be absorbed before one can begin to understand what homology might be good for. Although it might seem sensible to recommend that novices first read some brief and elementary introduction that provides basic geometric motivation, such readings seem hard to find, and in fact it is not so easy to do better than FAC itself in terms of providing a concrete setting in which this material finds interesting applications.

In many respects the framework here is as simple as possible, consistent with encompassing Serre's applications, and reader should be aware that the motivation for this is minimization of clutter, rather than any logical simplification. On the contrary, the definitions and most of the results extend without much modification to settings that are more general in many respects, and there are interesting points that could be mentioned in many directions. A particularly important point is that for a large fraction of the results the given ring, or rings, need not be commutative, and many of the methods extend to functors with an arbitrary number of covariant and contravariant arguments.

There is one other direction of generalization that is, perhaps, worth mentioning at this point. An *abelian category* is a category such that the set of morphisms between any pair of objects has the structure of an abelian group, the composition law is bilinear, finite direct sums exist, every morphism has a kernel and cokernel, every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel, and (finally!) every morphism can be factored into an epimorphism followed by a monomorphism. This notion was introduced by Buchsbaum in his thesis, but the term "abelian category" was coined in Grothendieck's famous Tohoku paper, which is a close companion of FAC in the history of algebraic geometry. The objects of an abelian category need not be sets, so all the concepts mentioned above need to be defined categorically in terms of universal properties, and while the results below mostly generalize to this level of abstraction, the arguments (which refer to elements of the objects) do not. For a discussion of the (mostly quite demanding) methods of graduating to this level of the subject, see Hartshorne (1977), p. 203.

B1 The Five and Snake Lemmas

We continue to work with a fixed ring R that is assumed to be commutative with unit. A sequence

$$A \xrightarrow{\psi} B \xrightarrow{\varphi} C$$

of R-modules and R-module homomorphisms is said to be *exact at* B if $\operatorname{Im}\psi = \operatorname{Ker}\varphi$. A *short exact sequence* is a sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

that is exact at A, B, and C, which is to say that i is injective, the image of i is the kernel of p, and p is surjective.

The following results, which have many applications, are technical and thus a bit ill suited to be our first topic, but there is no better spot for them. A proof that refers to elements is sometimes called a *diagram chase*, for reasons that are well illustrated by the arguments below.

Lemma B1.1. *Suppose that the diagram*

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[d, "\beta"'] & C \arrow[r, "p"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"'] \\ B' \arrow[r, "i'"] & C' \arrow[r, "p'"] & D' \end{tikzcd}$$

commutes. Then:

(a) *If the top row is exact and β, δ, and i' are injective, then γ is injective.*

(b) *If the bottom row is exact and β, δ, and p are surjective, then γ is surjective.*

Proof. (a) If $c \in \operatorname{Ker}\gamma$, then $c \in \operatorname{Ker}(p' \circ \gamma) = \operatorname{Ker}(\delta \circ p)$, so $c \in \operatorname{Ker} p$ because δ is injective. Exactness gives a preimage $b \in B$ of c. Now $i'(\beta(b)) = \gamma(i(b)) = 0$, so $b = 0$ because i' and β are injective, and consequently $c = 0$.

(b) Suppose that $c' \in C'$. Since δ and p are surjective there is a $c \in C$ such that $p'(c') = \delta(p(c)) = p'(\gamma(c))$. Now $c' - \gamma(c)$ is in $\operatorname{Ker} p'$, so it has a preimage in B' which in turn has preimage $b \in B$, and $\gamma(i(b) + c) = i'(\beta(b)) + \gamma(c) = c'$. $\square$

Lemma B1.2. *(Five Lemma) Suppose that in the commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

the rows are exact. If α is surjective and β and δ are injective, then γ is injective. If ε is injective and β and δ are surjective, then γ is surjective.

Proof. Let $\tilde{B} = B/\operatorname{Im}(A \to B)$, $\tilde{B}' = B/\operatorname{Im}(A' \to B')$, $\tilde{D} = \operatorname{Im}(C \to D)$, and $\tilde{D}' = \operatorname{Im}(C' \to D')$, and let $\beta : \tilde{B} \to \tilde{B}'$ and $\tilde{\delta} : \tilde{D} \to \tilde{D}'$ be the induced maps. (Since the diagram commutes, β maps the image of A into the image of A', so $\tilde{\beta}$ is well defined, and δ maps the image of C into the image of C', so $\tilde{\delta}$ is well defined.) It is easy to see that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \tilde{B} \arrow[r] \arrow[d, "\tilde{\beta}"'] & C \arrow[r] \arrow[d, "\gamma"'] & \tilde{D} \arrow[r] \arrow[d, "\tilde{\delta}"'] & 0 \\ 0 \arrow[r] & \tilde{B}' \arrow[r] & C' \arrow[r] & \tilde{D}' \arrow[r] & 0 \end{tikzcd}$$

is commutative, with exact rows.

Suppose that α is surjective and β is injective. Consider a $b \in B$ such that $\beta(b) \in \operatorname{Im}(A' \to B')$, say $\beta(b)$ is the image of a'. The surjectivity of α gives an $a \in A$ such that $\alpha(a) = a'$. Since the diagram commutes and β is injective, a is mapped to b. Therefore $\tilde{\beta}$ is injective. If, in addition, δ is injective, then so is the restriction $\tilde{\delta}$, and (a) above implies that γ is injective.

Suppose that δ is surjective and ε is injective. If $d' \in \tilde{D}'$, then it is the image of some $c' \in C'$, and also the image of some $d \in D$. Its image in E' is 0 (exactness) so the image of d in E is zero (because ε is injective) so d is the image of some $c \in C$ (exactness). Therefore $\tilde{\delta}$ is surjective. If, in addition, β is surjective, then $\tilde{\beta}$ is surjective, and (b) above implies that γ is surjective. $\square$

Lemma B1.3. *(Snake Lemma) If the diagram*

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

commutes and has exact rows, then there is a homomorphism

$$\partial : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$$

defined by letting $\partial(c) = a' + \operatorname{Im}\alpha$ *for any* $a' \in A'$ *and* $b \in B$ *such that* $f'(a') = \beta(b)$ *and* $g(b) = c$. *The sequence*

$$\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{\partial} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma$$

is exact.

Proof. We first show that ∂ is well defined. For any $c \in \operatorname{Ker}\gamma$ it is always possible to find satisfactory a' and b: c has a preimage $b \in B$, by exactness, and $\beta(b)$ is in the kernel of g', which is the image of f'.

If we also have $g(\tilde{b}) = c$ and $f'(\tilde{a}') = \beta(\tilde{b})$, then exactness gives an $a \in A$ such that $f(a) = \tilde{b} - b$, and commutativity implies that $f'(\alpha(a)) = \beta(\tilde{b} - b) = f'(\tilde{a}' - a')$. Since f' is injective, $\alpha(a) = \tilde{a}' - a'$, so $\tilde{a}' + \operatorname{Im}\alpha = a' + \operatorname{Im}\alpha$. Thus the definition of $\partial(c)$ does not depend on the choice of b and a'.

Commutativity and exactness imply that $f(\operatorname{Ker}\alpha) \subset \operatorname{Ker}\beta$, $g(\operatorname{Ker}\beta) \subset \operatorname{Ker}\gamma$, $f'(\operatorname{Im}\alpha) \subset \operatorname{Im}\beta$, and $g'(\operatorname{Im}\beta) \subset \operatorname{Im}\gamma$. Therefore the sequence in question is well defined.

To prove exactness we begin by observing that the composition of any two successive maps in the sequence is zero. For $\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma$ and $\operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma$ this follows from the exactness of the rows. From the definition of ∂ it is evident that $\operatorname{Ker}\beta \to \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$ vanishes, and also that if $\partial(c) = a' + \operatorname{Im}\alpha$, then $f'(a') \in \operatorname{Im}\beta$, so that $\operatorname{Ker}\gamma \to \operatorname{Coker}\alpha \to \operatorname{Coker}\beta$ vanishes.

Suppose that b is in the kernel of $\operatorname{Ker}\beta \to \operatorname{Ker}\gamma$. Then it is of course in $\operatorname{Ker} g = \operatorname{Im} f$, and if $f(a) = b$, then commutativity implies that $f'(\alpha(a)) = 0$, after which exactness implies that $\alpha(a) = 0$, so b is in the image of $\operatorname{Ker}\alpha \to \operatorname{Ker}\beta$.

Suppose that c is in the kernel of ∂. Then there are b and a' with $g(b) = c$, $f'(a') = \beta(b)$, and $a' = \alpha(a)$ for some a. Then $g(b - f(a)) = c$ and $\beta(b - f(a)) = \beta(b) - f'(a') = 0$, so c is in the image of $\operatorname{Ker}\beta \to \operatorname{Ker}\gamma$.

Suppose that $a' + \operatorname{Im}\alpha$ is in the kernel of $\operatorname{Coker}\alpha \to \operatorname{Coker}\beta$. Then $f'(a') \in \operatorname{Im}\beta$, say $f'(a') = \beta(b)$, and the definition of ∂ gives $a' + \operatorname{Im}\alpha = \partial(g(b))$.

Suppose that $b' + \operatorname{Im}\beta$ is in the kernel of $\operatorname{Coker}\beta \to \operatorname{Coker}\gamma$, so that for some $c \in C$ we have $g'(b') = \gamma(c)$. By exactness $c = g(b)$ for some b, and since $g'(b' - \beta(b)) = g'(b') - \gamma(g(b)) = 0$ exactness gives an a' such that $f(a') = b' - \beta(b)$. In particular $b' + \operatorname{Im}\beta$ is the image of $a' + \operatorname{Im}\alpha$. $\square$

B2 Complexes, Homology, and Cohomology

This section defines homology in general, and then specializes to the homology of a chain complex and the cohomology of a cochain complex. We assume familiarity with the category concept, the definition of a functor, and the definition of a natural transformation between functors.

Center stage will be occupied throughout by the category of R-modules and R-module homomorphisms. Note that the category of abelian groups is the special case $R = \mathbb{Z}$. If Q is a second ring, a univariate functor T from the category of R-modules to the category of Q-modules is *additive* if

$$T(f + f') = T(f) + T(f')$$

for all R-modules M and N and all $f, f' \in \mathrm{Hom}_R(M, N)$. We will have no reason to consider functors that are not additive.

An *R-module with differentiation* is an R-module X endowed with a homomorphism $d : X \to X$ such that $d \circ d = 0$. It is possible that $d = 0$, in which case we say that the differentiation is *trivial*. If X' is a second R-module with differentiation, a *morphism* from X to X' is an R-module homomorphism $f : X \to X'$ such that $d' \circ f = f \circ d$, where $d' : X' \to X'$ is the differentiation operator of X'. It is easy to see that R-modules with differentiation and their morphisms constitute a category.

Let X be an R-module with differentiation, and let $B(X)$ and $Z(X)$ be the image and kernel of d. Elements of $B(X)$ are called *boundaries* and elements of $Z(X)$ are called *cycles*. The quotient module

$$H(X) = Z(X)/B(X)$$

is the *homology module* of X. If $c \in Z(X)$, then $[c] = c + B(X)$ is the *homology class* of c, and c is a *representative* of this class. Homology modules arise in diverse contexts in mathematics, and stand in certain systematic relationships with each other. Broadly speaking, homological algebra develops these relationships in a systematically organized body of concepts and results.

If $f : X \to X'$ is as above, the condition $d' \circ f = f \circ d$ implies that $f(B(X)) \subset B(X')$ and $f(Z(X)) \subset Z(X')$, so there is an induced homomorphism $H(f) : H(X) \to H(X')$ taking $[c] \in H(X)$ to $[f(c)]$. It is easy to check that H is a functor from the category of R-modules with differentiation to the category of R-modules. We will sometime regard H as a functor whose range is the category of R-modules with differentiation by adopting the convention that the differentiation in $H(X)$ is always trivial.

A *homotopy* between two morphisms $f, g : X \to X'$ is a homomorphism $s : X \to X'$ such that $g - f = d' \circ s + s \circ d$. When such an s exists we say that f and g are *homotopic*. When this is the case $H(f) = H(g)$ because

$$g(x) - f(x) = d'(s(x)) - s(d(x)) = d'(s(x))$$

is a boundary whenever $x \in Z(X)$.

In many applications the module with differentiation is graded, with the grading corresponding usually to some notion of dimension, and the differentiation operator passing between homogeneous submodules of adjacent dimension. A *chain complex* X is a diagram

$$X : \cdots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots$$

where each X_n is an R-module, each d_n is an R-module homomorphism, and $d_n \circ d_{n+1} = 0$ for all n, which is to say that $\operatorname{Im} d_{n+1} \subset \operatorname{Ker} d_n$. Frequently the groups are all 0 for negative indices, or outside some finite range, in which case the diagram adjusts to reflect that.

We can identify the chain complex X with the R-module with differentiation obtained by setting $X = \bigoplus_n X_n$. Then

$$B(X) = \bigoplus_n B_n(X), \quad Z(X) = \bigoplus_n Z_n(X), \quad H(X) = \bigoplus_n H_n(X)$$

where, for each integer n,

$$B_n(X) = \operatorname{Im}(d_{n+1}), \quad Z_n(X) = \operatorname{Ker}(d_n), \quad H_n(X) = Z_n(X)/B_n(X).$$

Since $d_n \circ d_{n+1} = 0$, $B_n(X)$ is a submodule of $Z_n(X)$, so this makes sense. If $x \in Z_n(X)$, the associated *homology class* is $[x] = x + B_n(X)$. The chain complex is said to be *exact at* X_n if $B_n(X) = Z_n(X)$, which is to say that $H_n(X) = 0$, and it is *exact* of *acyclic* if it is exact at every X_n. Evidently $H_n(X)$ measures the extent to which X fails to be exact at X_n.

If X and X' are chain complexes, a *chain map* $f : X \to X'$ is a collection of homomorphisms $\{f_n\}$ such that $f_n \circ d_{n+1} = d'_{n+1} \circ f_{n+1}$ for all n. If $g : X' \to X''$ is a second chain map, then $g \circ f$ is defined to be the collection of homomorphisms $\{g_n \circ f_n\}$. Evidently $g \circ f$ is also a chain map, and it is easy to see that there is a category of chain complexes and chain maps between them.

If $f : X \to X'$ is a chain map, where X' is a second chain complex

$$\cdots \xrightarrow{d_{n+2}} X'_{n+1} \xrightarrow{d_{n+1}} X'_n \xrightarrow{d_n} X'_{n-1} \xrightarrow{d_{n-1}} \cdots$$

then for each n, $f_n(B_n(X)) \subset B_n(X')$ and $f_n(Z_n(X)) \subset Z_n(X')$, so there are induced homomorphisms

$$H_n(f) : H_n(X) \to H_n(X'), \quad x + B_n(X) \mapsto f_n(x) + B_n(X').$$

It is now easy to see that H_n is a covariant functor from the category of chain complexes of R-modules to the category of R-modules: $H_n(\mathbf{1}_X) = \mathbf{1}_{H_n(X)}$ obviously, and if $f' : X' \to X''$ is a second chain map, then simply plugging in the definitions gives

$$H_n(f' \circ f) = H_n(f') \circ H_n(f).$$

In this subject it works well to keep notation spare, so when there seems to be little danger of confusion we will often write f in place of $H_n(f)$.

The homotopy concept specializes to chain complexes as follows. A *chain homotopy* between two chain maps $f, g : X \to X'$ is a collection of homomorphisms $s_n : X_n \to X'_{n+1}$ such that

$$g_n - f_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$$

for all n. When such a thing exists we say that f and g are *chain homotopic.* If this is the case, then the associated morphisms of R-modules are homotopic, so $\bigoplus_n H_n(f) = \bigoplus_n H_n(g)$ and consequently $H_n(f) = H_n(g)$ for each n. Concretely,

$$g_n(x) - f_n(x) = d'_{n+1}(s_n(x)) - s_{n-1}(d_n(x)) = d'_{n+1}(s_n(x))$$

is a boundary whenever $x \in Z_n(X)$.

The definitions related to cohomology are obvious modications of the definitions above for homology. A *cochain complex* is a diagram

$$X : \cdots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \cdots$$

where each X^n is an R-module, each d^n is an R-module homomorphism, and $d^n \circ d^{n-1} = 0$ for all n. That is, a cochain complex is just a chain complex with superscripts instead of subscripts and the directions of the morphisms reversed or, if you prefer, with the numerical ordering of the indices reversed.

For each $n \in \mathbb{Z}$ the modules of *n-coboundaries* and *n-cocycles* are $B^n(X) = \operatorname{Im}(d^{n-1})$ and $Z^n(X) = \operatorname{Ker}(d^n)$ respectively. The n^{th} *cohomology module* of X is

$$H^n(X) = Z^n(X)/B^n(X).$$

We say that X is *exact* at X^n if $H^n(X) = 0$. If it is exact at each X^n we say simply that X is *exact* or *acyclic.*

There is a category of cochain complexes and chain maps between them, where a chain map $f : X \to X'$ is a collection of homomorphisms $\{f^n\}$ such that $f^n \circ d^{n-1} = d'^{n-1} \circ f^{n-1}$ for all n. If $f : X \to X'$ is a chain map, then $f^n(B^n(X)) \subset B^n(X')$ and $f^n(Z^n(X)) \subset Z^n(X')$, so there is an induced homomorphism

$$H^n(f) : H^n(X) \to H^n(X'), \quad x + B^n(X) \mapsto f^n(x) + B^n(X').$$

It is easy to show that if $f' : X' \to X''$ is a second chain map, then

$$H^n(f' \circ f) = H^n(f') \circ H^n(f).$$

That is, H^n is a *covariant* functor from cochain complexes to R-modules. As with homology, we will often write f in place of $H^n(f)$ when confusion seems unlikely.

If X and X' are cochain complexes, a *chain homotopy* between two chain maps $f, g : X \to X'$ is a collection of homomorphisms $s_n : X_n \to X'_{n-1}$ such that

$$g_n - f_n = d'_{n-1} \circ s_n + s_{n+1} \circ d_n$$

for all n. As above, if f and g are chain homotopic, then $H^n(f) = H^n(g)$.

As this discussion reflects, at this level of generality cohomology differs from homology in only trivial ways, but in their applications homology and cohomology differ significantly, with cohomology being in some ways more potent. To some extent the difference arises out of a general tendency (or perhaps convention) that covariant functors going to the category of chain complexes are treated homologically, while contravariant functors are dealt with using cohomology.

B3 Direct and Inverse Limits

This section (which is included for reference, and is not part of the logical flow of this chapter) discusses direct limits of direct systems of R-modules with differentiation. The main result is that passage to direct limits commutes with homology. We will also present the definition of the inverse limit of an inverse system of R-modules with differentiation. It turns out that passage to inverse limits does *not* commute with homology.

Because the differentiation may be trivial, and R may be $\mathbb{Z}$, the discussion encompasses the definition of direct limits of direct systems of R-modules and abelian groups. In fact it will be evident that these definitions are in fact applicable to a wide range of categories. We will feel free to cite this section as a reference for direct and inverse limits in this broader sense.

A *directed set* is a pair $(I, \leq)$ in which I is a set and $\leq$ is a partial ordering of I such that for all $U, V \in I$ there is $W \in I$ with $U \leq W$ and $V \leq W$. The most obvious example is the natural numbers. In another important example the elements of I are the neighborhoods of a point in a topological space, and $\leq$ is reverse inclusion, so that $U \leq V$ when $V \subset U$. In the example that underlies the definition of sheaf cohomology in FAC, I is the set of open covers of a space and $U \leq V$ if V is finer than U, in the sense that every element of V is a subset of some element of U.

Suppose that for each $U \in I$ we have an R-module X_U with differentiation $d_U : X_U \to X_U$, and there is a system of morphisms $f^U_V : X_U \to X_V$ for those U and V with $U \leq V$ such that f^U_U is always the identity and $f^U_V \circ f^V_W = f^U_W$ whenever $U \leq V \leq W$. These objects constitute a *direct system* of R-modules with differentiation.

We say that $x_U \in X_U$ and $x_V \in X_V$ are *equivalent* if $f^U_W(x_U) = f^V_W(x_V)$ for some W with $U \leq W$ and $V \leq W$. (You should think through the verification that equivalence is transitive.) Let $[x_U]$ denote the equivalence class of x_U. The *direct limit* (sometimes called the *inductive limit*) of the direct system is the set of all such equivalence classes. We define addition on this set by specifying that if $x_U \in X_U$, $x_V \in X_V$, and $U, V \leq W$, then

$$[x_U] + [x_V] := [f_W^U(x_U) + f_W^V(x_V)].$$

(Make sure you see why this definition is independent of the choice of representatives.) Scalar multiplication and differentiation are defined by requiring that

$$r[x_U] = [rx_U] \quad \text{and} \quad d([x_U]) = [d_U(x_U)].$$

It is easy to verify that these operations make the direct limit an R-module with differentiation, which we denote by

$$\varinjlim X_U.$$

For each V let $f_V : X_V \to \varinjlim X_U$ be the homomorphism $x_V \mapsto [x_V]$.

The direct limit is characterized up to isomorphism by the following universal property: if Y is an R-module with differentiation and there is a system of homomorphisms $g_U : X_U \to Y$ such that $g_V \circ f_V^U = g_U$ whenever $U \leq V$, then there is a unique homomorphism $g : \varinjlim X_U \to Y$ such that $g_V = g \circ f_V$ for all V. To see this observe that, if it exists, g must satisfy $g([x_V]) = g_V(x_V)$ for all V and $x_V \in X_V$. There is no difficulty showing that this formula defines g unambiguously, in the sense that it does not depend on the choice of the representative x_V of the equivalence class $[x_V]$, and that this function is a homomorphism. In addition, if X' is a second R-module with differentiation for which there are homomorphisms $f'_U : X_U \to X'$ satisfying this condition, then this condition gives homomorphisms $\varinjlim X_U \to X'$ and $X' \to \varinjlim X_U$ that are (by virtue of the uniqueness requirement) inverse isomorphisms.

Now note that, because homology is a functor, the R-modules (with trivial differentiation) $H(X_U)$ and homomorphisms $H(f_V^U)$ are also a direct system.

Proposition B3.1. *If $\{X_U : U \in I\}$ and $\{f_V^U : U \leq V\}$ is a direct system of R-modules with differentiation, then (up to isomomorphism)*

$$\varinjlim H(X_U) = H(\varinjlim X_U).$$

Proof. Suppose we have an R-module with differentiation Y and a system of homomorphisms $g_U : H(X_U) \to Y$ such that $g_V \circ H(f_V^U) = g_U$ whenever $U \leq V$. It is straightforward to check that the formula

$$g([z_U] + B(X)) = g_U(z_U + B(X_U))$$

gives a well defined (in the sense of independence of choice of representatives) homomorphism $g : H(X) \to Y$ satisfying $g_U = g \circ H(f_U)$ for all U. $\square$

We now turn to the definitions of inverse systems and inverse limits. Suppose that for each $U \in I$ we have an R-module with differentiation X_U, and there is a system of homomorphisms $f_U^V : X_V \to X_U$ for those U and V with $U \leq V$ such that f_U^U is always the identity and $f_U^V \circ f_V^W = f_U^W$ whenever $U \leq V \leq W$. These objects constitute a *inverse system* of R-modules.

The *inverse limit* (or *projective limit*)

$$\varprojlim X_U$$

of this inverse system is the R-module with differentiation consisting of those $(x_U)_{U \in I} \in \prod_{U \in I} X_U$ such that $f_U^V(x_V) = x_U$ whenever $U \leq V$. (Addition, scalar multiplication, and differentiation are defined componentwise, of course.) For each V let $f^V : \varprojlim X_U \to X_V$ be the projection $(x_U)_{U \in I} \to x_V$.

The inverse limit can also be characterized by a universal property. Specifically, if Y is an R-module and there is a system of homomorphisms $g_U : Y \to X_U$ such that $f_U^V \circ g_V = g_U$ whenever $U \leq V$, then there is a unique homomorphism $g : Y \to \varprojlim X_U$ such that $g_V = g \circ f_V$ for all V. Of course $g(y) = (g_U(y))_{U \in I}$.

Since passage to inverse limits does not commute with homology, there is nothing to prove, but we can say a few words about why this fails. First, there is an obvious homomorphism $B(\varprojlim X_U) \to \varprojlim B(X_U)$ that is easily seen to be injective, but may fail to be surjective. Second, suppose that for each U, Y_U is a submodule of X_U, and that $f_U^V(Y_V) \subset Y_U$ whenever $U \leq V$. (If we like we can require that $d_U(Y_U) \subset Y_U$ for all U, but this doesn't have anything to do with the point.) Then there is a natural homomorphism $\varprojlim X_U / \varprojlim Y_U \to \varprojlim X_U / Y_U$ that is easily seen to be injective, but again it may fail to be surjective.

B4 The Long Exact Sequence

Long exact sequences probably strike the uninitiated as a bit baroque, but one quickly becomes accustomed to them because they are very useful in computing many things, and figure in axiomatic characterizations of various concepts. We first develop the idea in the context of modules with differentiation. This is more general, and in addition, for certain topics it presents a framework that is simpler, notationally and in other senses. In this way we will benefit from a general result to support this style of analysis.

We say that a diagram

$$\begin{tikzcd} A \arrow[dr, "f"'] & & C \arrow[ll, "i"'] \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

of R-modules and homomorphisms is an *exact triangle* if:

$$\operatorname{Im} f = \operatorname{Ker} g, \quad \operatorname{Im} g = \operatorname{Ker} h, \quad \operatorname{Im} h = \operatorname{Ker} f.$$

Proposition B4.1. *If* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *is a short exact sequence of* R*-modules with differentiation, there is an exact triangle*

$$\begin{tikzcd} H(A) \arrow[dr, "H(i)"] & & H(C) \arrow[ll, "\Delta"'] \\ & H(B). \arrow[ur, "H(p)"] & \end{tikzcd}$$

in which the connecting homomorphism Δ *has the following description: the image of* $[c] \in H(C)$ *in* $H(A)$ *is the homology class* $[a]$ *of* $a = i^{-1}(d(b))$ *for some* $b \in p^{-1}(c)$.

Proof. The snake lemma, applied to the diagram

$$\begin{tikzcd} & A \arrow[r, "i"] \arrow[d, "d"'] & B \arrow[r, "p"] \arrow[d, "d"'] & C \arrow[r] \arrow[d, "d"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C & \end{tikzcd}$$

asserts that we can define $\partial : Z(C) \to A/B(A)$ by letting $\partial(c) = a + B(A)$ for some $a \in A$ and $b \in B$ such that $i(a) = d(b)$ and $p(b) = c$. We recall that this map is well defined because: 1) $p(d(b)) = d(p(b)) = p(c) = 0$, so $b \in i(A)$, and 2) if b' is another element of $p^{-1}(c)$, and $a' = i^{-1}(d(b'))$, then $b' - b = i(\hat{a})$ for some $\hat{a} \in A$ because $p(b' - b) = 0$, so $d(\hat{a}) = i^{-1}(d(i(\hat{a}))) = i^{-1}(d(b' - b)) = a' - a$, whence $a' + B(A) = a + B(A)$.

Moreover, the snake lemma asserts that the sequence

$$Z(A) \to Z(B) \to Z(C) \xrightarrow{\partial} A/B(A) \to B/B(B) \to C/B(C)$$

is exact. We claim that the first three maps induce maps of homology modules, and that we can restrict the final three spaces to homology modules, thereby obtaining.

$$H(A) \to H(B) \to H(C) \xrightarrow{\Delta} H(A) \to H(B) \to H(C).$$

Specifically, the derived sequence is well defined because

$$i(B(A)) \subset B(B), \qquad p(B(B)) \subset B(C), \qquad \partial(B(C)) = 0,$$

$$\partial(Z(C)) \subset H(A), \quad i(Z(A)) \subset Z(B), \quad \text{and} \quad p(Z(B)) \subset Z(C).$$

The first, second, fifth, and sixth containment are consequences of commutativity. To see the third observe that if $c = d(c')$, $p(b') = c'$, then $p(d(b')) = c$ and $d(d(b')) = 0$, so that we can take $b = d(b')$ and $a = 0$ in the definition of $\partial(c)$. For the fourth, observe that if $i(a) = d(b)$ and $p(b) = c \in Z(C)$, then $d(a) = i^{-1}(d(i(a))) = i^{-1}(d^2(b)) = 0$.

Clearly the composition of any two successive maps in the derived sequence is zero, so exactness follows from the following observations:

- If $b \in Z(B)$ and $p(b) \in B(C)$, say because $p(b) = d(c')$, then there is b' such that $p(b') = c'$. Since $p(b - d(b')) = 0$ there is an $a \in A$ such that $i(a) = b - d(b')$. Now $i(d(a)) = d(i(a)) = 0$, so $a \in Z(A)$ because i is injective, and we have $[b] = i([a])$.
- If $c \in Z(C)$, $p(b) = c$, $i(a) = d(b)$, and $a = d(a')$, then $d(b - i(a')) = 0$ and $[c] = p([b - i(a)])$.
- If $a \in Z(A)$ and $i(a) \in B(B)$, say because $i(a) = d(b)$, then $a = \partial(p(b))$. In addition, $p(b) \in Z(C)$ because $d(p(b)) = p(d(b)) = p(i(a)) = 0$, so $[a] = \Delta([p(b)])$.

□

There is a category of exact triangles in which a morphism from the triangle above to a second one

$$\begin{tikzcd} A' \arrow[dr, "f'"] & & C' \arrow[ll, "h'"'] \\ & B' \arrow[ur, "g'"] & \end{tikzcd}$$

is a triple of homomorphisms $\alpha : A \to A'$, $\beta : B \to B'$, $\gamma : C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\gamma"'] & A \arrow[d, "\alpha"'] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'. \end{tikzcd}$$

Theorem B4.2. *There is a covariant functor from the category of short exact sequences of R-modules with differentiation to the category of exact triangles that takes $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ to*

$$\begin{tikzcd} H(A) \arrow[dr, "H(i)"] & & H(C) \arrow[ll, "\Delta"'] \\ & H(B). \arrow[ur, "H(p)"] & \end{tikzcd}$$

The functor maps a morphism

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

of short exact sequences of R-modules with differentiation to the morphism of exact triangles given by $H(f)$, $H(g)$, and $H(h)$.

Proof. What remains after the last result is to show that we have defined a functor from short exact sequences of R-modules with differentiation to exact triangles. Much of this follows from the fact that H is a functor. First, the image of a composition of morphisms in the domain category agrees with the composition of the image morphisms. We have to show that for a given morphism of short exact sequences, the diagram

$$\begin{tikzcd} H(A) \arrow[r, "i"] \arrow[d, "f"'] & H(B) \arrow[r, "p"] \arrow[d, "g"'] & H(C) \arrow[r, "\Delta"] \arrow[d, "h"'] & H(A) \arrow[d, "f"'] \\ H(A') \arrow[r, "i'"] & H(B') \arrow[r, "p'"] & H(C') \arrow[r, "\Delta'"] & H(A'). \end{tikzcd}$$

commutes. For the first two squares this follows from the commutativity of the corresponding diagram for the given morphism and the fact that H is a functor.

We now show that the third square commutes. Suppose that $c \in Z(C)$, with $p(b) = c$. Then $p(d(b)) = d(p(b)) = 0$, so $d(b) = i(a)$ for some $a \in A$. Let $a' = f(a)$, $b' = g(b)$, and $c' = h(c)$. Then a' and c' are cycles because a and c are. In addition,

$$p'(b') = p'(g(b)) = h(p(b)) = h(c) = c'$$

and

$$i'(a') = i'(f(a)) = g(i(a)) = g(d(b)) = d'(g(b)) = d'(b'),$$

so $[a'] = \Delta'([c'])$. Therefore

$$f(\Delta([c])) = f([a]) = [a'] = \Delta'([c']) = \Delta'(h([c])).$$

□

The next two results are immediate consequences of Theorem B4.2 once one recognizes that the images of $H_n(C)$ and $H^n(C)$ under the respective connecting homomorphisms are contained in $H_{n-1}(A)$ and $H^{n+1}(A)$ respectively, by virtue of the connecting homomorphism's description.

Proposition B4.3. *There is a functor from the category of short exact sequences* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *of chain complexes to the category of exact sequences of R-modules with the following properties:*

(a) *The image of* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *is the* long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i} H_n(B) \xrightarrow{p} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow \cdots$$

where the connecting homomorphism ∂_n *has the following description: the image of* $[c] \in H_n(C)$ *in* $H_{n-1}(A)$ *is the cohomology class of the preimage, in* A_{n-1}*, of the image, in* B_{n-1}*, of a preimage of* c *in* B_n.

(b) *The functor maps a morphism*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences of chain complexes to the chain map

$$\begin{tikzcd}
\to & H_n(A) \arrow[r] \arrow[d, "f"'] & H_n(B) \arrow[r] \arrow[d, "g"'] & H_n(C) \arrow[r, "\partial_n"] \arrow[d, "h"'] & H_{n-1}(A) \arrow[d, "f"'] & \to \\
\to & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial'_n"] & H_{n-1}(A') & \to
\end{tikzcd}$$

Proposition B4.4. *There is a functor from the category of short exact sequences* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *of cochain complexes to the category of exact sequences of R-modules with the following properties:*

(a) *The image of* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *is the* long exact sequence

$$\cdots \longrightarrow H^n(A) \xrightarrow{i} H^n(B) \xrightarrow{p} H^n(C) \xrightarrow{\partial^n} H^{n+1}(A) \longrightarrow \cdots$$

where the connecting homomorphism ∂^n *has the following description: the image of $[c] \in H^n(C)$ in $H^{n+1}(A)$ is the cohomology class of the preimage, in A^{n+1}, of the image, in B^{n+1}, of a preimage of c in B^n.*

(b) *The functor maps a morphism*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences of cochain complexes to the chain map

$$\begin{tikzcd}
\to & H^n(A) \arrow[r] \arrow[d, "f"'] & H^n(B) \arrow[r] \arrow[d, "g"'] & H^n(C) \arrow[r, "\partial^n"] \arrow[d, "h"'] & H^{n+1}(A) \arrow[d, "f"'] & \to \\
\to & H^n(A') \arrow[r] & H^n(B') \arrow[r] & H^n(C') \arrow[r, "\partial'^n"] & H^{n+1}(A') & \to
\end{tikzcd}$$

B5 Left and Right Exact Functors

In this section we will study ways in which exactness may be preserved, in whole or in part, when an additive functor is applied to a short exact sequence. We begin with a situation that is guaranteed to work out nicely.

Lemma B5.1. *For a short exact sequence* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *the following are equivalent:*

(a) there is a homomorphism $j : B \to A$ *such that* $j \circ i = \mathbf{1}_A$*;*

(b) there is a homomorphism $q : C \to B$ *such that* $p \circ q = \mathbf{1}_C$*;*

(c) there are homomorphisms $j : B \to A$ *and* $q : C \to B$ *such that* $i \circ j + q \circ p = \mathbf{1}_B$*.*

When these conditions hold we say that the sequence *splits*, and that j and q are *splitting maps*.

Proof. We first show that (a) and (b) each imply (c). If (a) holds, then $j \circ (\mathbf{1}_B - i \circ j) = j - \mathbf{1}_A \circ j = 0$, so the image of $\mathbf{1}_B - i \circ j$ cannot contain any nonzero element of the image of i, which is the kernel of p. Thus the restriction of p to the image of $\mathbf{1}_B - i \circ j$ is injective, and it is surjective because $p \circ (\mathbf{1}_B - i \circ j) = p$. Let q be its inverse. Now

$$\mathbf{1}_B - i \circ j = q \circ p \circ (\mathbf{1}_B - i \circ j) = q \circ p.$$

If (b) holds, then $p \circ (\mathbf{1}_B - q \circ p) = 0$, so the image of $\mathbf{1}_B - q \circ p$ is contained in the image of i, and since i is injective we can define j to be the composition of $\mathbf{1}_B - q \circ p$ with the inverse of i, so that $i \circ j = \mathbf{1}_B - q \circ p$.

Now suppose that (c) holds. Then

$$i = (i \circ j + q \circ p) \circ i = i \circ (j \circ i) \text{ and } p = p \circ (i \circ j + q \circ p) = (p \circ q) \circ p.$$

Since i is injective, $j \circ i = \mathbf{1}_A$, and since p is surjective, $p \circ q = \mathbf{1}_C$. That is, (a) and (b) hold. □

Fix a second ring Q, and let T be an additive covariant functor from the category of R-modules to the category of Q-modules.

Proposition B5.2. *If the exact sequence* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *splits, then* $0 \to T(A) \xrightarrow{T(i)} T(B) \xrightarrow{T(p)} T(C) \to 0$ *is an exact sequence that splits.*

Proof. Of course $T(p) \circ T(i) = T(p \circ i) = 0$, which is to say that the image of $T(i)$ is contained in the kernel of $T(p)$. Let j and q be splitting maps. Then $T(j) \circ T(i) = \mathbf{1}_{T(A)}$ and $T(p) \circ T(q) = \mathbf{1}_{T(C)}$, so $T(i)$ is injective and $T(p)$ is surjective. We also have

$$\mathbf{1}_{T(B)} = T(\mathbf{1}_B) = T(i \circ j + q \circ p) = T(i) \circ T(j) + T(q) \circ T(p).$$

This will show that the sequence splits, but first we need to observe that it also implies that the kernel of $T(p)$ is contained in the image of $T(i)$, so that the sequence is exact. □

We say that T is *exact* if $0 \to T(A) \to T(B) \to T(C) \to 0$ is exact whenever $0 \to A \to B \to C \to 0$ is exact. A number of important functors are exact, perhaps most notably localization (Proposition A5.1), but the properties of such functors are not an important motivation for homological algebra, perhaps precisely because they are unproblematic.

Instead, homological algebra is principally concerned with certain functors that satisfy weaker conditions, specifically the bifunctors $\mathrm{Hom}_R(-,-)$ and $-\otimes_R -$ that will be introduced in Section B7. We say that T is *left exact* (resp. *right exact*) if $0 \to T(A) \to T(B) \to T(C)$ (resp. $T(A) \to T(B) \to T(C) \to 0$) is exact whenever $0 \to A \to B \to C \to 0$ is exact. Note that T is exact if and only if it is both left and right exact.

Exactness, and left and right exactness, imply superficially stronger conditions.

Lemma B5.3. *The covariant functor T is exact (resp. left exact, right exact) if and only if $T(A) \to T(B) \to T(C)$ (resp. $0 \to T(A) \to T(B) \to T(C)$, $T(A) \to T(B) \to T(C) \to 0$) is exact whenever $A \to B \to C$ (resp. $0 \to A \to B \to C$, $A \to B \to C \to 0$) is exact.*

Proof. In all three cases the 'if' is immediate. To prove the converses let $A \to B \to C$ be exact. If we set

$$A' = \mathrm{Ker}(A \to B), \quad B' = \mathrm{Im}(A \to B) = \mathrm{Ker}(B \to C), \quad C' = \mathrm{Im}(B \to C),$$

then the sequences

$$0 \to A' \to A \to B' \to 0, \quad 0 \to B' \to B \to C' \to 0,$$

and

$$0 \to C' \to C \to C/C' \to 0$$

are all exact.

First suppose that T is exact. Application of T gives three short exact sequences, from which we extract, respectively,

$$T(A) \to T(B') \to 0, \quad T(B') \to T(B) \to T(C'),$$

and

$$0 \to T(C') \to T(C).$$

In view of the second of these, it suffices to show that the image of $T(A) \to T(B)$ is the image of $T(B') \to T(B)$ and the kernel of $T(B) \to T(C)$ is the kernel of $T(B) \to T(C')$. Since $A \to B$ is the composition $A \to B' \to B$, $T(A) \to T(B)$ is the composition $T(A) \to T(B') \to T(B)$, so the first of the derived sequences implies that the image of $T(A) \to T(B)$ is the image of $T(B') \to T(B)$. Similarly, $B \to C$ is the composition $B \to C' \to C$, so the third implies that the kernel of $T(B) \to T(C)$ is the kernel of $T(B) \to T(C')$.

Now suppose that T is left exact. We may suppose that $0 \to A \to B \to C$ is exact, which is to say that $A' = 0$. Applying T to the exact sequences

$$0 \to A \to B' \to 0, \quad 0 \to B' \to B \to C' \to 0, \quad 0 \to C' \to C \to C/C' \to 0,$$

left exactness is enough to give the same exact sequences as above, and the argument of the last paragraph shows that $T(A) \to T(B) \to T(C)$ is exact. In addition, we actually have exactness of $0 \to T(A) \to T(B')$ and $0 \to T(B') \to T(B)$, so the composition $T(A) \to T(B') \to T(B)$ is injective.

Similarly, if T is right exact we may suppose that $A \to B \to C \to 0$ is exact, which is to say that $C' = C$. Applying T to the exact sequences

$$0 \to A \to B' \to B \to 0, \quad 0 \to B' \to B \to C' \to 0, \quad 0 \to C' \to C \to 0,$$

we again obtain the exact sequences above and can then show that $T(A) \to T(B) \to T(C)$ is exact. Furthermore, since $C' = C$, right exactness implies that $T(B) \to T(C) \to 0$ is exact. $\square$

Now let U be a contravariant functor. The proof of the following assertion is dual to the corresponding argument above, hence omitted.

Proposition B5.4. *If the exact sequence* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *splits, then* $0 \to U(C) \xrightarrow{p} U(B) \xrightarrow{i} U(A) \to 0$ *is an exact sequence that splits.*

We say that U is *exact* if $0 \to U(C) \to U(B) \to U(A) \to 0$ is exact whenever $0 \to A \to B \to C \to 0$ is exact. If, for any exact sequence of R-modules $0 \to A \to B \to C \to 0$, the sequence $0 \to U(C) \to U(B) \to U(A)$ $(U(C) \to U(B) \to U(A) \to 0)$ is exact, then we say that U is a *left exact* (*right exact*). The proofs of the following assertions are again dual to those above and omitted.

Lemma B5.5. *A contravariant functor U is exact (resp. left exact, right exact) if and only if $U(C) \to U(B) \to U(A)$ (resp. $0 \to U(C) \to U(B) \to U(A)$, $U(C) \to U(B) \to U(A) \to 0$) is exact whenever $A \to B \to C$ (resp. $A \to B \to C \to 0$, $0 \to A \to B \to C$) is exact.*

One important consequence of this result and Lemma B5.3 above is that half exactness can be inherited by compositions of functors. For example, if $Q = R$, T is left exact, and U is right exact, then the composition $M \mapsto U(T(M))$ is a right exact contravariant functor.

B6 The Two Main Bifunctors

A *bifunctor* F whose first argument takes values in the category $\mathcal{A}$, whose second argument takes values in the category $\mathcal{B}$, and whose range is the category $\mathcal{C}$, associates a functor $F(A,–)$ from $\mathcal{B}$ to $\mathcal{C}$ with each object in $\mathcal{A}$ and

and a functor $F(-,B)$ from $\mathcal{A}$ to $\mathcal{C}$ with each object in $\mathcal{B}$. These must agree in the sense that the object assigned to B by $F(A,-)$ must be the same as the object assigned to A by $F(-,B)$. We require that either $F(A,-)$ is covariant for all A, or it is covariant for all A, and similarly for the second argument. Thus there are four types of bifunctors, according to the variances. If F is covariant in both variables, then we insist that

$$F(f,B') \circ F(A,g) = F(A',g) \circ F(f,B)$$

for all morphisms $f : A \to A'$ and $g : B \to B'$, and we define $F(f,g)$ to be the common value of these two compositions. For each of the three other types of bifunctor the obvious analogous requirement is imposed. Trifunctors, quadrafunctors, and so forth, are defined analogously, but will rarely figure in our discussion. We say that F is *additive* if, for all A and B, $F(A,-)$ and $F(-,B)$ are additive univariate functors. We will have no reason to consider bifunctors that are not additive. indexadditive bifunctor

If F is a bifunctor taking pairs of R-modules to Q-modules, we say that F is *right exact* if $F(M,-)$ and $F(-,N)$ are right exact for all M and N, and F is *left exact* is $F(M,-)$ and $F(-,N)$ are always left exact. In these definitions F could be covariant or contravariant in either variable. It is possible that F could be right exact in one variable and left exact in another, but such functors are uncommon in practice, so we do not specify terminology for these cases.

The central focus of homological algebra is a pair of bifunctors whose domain and range categories are categories of modules over a ring. For R-modules M and N, $\mathrm{Hom}_R(M,N)$ is the set of R-homomorphisms from M to N. For a given R-module M, there is an additive covariant functor $\mathrm{Hom}_R(M,-)$ where, if $g : N \to N'$ is a morphism,

$$\mathrm{Hom}_R(M,g) : \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N')$$

is the function $\varphi \mapsto g \circ \varphi$. Similarly, there is an additive contravariant functor $\mathrm{Hom}_R(-,M)$ where, for $f \in \mathrm{Hom}_R(M,M')$,

$$\mathrm{Hom}_R(f,N) : \mathrm{Hom}_R(M',N) \to \mathrm{Hom}_R(M,N)$$

is the function $\varphi \mapsto \varphi \circ f$. Evidently

$$\mathrm{Hom}_R(M',g) \circ \mathrm{Hom}_R(f,N) = \mathrm{Hom}_R(f,N') \circ \mathrm{Hom}_R(M,g)$$

because they both take φ to $f \circ \varphi \circ g$, so $\mathrm{Hom}_R(-,-)$ is a bifunctor.

Proposition B6.1. $\mathrm{Hom}_R(-,-)$ *is left exact.*

Proof. First consider a short exact sequence $0 \to N' \xrightarrow{i} N \xrightarrow{p} N'' \to 0$. For an R-module M the sequence

$$0 \to \mathrm{Hom}_R(M,N') \xrightarrow{i'} \mathrm{Hom}_R(M,N) \xrightarrow{p'} \mathrm{Hom}_R(M,N'')$$

is exact at $\mathrm{Hom}_R(M, N')$ because the injectivity of i implies that $i \circ f \neq 0$ whenever $0 \neq f \in \mathrm{Hom}_R(M, N')$. Since $\mathrm{Hom}_R(M, –)$ is a functor, $p \circ i = 0$ implies $p' \circ i' = 0$. On the other hand, if $g \in \mathrm{Hom}_R(M, N)$ and $p'(g) = p \circ g = 0$, then the given exactness implies that $g = i \circ f$ for some $f \in \mathrm{Hom}_R(M, N')$.

Now consider a short exact sequence $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ and an R-module N. The sequence

$$0 \to \mathrm{Hom}_R(M'', N) \xrightarrow{p'} \mathrm{Hom}_R(M, N) \xrightarrow{i'} \mathrm{Hom}_R(M', N)$$

is exact at $\mathrm{Hom}_R(M'', N)$ because the surjectivity of p implies that $h \circ p \neq 0$ whenever $0 \neq h \in \mathrm{Hom}_R(M'', N)$. Since $\mathrm{Hom}_R(–, N)$ is a functor, $p \circ i = 0$ implies $i' \circ p' = 0$. On the other hand, if $g \in \mathrm{Hom}_R(M, N)$ and $i'(g) = g \circ i = 0$, then the given exactness implies that $g = h \circ p$ for some $h \in \mathrm{Hom}_R(M'', N)$. □

The second main example is the tensor product, which has already been introduced in Chapter A. In addition to the discussion there, at this point it is only necessary to say that we did verify that $– \otimes_R –$ is a bifunctor (without using that term) and that this bifunctor is obviously additive.

Proposition B6.2. *$– \otimes_R –$ is right exact.*

Proof. By symmetry, it suffices to show that $M \otimes_R –$ is right exact. Consider a short exact sequence $0 \to N' \xrightarrow{i} N \xrightarrow{p} N'' \to 0$. The sequence

$$M \otimes_R N' \xrightarrow{i'} M \otimes_R N \xrightarrow{p'} M \otimes_R N'' \to 0$$

is exact at $M \otimes_R N''$ because the surjectivity of p implies that every $m \otimes n''$ is in the image of p'. Since $p \circ i = 0$ and $M \otimes_R –$ is a functor, $p' \circ i' = 0$.

It remains to show that $\mathrm{Ker}\, p' = \mathrm{Im}\, i'$. Since $p' \circ i' = 0$, p' induces a homomorphism $u : \mathrm{Coker}\, i' \to M \otimes N''$. If we can show that u is injective, we will be done.

Since $M \otimes_R \mathrm{Im}\, i \subset \mathrm{Im}\, i'$, there is a surjection $v : M \otimes \mathrm{Coker}\, i \to \mathrm{Coker}\, i'$.

For $m \in M$ and $n'' \in N''$ choose $n \in p^{-1}(n'')$ and let $\varphi(m, n'')$ denote the image of $m \otimes n$ in $M \otimes_R \mathrm{Coker}\, i$. The difference between two preimages of n'' is an image of i, so $\varphi(m, n'')$ does not depend on the choice of n. Evidently φ is R-bilinear. Therefore the universal property characterization of the tensor product (Proposition A6.1) gives a unique homomorphism $w : M \otimes_R N'' \to M \otimes_R \mathrm{Coker}\, i$ such that $w(m \otimes n'') = \varphi(m, n'')$.

Starting with $m \otimes n''$, pick some $n \in p^{-1}(n'')$. Then $w(m \otimes n'')$ is the image of $m \otimes n$ in $M \otimes_R \mathrm{Coker}\, i$, $v(w(m \otimes n''))$ is the image of this in $\mathrm{Coker}\, i'$, and $u(v(w(m \otimes n''))) = m \otimes n''$. Thus $u \circ v \circ w$ is the identity, so u must be injective. □

Although the univariate functors derived from $\mathrm{Hom}_R(–, –)$ and $– \otimes_R –$ are not exact in general, for any particular M the derived univariate functors can be exact. This gives three very important properties of M-modules. Specifically, M is:

(a) *projective* if $\mathrm{Hom}_R(M,-)$ is exact;

(b) *injective* if $\mathrm{Hom}_R(-,M)$ is exact;

(c) *flat* if $M \otimes_R -$ and $- \otimes_R M$ are exact.

These are the respective topics of the next three sections.

Historically, injective modules appeared first, in work by Baer in 1940. Projective modules were introduced in CE. Flat modules were introduced by Serre (1956) in the famous paper *Géométrie Algébrique et Géométrie Analytique*, which came to be known as GAGA, and they have been important in homological algebra and algebraic geometry since then. Projective and injective modules figure prominently in the further development of the subject, and will be treated in a parallel manner, both in the next two sections and in the first two section of Chapter C, even though there are some important differences. Eventually we will see that when R is Noetherian, a finitely generated R-module is projective if and only if it is flat, but nevertheless flatness will have independent significance throughout the remainder.

B7 Projective Modules

Let M and N be R-modules. We say that N is a *direct factor* of M if there is a third R-module L such that $L \oplus N$ and M are isomorphic. Two other characterizations of this situation occur frequently.

Lemma B7.1. *For R-modules M and N the following are equivalent:*

(a) N is a direct factor of M.

(b) there are homomorphisms $p : M \to N$ and $q : N \to M$ such that $p \circ q = 1_N$.

(c) there is a short exact sequence $0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$ that splits.

Proof. If (a) holds, then $p : (l, n) \mapsto n$ and $q : n \mapsto (0, n)$ satisfy (b). If (b) holds, then q is a splitting map for $0 \to \mathrm{Ker}(p) \to M \to N \to 0$, so (c) holds. If (c) holds, and $q : N \to M$ splits the sequence, then $i \oplus q : L \oplus N \to M$ is an isomorphism, so (a) holds. ☐

Proposition B7.2. *For an R-module P the following are equivalent:*

(a) P is projective.

(b) *Whenever $f : P \to N$ and $g : M \to N$ are homomorphisms, with g surjective, there is a homomorphism $h : P \to M$ such that $f = g \circ h$.*

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "f"] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

(c) *Any exact sequence $0 \to K \xrightarrow{i} M \xrightarrow{p} P \to 0$ splits.*

(d) *P is a direct factor of a free module.*

Proof. Since $\mathrm{Hom}_R(P, -)$ is left exact, P is projective if and only if

$$\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N) \to 0$$

is exact for each short exact sequence $0 \to L \to M \to N \to 0$. That is, (a) and (b) are equivalent.

If $0 \to K \xrightarrow{i} M \xrightarrow{p} P \to 0$ is exact, applying (b) with $N = P$, $f = \mathbf{1}_P$, and $g = p$, gives a map $q : P \to M$ such that $p \circ q = \mathbf{1}_P$. Thus (b) implies (c).

Let F be a free module on any set of generators of P, let $p : F \to P$ be the natural projection, let K be the kernel of p, and let $i : K \to F$ be the inclusion. Now (c) implies that P satisfies the last condition in the result above, so P is a direct factor of F. Thus (c) implies (d).

Suppose that P is a direct factor of a free module F, so that there are homomorphisms $p : F \to P$ and $q : P \to F$ with $p \circ q = 1_P$. Let $f : P \to N$ and $g : M \to N$ be as in (b). A homomorphism $j : F \to M$ such that $f \circ p = g \circ j$ is induced by any assignment of an element of $g^{-1}(f(p(x)))$ to each x in a basis of F. Let $h = j \circ q$. Then

$$g \circ h = g \circ j \circ q = f \circ p \circ q = f.$$

Thus (d) implies (b). ☐

Criterion (d) allows us to easily prove that projectivity is preserved by a change of ring:

Proposition B7.3. *If S is an R-algebra and P is a projective R-module, then $S \otimes_R P$ is a projective S-module.*

Proof. If F is a free R-module, then $S \otimes_R F$ is a free S-module. Concretely, if $\{f_\alpha\}$ is a basis for F, let N be the free S-module on the set of generators $\{1 \otimes f_\alpha\}$. Then there is an obvious bilinear $\phi : S \times F \to N$, and any bilinear ϕ with domain $S \times F$ is $\kappa \circ \phi$ where κ is the obvious homomorphism. Therefore (Proposition A6.1) $S \otimes_R F = N$.

If $i : P \to F$ and $p : F \to P$ are homomorphisms with $p \circ i = \mathbf{1}_P$, then $(\mathbf{1}_S \otimes_R p) \circ (\mathbf{1}_S \otimes_R i) = \mathbf{1}_{S \otimes_R P}$. Thus $S \otimes_R P$ is a direct factor of $S \otimes_R F$. ☐

Let P be a projective R-module. This result implies that if I is an ideal, then P/IP is a projective R/I-module, and if $S \subset R$ is a multiplicatively closed set, then $S^{-1}P$ is a projective $S^{-1}R$-module. But note that if P is a projective R-module, then $S^{-1}P$ may *not* be a projective R-module. For example, $\mathbb{Z}$ is a projective $\mathbb{Z}$-module, of course, but $\mathbb{Q} = (\mathbb{Z} \setminus \{0\})^{-1}\mathbb{Z}$ cannot be a projective $\mathbb{Z}$-module because a homomorphism from $\mathbb{Q}$ to a free $\mathbb{Z}$-module is, in effect, a cartesian product of homomorphisms from $\mathbb{Q}$ to $\mathbb{Z}$, and the only such homomorphism is zero.

There is also the following intriguing result.

Proposition B7.4. *A projective R-module is flat.*

Proof. Let P be projective. By symmetry, proving that $P \otimes_R -$ is exact is enough, and since it is right exact, it suffices to prove that

$$\mathbf{1}_P \otimes f : P \otimes_R A \to P \otimes_R B$$

is injective whenever $f : A \to B$ is an injective homomorphism.

Proposition B7.2 gives a free module F and homomorphisms $q : P \to F$ and $p : F \to P$ such that $p \circ q$ is the identity. Let $F = \oplus_{i \in I} R_i$ where each R_i is a copy of R. Since tensor product commutes with direct sum (Lemma A6.2) we have

$$F \otimes_R A = (\oplus_i R_i) \otimes_R A = \oplus_i (R_i \otimes A) = \oplus_i A_i$$

where each A_i is a copy of A. Similarly, $F \otimes_R B = \oplus_i B_i$, and evidently $\mathbf{1}_F \otimes f : F \otimes_R A \to F \otimes_R B$ is $\oplus_i f_i : \oplus_i A_i \to \oplus_i B_i$ where each $f_i : A_i \to B_i$ is a copy of f. In particular, $\mathbf{1}_F \otimes f$ is injective.

Now observe that

$$q \otimes \mathbf{1}_A : P \otimes_R A \to F \otimes_R A \quad \text{and} \quad q \otimes \mathbf{1}_B : P \otimes_R B \to F \otimes_R B$$

are injective because in each case $(p \otimes \mathbf{1}) \circ (q \otimes \mathbf{1}) = (p \circ q) \otimes \mathbf{1}$ is the identity. We now have the commutative diagram

$$\begin{tikzcd}
P \otimes_R A \arrow[r, "\mathbf{1}_P \otimes f"] \arrow[d, "q \otimes \mathbf{1}_A"'] & P \otimes_R B \arrow[d, "q \otimes \mathbf{1}_B"] \\
F \otimes_R A \arrow[r, "\mathbf{1}_F \otimes f"] & F \otimes_R B
\end{tikzcd}$$

in which the vertical maps and the lower map are injections, so the upper map must also be an injection. □

It is easy to see that if $f : A \to B$ is an injective $\mathbb{Z}$-module homomorphism, then $\mathbf{1}_{\mathbb{Q}} \otimes f : \mathbb{Q} \otimes_{\mathbb{Z}} A \to \mathbb{Q} \otimes_{\mathbb{Z}} B$ is injective, so $\mathbb{Q}$ is $\mathbb{Z}$-flat. But the only homomorphism from $\mathbb{Q}$ to $\mathbb{Z}$, or from $\mathbb{Q}$ to any free $\mathbb{Z}$-module, is zero, so $\mathbb{Q}$ is not $\mathbb{Z}$-projective. However, later in this chapter we will see that a flat module is projective if it is finitely presented.

B8 Injective Modules

Paralleling the description of projective modules above, there are the following alternative characterizations of injective modules. Another important characterization will be developed later, after we have shown that every R-module can be embedded in an injective R-module.

Proposition B8.1. *For an R-module J the following are equivalent:*

(a) J is injective.

(b) Whenever $f : L \to J$ and $g : L \to M$ are homomorphisms, with g injective, there is a homomorphism $h : M \to J$ such that $f = h \circ g$.

$$\begin{tikzcd} L \arrow[r, "g"] \arrow[d, "f"'] & M \arrow[dl, dashed, "h"] \\ J & \end{tikzcd}$$

(c) Whenever $f : L \to J$ and $g : L \to M$ are homomorphisms with $\mathrm{Ker}(g) \subset \mathrm{Ker}(f)$*, there is a homomorphism $h : M \to J$ such that $f = h \circ g$.*

(d) (Baer's Criterion) For any ideal I of R, every R-homomorphism $I \to J$ can be extended to an R-homomorphism $R \to J$.

Proof. Since $\mathrm{Hom}_R(-, J)$ is a left exact contravariant functor, J is injective if and only if
$$\mathrm{Hom}_R(M, J) \to \mathrm{Hom}_R(L, J) \to 0$$
is exact for each short exact sequence $0 \to L \to M \to N \to 0$, which is precisely (b). Thus (a) and (b) are equivalent.

Clearly (c) implies (b). For the converse suppose that $\mathrm{Ker}(g) \subset \mathrm{Ker}(f)$, and let $\tilde{f} : L/\mathrm{Ker}(g) \to J$ and $\tilde{g} : L/\mathrm{Ker}(g) \to M$ be the obvious derived maps. Then $\tilde{g}$ is injective, so there is an $h : M \to J$ such that $\tilde{f} = h \circ \tilde{g}$. Let $\pi : L \to L/\mathrm{Ker}(g)$ be the map $\ell \mapsto \ell + \mathrm{Ker}(g)$. Then
$$f = \tilde{f} \circ \pi = h \circ \tilde{g} \circ \pi = h \circ g.$$

It now suffices to show that (b) and Baer's criterion are equivalent, and it is easy to see that (b) implies Baer's criterion: put $L = I$ and $M = R$. So, suppose that Baer's criterion holds, let $f : L \to J$ be given, and suppose that L is a submodule of M. (This formulation of the given data of (b) eases the discussion.)

Let A be the set of ordered pairs (N, j) where N is a submodule of M that contains L and $j : N \to J$ is a homomorphism such that $j|_L = f$. We write

$(N, j) \le (N', j')$ if $N \subset N'$ and $j'|_N = j$; this is a partial ordering of A, and the union (in the obvious sense) of any chain in this ordering is an element of A that is an upper bound of the chain. Therefore Zorn's lemma implies that A has a maximal element $(\overline{N}, \overline{j})$.

Aiming at a contradiction, suppose there is some $m \in M \setminus \overline{N}$. Let

$$I = (\overline{N} : m) = \{ r \in R : rm \in \overline{N} \},$$

let $\alpha : I \to J$ be the homomorphism $\alpha(r) = \overline{j}(rm)$, and let $\beta : R \to J$ be an extension. We can now set $N = \overline{N} + Rm$ and define an extension $j : N \to J$ of $\overline{j}$ by setting

$$j(n + rm) = \overline{j}(n) + \beta(r).$$

This is unambiguous because if $n' + r'm = n + rm$, then $r' - r \in I$, and

$$\overline{j}(n') + \beta(r') - \overline{j}(n) - \beta(r) = \overline{j}(n' - n) + \overline{j}((r' - r)m) = 0.$$

Since (N, j) contradicts the maximality of $(\overline{N}, \overline{j})$, the proof is complete. □

Direct sums and products of injective modules are injective.

Lemma B8.2. *A direct sum $J = \bigoplus_\alpha J_\alpha$ (or a direct product $J = \prod_\alpha J_\alpha$) of R-modules is injective if and only if each J_α is injective.*

Proof. The proof will be based on (b) of Proposition B8.1 as the criterion for injectivity. Let homomorphisms $f : L \to J$ and $g : L \to M$ be given, with g injective.

The proofs for the two constructions are exactly the same. For each α let $i_\alpha : J_\alpha \to J$ and $p_\alpha : J \to J_\alpha$ be the usual inclusion and projection. First suppose that each J_α is injective. Then for each α there is $h_\alpha : M \to J_\alpha$ such that $h_\alpha \circ g = p_\alpha \circ f$. These combine to give $h : M \to J$ with $p_\alpha \circ h \circ g = p_\alpha \circ f$ for all α, which means (by virtue of the universal property characterizing the direct product, if you like) that $h \circ g = f$.

Now suppose that J is injective, and for some α consider a homomorphism $f_\alpha : L \to J_\alpha$. There is a homomorphism $h : M \to J$ such that $h \circ g = i_\alpha \circ f_\alpha$, and $p_\alpha \circ h$ has the desired property: $(p_\alpha \circ h) \circ g = f_\alpha$. □

An abelian group G is *divisible* if, for each $g \in G$ and nonzero integer n, there is some $g' \in G$ with $ng' = g$. An abelian group is the same thing as a $\mathbb{Z}$-module, so divisibility amounts to Baer's criterion: any homomorphism from an ideal (n) to G has an extension to $\mathbb{Z}$. Thus an abelian group is divisible if and only if it is an injective $\mathbb{Z}$-module. The prototypical injective $\mathbb{Z}$-module is $\mathbb{Q}$. As this example suggests, in contrast to projective modules (e.g., R itself is always projective) for many rings nonzero injective modules are necessarily quite "large" in relation to the ring, and are rarely finitely generated.

We will need to know that any quotient of a divisible group is divisible. The logic of the proof is a bit more general.

Proposition B8.3. *If R is a PID, J is an injective R-module, and L is a submodule of J, then J/L is injective.*

Proof. We show that J/L satisfies Baer's criterion. Let $\varphi : I \to J/L$ be a homomorphism whose domain is an ideal $I \subset R$, and let $\pi : J \to J/L$ be the quotient map. Choose $i \in R$ and $j \in J$ such that $I = (i)$ and $\varphi(i) = j + L$. A PID is an integral domain, so the formula $\varphi'(ri) = rq$ unambiguously defines a homomorphism $\varphi' : I \to J$ such that $\pi \circ \varphi' = \varphi$. Baer's criterion for J gives an extension $\psi' : R \to J$ of φ', and $\psi = \pi \circ \psi'$ is an extension of φ. ☐

Corollary B8.4. *$\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, and consequently it is divisible.*

B9 Flat Modules

The geometric interest of flatness depends in large part on it being a *local property* in the sense of the following result.

Lemma B9.1. *For an R-module M the following are equivalent:*

(a) M is flat;

(b) for any multiplicatively closed set $S \subset R$, $S^{-1}M$ is a flat $S^{-1}R$-module;

(c) for any maximal ideal $\mathfrak{m}$, $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module.

Proof. That (a) implies (b) follows from the fact (Lemma A6.6) that for any $S^{-1}R$-module N, $S^{-1}M \otimes_{S^{-1}R} N = M \otimes_R N$. Of course (b) implies (c).

Suppose that (c) holds, let $N' \to N$ be an injection, and let L be the kernel of $N' \otimes_R M \to N \otimes_R M$, so there is an exact sequence

$$0 \to L \to N' \otimes_R M \to N \otimes_R M.$$

For any maximal ideal $\mathfrak{m}$ the sequence

$$0 \to L \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{m}} \to (N' \otimes_R M) \otimes_R R_{\mathfrak{m}} \to (N \otimes_R M) \otimes_R R_{\mathfrak{m}}$$

is exact because (Proposition A6.8) $R_{\mathfrak{m}}$ is a flat R-module. Now Lemma A6.6 and Proposition A6.5 give

$$(N' \otimes_R M) \otimes_R R_{\mathfrak{m}} = (N' \otimes_R M)_{\mathfrak{m}} = N'_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}},$$

and similarly for N. The hypothesis implies that $N'_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \to N_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ is injective, so $L \otimes_R R_{\mathfrak{m}} = L_{\mathfrak{m}} = 0$. Since this is true for any $\mathfrak{m}$, Lemma A5.8 implies that $L = 0$. ☐

We now give a minimal sufficient condition for flatness that is an analogue of Baer's criterion for injectivity.

Proposition B9.2. *For an R-module M the following are equivalent:*

(a) M is flat.

(b) Whenever $i : K \to L$ is an injection, $i \otimes \mathbf{1}_M : K \otimes_R M \to L \otimes_R M$ is injective.

(c) For every ideal I, $I \otimes_R M \to R \otimes_R M = M$ is injective.

Proof. Because $- \otimes_R M$ is right exact, (a) and (b) are equivalent. That (b) implies (c) is trivial, so we only need to show that (c) implies (b).

First suppose that L is free of finite rank, i.e., $L = R^n$ for some n. If $n = 1$ then K is an ideal, and injectivity holds by assumption. By induction we may assume that $n > 1$, so $L = L_1 \oplus L_2$ where L_1 and L_2 are free modules of smaller rank, for which the claim may be assumed to hold. Let $K_1 = K \cap L_1$, and let K_2 be the image of K in $L/L_1 = L_2$. We have the following diagram with exact rows:

$$\begin{tikzcd} K_1 \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & 0 \\ L_1 \arrow[r] & L \arrow[r] & L_2 \arrow[r] & 0 \end{tikzcd}$$

Since $- \otimes_R M$ is right exact, the diagram

$$\begin{tikzcd} K_1 \otimes_R M \arrow[r] \arrow[d, "\beta"'] & K \otimes_R M \arrow[r] \arrow[d, "\gamma"'] & K_2 \otimes_R M \arrow[r] \arrow[d, "\delta"'] & 0 \\ L_1 \otimes_R M \arrow[r, "\iota"] & L \otimes_R M \arrow[r] & L_2 \otimes_R M \arrow[r] & 0 \end{tikzcd}$$

also has exact rows. The induction hypothesis implies that β and δ are injective, and ι is injective because L_1 is a direct factor of L. (If the composition $L_1 \xrightarrow{i} L \xrightarrow{p} L_1$ is the identity, then $(p \otimes \mathbf{1}_M) \circ (i \otimes \mathbf{1}_M) = \mathbf{1}_{L_1 \otimes_R M}$.) Therefore Lemma B1.1 (a) implies that γ is injective.

Now suppose that L is free, but not necessarily of finite rank, and that $\{\ell_i\}_{i \in I}$ is a basis. Any $x \in K \otimes_R M$ is a finite sum $\sum_j k_j \otimes m_j$, and each k_j is a linear combination of the ℓ_i, so there is a finite $I_0 \subset I$ such that x is in the image of $(K \cap L_0) \otimes_R M \to K \otimes_R M$, where L_0 is the submodule of L generated by $\{\ell_i\}_{i \in I_0}$. From above we know that $(K \cap L_0) \otimes_R M \to L_0 \otimes_R M$ is injective, and $L_0 \otimes_R M \to L \otimes_R M$ is injective because L_0 is a direct factor, so if x goes to zero in $L \otimes_R M$, then 0 is its only preimage in $(K \cap L_0) \otimes_R M$.

Now let L be arbitrary, and let $0 \to Z \to F \to L \to 0$ be exact, where F is free. (For example F could have the elements of L as its set of generators.) Let $E \subset F$ be the preimage of K. There is a commutative diagram with exact rows which, when tensored with M, gives

$$\begin{tikzcd} Z \otimes_R M \arrow[r] \arrow[d, "\alpha"'] & E \otimes_R M \arrow[r] \arrow[d, "\beta"'] & K \otimes_R M \arrow[r] \arrow[d, "\gamma"'] & 0 \\ Z \otimes_R M \arrow[r] & F \otimes_R M \arrow[r] & L \otimes_R M \arrow[r] & 0. \end{tikzcd}$$

The rows here are also exact, α is the identity, and β is injective, so (after adding a pair of zeros on the right) the five lemma implies that γ is injective. $\square$

There is also a characterization of flatness in terms of what we thought algebra was when we were in high school. A relation $\sum_i r_i m_i = 0$ in M is *trivial* if there are $m'_1, \dots, m'_m \in M$ and ring elements a_{ij} such that $x_i = \sum_j a_{ij} m'_j$ for all i and $\sum_i r_i a_{ij} = 0$ for all j.

Proposition B9.3 (Equational Criterion for Flatness)**.** *M is flat if and only if every relation in M is trivial.*

Proof. Suppose that M is flat, so $I \otimes_R M \to R \otimes_R M$ is injective for every ideal I. Let $\sum_{i=1}^n r_i m_i = 0$ be a relation. Let $I = (r_1, \dots, r_n)$, let $e_1, \dots, e_n$ be generators for R^n, let $R^n \to I$ be the map taking each e_i to r_i. and let K be the kernel. Then $0 \to K \to R^n \to I \to 0$ is exact. Since $\sum_i r_i \otimes m_i$ maps to zero in $R \otimes_R M = M$, it is zero in $I \otimes_R M$, so $\sum_i e_i \otimes m_i$ maps to zero in $I \otimes_R M$, and consequently there is a $\sum_j k_j \otimes m'_j \in K \otimes_R M$ that maps to $\sum_i e_i \otimes m_i$. Setting $k_j = \sum_i a_{ij} e_i$, the equations of triviality are satisfied.

Now suppose that every relation in M is trivial. In view of the last result it suffices to show that for a given ideal I the map $I \otimes_R M \to R \otimes_R M = M$ is injective. Let $\sum_i r_i \otimes m_i$ be an element of $I \otimes_R M$ that goes to zero. This means that $\sum_i r_i m_i = 0$, so there are $m'_1, \dots, m'_m \in M$ and ring elements a_{ij} such that $x_i = \sum_j a_{ij} m'_j$ for all i and $\sum_i r_i a_{ij} = 0$ for all j. Now

$$\sum_i r_i \otimes m_i = \sum_i r_i \otimes \Big(\sum_j a_{ij} m'_j\Big) = \sum_j \Big(\sum_i r_i a_{ij}\Big) \otimes m'_j = 0.$$

$\square$

The main result that will be carried forward from the remainder of this section is that a finitely presented flat R-module is projective. (Recall (Proposition B7.4) that any projective R-module is flat.) The tool used to prove this has quite general interest and significance.

Let G and H be abelian groups, and let $\alpha : G \to H$ be a homomorphism. In general, a *character* of G is a homomorphism from G to the circle group $\mathbb{R}/\mathbb{Z}$. We will restrict attention to characters that take values in $\mathbb{Q}/\mathbb{Z}$. In a nutshell, *Pontryagin duality* is the additive contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Q}/\mathbb{Z})$. Let $G^* = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$ and $\alpha^* = \mathrm{Hom}_{\mathbb{Z}}(\alpha, \mathbb{Q}/\mathbb{Z}) : H^* \to G^*$.

Lemma B9.4. *$G^* = 0$ if and only if $G = 0$, and $\alpha^* = 0$ if and only if $\alpha = 0$.*

Proof. Of course $G^* = 0$ if $G = 0$ and $\alpha^* = 0$ if $\alpha = 0$, because Pontryagin duality is an additive functor. Suppose $0 \neq g \in G$. If the order of g is finite, we can induce a nonzero map from $\mathbb{Z}g$ to $\mathbb{Q}/\mathbb{Z}$ by mapping g to any element of the same order, while if the order of g is infinite, then for any nonzero element of $\mathbb{Q}/\mathbb{Z}$ there is a homomorphism $\mathbb{Z}g \to \mathbb{Q}/\mathbb{Z}$ taking g to that element. Since $\mathbb{Q}/\mathbb{Z}$ is injective, this homomorphism extends to a nonzero element of G^*.

Saying that $\alpha^* \neq 0$ whenever $\alpha \neq 0$ amounts to a rephrasing of the injectivity of $\mathbb{Q}/\mathbb{Z}$. □

Now let $\beta : H \to I$ be a second homomorphism.

Lemma B9.5. $\mathrm{Ker}(\beta) \subset \mathrm{Im}(\alpha)$ *if and only if* $\mathrm{Ker}(\alpha^*) \subset \mathrm{Im}(\beta^*)$.

Proof. Suppose that $\mathrm{Ker}(\beta) \subset \mathrm{Im}(\alpha)$. If $\varphi \in \mathrm{Ker}(\alpha^*)$, which is to say that $\mathrm{Im}(\alpha) \subset \mathrm{Ker}(\varphi)$, then $\mathrm{Ker}(\beta) \subset \mathrm{Ker}(\varphi)$, so there is a $\gamma' : \mathrm{Im}(\beta) \to \mathbb{Q}/\mathbb{Z}$ such that $\varphi = \gamma' \circ \beta$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, γ' has an extension $\gamma : I \to \mathbb{Q}/\mathbb{Z}$, and $\varphi = \beta^*(\gamma)$. Thus $\mathrm{Ker}(\alpha^*) \subset \mathrm{Im}(\beta^*)$.

If $g \in \mathrm{Ker}(\beta) \setminus \mathrm{Im}(\alpha)$, there is a homomorphism $\gamma' : H/\mathrm{Im}(\alpha) \to \mathbb{Q}/\mathbb{Z}$ with $\gamma'(g + \mathrm{Im}(\alpha)) \neq 0$. If γ is the composition $H \to H/\mathrm{Im}(\alpha) \xrightarrow{\gamma'} I$, then $\gamma \in \mathrm{Ker}(\alpha^*) \setminus \mathrm{Im}(\beta^*)$. □

Combining these two results, we have:

Proposition B9.6. *The sequence* $G \xrightarrow{\alpha} H \xrightarrow{\beta} I$ *is exact if and only if* $I^* \xrightarrow{\beta^*} H^* \xrightarrow{\alpha^*} G^*$ *is exact.*

This can be applied to $0 \to G \xrightarrow{\alpha} H \to \mathrm{Coker}(\alpha) \to 0$ when α is injective, and to $0 \to \mathrm{Ker}(\alpha) \to G \xrightarrow{\alpha} H \to 0$ when α is surjective, so:

Corollary B9.7. *A homomorphism* $\alpha : G \to H$ *is injective (surjective) if and only if* α^* *is surjective (injective).*

For any R-module M we can endow M^* with a scalar multiplication, so that M^* is an R-module, by letting rf be $m \mapsto f(rm)$. If $\beta : M \to N$ is an R-module homomorphism, then β^* is in fact an R-module homomorphism: if $g \in N^*$, then

$$\beta^*(rg)m = (rg)(\beta(m)) = g(r\beta(m)) = g(\beta(rm))$$

$$= \beta^*(g)rm = r(\beta^*(g)m) = (r\beta^*(g))m,$$

so $\beta^*(rg) = r\beta^*(g)$. In this sense we may regard the Pontryagin dual as a covariant functor from the category of R-modules to itself.

For any R-modules M and N there is a homomorphism

$$\sigma_{(M,N)} : M \otimes_R N^* \to \mathrm{Hom}_R(M, N)^*$$

given by letting $\sigma_{(M,N)}(m \otimes g)$ be the map that takes $\alpha \in \mathrm{Hom}_R(M, N)$ to $g(\alpha(m))$.

Proposition B9.8. *The system of homomorphisms* $\sigma_{(M,N)}$ *is a natural transformation of bifunctors.*

Proof. That the relevant diagrams commute is shown just by using the definitions to evaluate the two compositions. If $\varphi : M \to M'$ is a homomorphism, then $\sigma_{(M',N)}(\varphi(m)\otimes g)$ is the map taking $\alpha' \in \mathrm{Hom}_R(M',N)$ to $g(\alpha'(\varphi(m)))$. On the other hand

$$\mathrm{Hom}_R(\varphi,N)^*(\sigma_{(M,N)}(m\otimes g)) = \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(\varphi,N),\mathbb{Q}/\mathbb{Z}\big)(\sigma_{(M,N)}(m\otimes g))$$
$$= \sigma_{(M,N)}(m\otimes g)\circ \mathrm{Hom}_R(\varphi,N) = (\alpha\mapsto g(\alpha(m)))\circ(\alpha'\mapsto \alpha'\circ\varphi).$$

Thus the diagram

$$\begin{array}{ccc} M\otimes_R N^* & \xrightarrow{\varphi} & M'\otimes_R N^* \\ {\scriptstyle\sigma_{(M,N)}}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_{(M',N)}} \\ \mathrm{Hom}_R(M,N)^* & \xrightarrow{\varphi} & \mathrm{Hom}_R(M',N)^* \end{array}$$

commutes.

If $\psi : N \to N'$ is a homomorphism, then

$$\sigma_{(M,N)}(m\otimes\psi^*(g')) = \sigma_{(M,N)}(m\otimes \mathrm{Hom}_{\mathbb{Z}}(\psi,\mathbb{Q}/\mathbb{Z})g') = \sigma_{(M,N)}(m\otimes(g'\circ\psi))$$

is the map $\alpha\mapsto g'(\psi(\alpha(m)))$, and

$$\mathrm{Hom}_R(M,\psi)^*(\sigma_{(M,N')}(m\otimes g') = \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(M,\psi),\mathbb{Q}/\mathbb{Z}\big)(\sigma_{(M,N')}(m\otimes g'))$$
$$= \sigma_{(M,N')}(m\otimes g')\circ\mathrm{Hom}_R(M,\psi) = (\alpha\mapsto g'(\alpha(m)))\circ(\alpha\mapsto\psi\circ\alpha).$$

Thus the diagram

$$\begin{array}{ccc} M\otimes_R N'^* & \xrightarrow{\psi} & M\otimes_R N^* \\ {\scriptstyle\sigma_{(M,N')}}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_{(M,N)}} \\ \mathrm{Hom}_R(M,N')^* & \xrightarrow{\psi} & \mathrm{Hom}_R(M,N)^* \end{array}$$

commutes. □

We will also need the following simple fact

Lemma B9.9. *For any R-modules M_1, M_2, and N,*

$$\sigma_{(M_1\oplus M_2,N)} = \sigma_{(M_1,N)}\oplus\sigma_{(M_2,N)}.$$

Proof. By definition, $\sigma_{(M_1\oplus M_2,N)}((m_1,m_2)\otimes g)$ is the map taking $\alpha=\alpha_1\oplus\alpha_2\in\mathrm{Hom}_R(M_1\oplus M_2,N)$ to $g(\alpha(m_1,m_2)) = g(\alpha_1(m_1))+g(\alpha_2(m_2))$. □

Recall that an R-module M is finitely presented if, for some integers m and n, there is a short exact sequence $R^m\to R^n\to M\to 0$. That is, M is finitely generated, and for some system of generators the *module of relations* $\mathrm{Ker}(R^n\to M)$ is also finitely generated.

Proposition B9.10. *If M and N are R-modules and M is finitely presented, then $\sigma_{(M,N)}$ is an isomorphism.*

Proof. This is true when $M = R$ because $g \in N^*$ is mapped to $1 \otimes g$ by the isomorphism between N^* and $R \otimes_R N^*$, then to $\alpha \mapsto g(\alpha(1))$ by $\sigma_{(R,N)}$, then back to g by the isomorphism between $\mathrm{Hom}(R,N)^*$ and N^*. In view of Lemma B9.9 it is also true when $M = R^m$.

Now let $R^m \to R^n \to M \to 0$ be exact, and consider the diagram

$$\begin{tikzcd}
R^m \otimes_R N^* \arrow[r] \arrow[d, "\sigma_{(R^m,N)}"'] & R^n \otimes_R N^* \arrow[r] \arrow[d, "\sigma_{(R^n,N)}"'] & M \otimes_R N^* \arrow[d, "\sigma_{(M,N)}"'] & \to 0 \\
\mathrm{Hom}_R(R^m,N)^* \arrow[r] & \mathrm{Hom}_R(R^n,N)^* \arrow[r] & \mathrm{Hom}_R(M,N)^* & \to 0.
\end{tikzcd}$$

This diagram commutes because of the last result. The first row is exact because $- \otimes_R N^*$ is a right exact functor. The second row is exact because $\mathrm{Hom}(-,N)$ is left exact and the Pontryagin dual is exact. We have shown that the first two vertical maps are isomorphisms, so adding another pair of zeros on the right, then applying the five lemma, gives the result. □

Theorem B9.11. *A flat finitely presented R-module is projective.*

Proof. Suppose that M is flat. We wish to show that $\mathrm{Hom}_R(M,-)$ is exact, and since it is always left exact, this amounts to $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N')$ being surjective whenever $N \to N'$ is surjective. If $N \to N'$ is surjective, then $N'^* \to N^*$ is injective (Corollary B9.7) and since M is flat, $M \otimes_R N'^* \to M \otimes_R N^*$ is injective. The diagram

$$\begin{tikzcd}
M \otimes_R N'^* \arrow[r] \arrow[d, "\sigma_{(M,N')}"'] & M \otimes_R N^* \arrow[d, "\sigma_{(M,N)}"] \\
\mathrm{Hom}_R(M,N')^* \arrow[r] & \mathrm{Hom}_R(M,N)^*
\end{tikzcd}$$

commutes because the $\sigma_{(M,N)}$ are natural, and both vertical homomorphisms are isomorphisms, so it follows that $\mathrm{Hom}_R(M,N')^* \to \mathrm{Hom}_R(M,N)^*$ is injective. Now Corollary B9.7 implies that $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,N')$ is surjective. □

Chapter C

Univariate Derived Functors

The failure of a functor to be exact can be "measured" by a sequence of derived functors. The result of applying these functors to an R-module will be defined as the homology of the application of the functor to projective and injective resolutions of the module. (Later there will be axiomatic characterizations of the derived functors, but such concrete calculations will continue to dominate our reasoning concerning derived functors.) Naturally we will need to show that the choice of resolution does not affect the result. Our first task is to show that projective and injective resolutions exist.

C1 Projective Resolutions

If M is an R-module, a *left complex* over M is a chain complex

$$\cdots \to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} M \to 0.$$

This complex will often be denoted simply by X. There is also the truncated complex

$$\cdots \to X_3 \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \to 0$$

which will be denoted by the corresponding bold faced letter, in this case $\mathbf{X}$. We say that X is *projective* (*free, flat*) if $X_0, X_1, \ldots$ are projective (free, flat). If it is both projective (free, flat) and acyclic, it is a *projective* (*free, flat*) *resolution* of M. Since a free module is projective (Proposition B7.2) and a projective module is flat (Proposition B7.4) a free resolution is a projective resolution and a projective resolution is a flat resolution.

Usually we think of M as the given object, and for this reason it is intuitive to state results in terms that highlight it. However, many of the arguments, both here and later, use a different system of notation, called *homogeneous notation*, obtained by setting $X_{-1} = M$, $X_j = 0$ for all $j < -1$, $d_0 = \epsilon : X_0 \to X_{-1}$, and $d_j = 0 : X_j \to X_{j-1}$ for all $j < 0$. This will spare us a certain amount of redundancy arising from special treatments of the initial step in inductive arguments and constructions. Almost always the only thing distinctive about the initial case is that more things are zero, which does not impair the logic of the general case.

Lemma C1.1. *Every R-module M has a free resolution.*

Proof. The construction is inductive. Suppose that $d_n : X_n \to X_{n-1}$ is given or has already been constructed. (This may be $d_{-1} : X_{-1} = M \to 0$.) We let X_{n+1} be the free module whose generators are a set of generators of the kernel of d_n, and we let $d_{n+1} : X_{n+1} \to \operatorname{Ker} d_n$ be the homomorphism taking each generator to itself. □

Sometimes one would like resolutions with additional properties.

Proposition C1.2. *If R is Noetherian and M is a finitely generated R-module, then M has a free resolution whose modules are all finitely generated.*

Proof. As in the proof of C1.1, the construction is inductive. Suppose that $d_n : X_n \to X_{n-1}$ is given or has already been constructed, with X_n free and finitely generated. Then X_n is Noetherian (Proposition A4.6) so $\operatorname{Ker} d_n$ is finitely generated. Let X_{n+1} be the free module on some finite system of generators, and let $d_{n+1} : X_{n+1} \to X_n$ be the homomorphism taking each generator to itself. □

C2 Injective Resolutions

Let M be an R-module. A *right complex* over M is a cochain complex

$$0 \to M \xrightarrow{\eta} I_0 \xrightarrow{d^0} I_1 \to \cdots \to I_n \xrightarrow{d^n} I_{n+1} \to \cdots .$$

Our notational conventions are the same as for left complexes: this complex will often be denoted simply by I, and the truncated complex

$$0 \to I_0 \xrightarrow{d^0} I_1 \xrightarrow{d^1} I_2 \xrightarrow{d^2} I_3 \to \cdots$$

will be denoted by $\mathbf{I}$. In arguments using induction we will frequently use homogeneous notation, setting $I_{-1} = M$ and $d^{-1} = \eta$.

We say that I is *injective* if $I_0, I_1, \ldots$ are injective. If I is both injective and acyclic, it is an *injective resolution* of M. Our constructions will be based on selecting an injective resolution of each R-module, and their validity depends on each R-module having at least one injective resolution. We would like to construct one inductively, so suppose that we have already constructed

$$0 \to M \xrightarrow{\eta} I_0 \xrightarrow{d^0} I_1 \to \cdots \to I_{n-1} \xrightarrow{d^{n-1}} I_n.$$

We can continue the construction if there is an injection $I_n/\operatorname{Im}(d^{n-1}) \to I_{n+1}$ with I_{n+1} injective. The category of R-modules is said to *have enough injectives* because of the following result.

Theorem C2.1. *Any R-module can be embedded in an injective R-module. Consequently every R-module has an injective resolution.*

This terminology makes sense for any abelian category, and in fact one of the major accomplishments of Grothendieck's Tohoku paper was to show that the category of sheaves over a topological space has enough injectives.

Curiously, the proof of Theorem C2.1 is a bootstrap, first establishing the case $R = \mathbb{Z}$.

Lemma C2.2. *Any abelian group G can be embedded in a divisible abelian group.*

Proof. Let Γ be a system of generators for G. Then G is isomorphic to the quotient F/K of the free abelian group F on Γ by the kernel K of $F \to G$. The map $\varphi : f \mapsto f \otimes 1$ embeds F in $F \otimes_{\mathbb{Z}} \mathbb{Q}$, which is a direct sum of divisible groups and consequently divisible itself. There is an induced embedding of G in $(F \otimes_{\mathbb{Z}} \mathbb{Q})/\varphi(K)$, and the latter group is divisible by Proposition B8.3. $\square$

Thus the additive group of any R-module can be embedded in a divisible abelian group. Things now get a bit magical. For an R-module M there is a map $\alpha : M \to \mathrm{Hom}_{\mathbb{Z}}(R, M)$ taking each m to $\alpha_m : a \mapsto am$. An embedding $\varphi : M \to G$ of the additive group of M in a divisible group G induces a map

$$\beta = \mathrm{Hom}_{\mathbb{Z}}(R, \varphi) : \mathrm{Hom}_{\mathbb{Z}}(R, M) \to \mathrm{Hom}_{\mathbb{Z}}(R, G).$$

Obviously α and β are embeddings. The trick is to endow $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ and $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ with R-module structures, so that $\beta \circ \alpha$ is an injective R-module homomorphism, and to show that $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ is injective.

In order to fully capture the generality of certain aspects of the construction we now assume that R is an Q-algebra. Specifically, Q is another commutative ring with unit and there is a homomorphism $Q \to R$ taking $1 \to 1$. For the sake of less cumbersome notation we treat Q as a subset of R, even though $Q \to R$ need not be injective; careful examination of the argument below shows that it does not make use of this implicit injectivity.

Let N be an Q-module. If $\varphi \in \mathrm{Hom}_Q(R, N)$ and $r \in R$, there is a Q-module homomorphism $r\varphi \in \mathrm{Hom}_Q(R, N)$ given by $a \mapsto \varphi(ra)$, and it is easy to verify that this scalar multiplication makes $\mathrm{Hom}_Q(R, N)$ into an R-module. Note that *the Q-module structure of* $\mathrm{Hom}_Q(R, N)$ *induced by this R-module structure agrees with the usual Q-module structure because $\varphi(qa) = q\varphi(a)$.*

Lemma C2.3. *Suppose that M is an R-module, N is a Q-module, and $\psi : M \to N$ is a Q-module homomorphism. For $m \in M$ let $\psi_m : R \to N$ be the function $\psi_m(a) = \psi(am)$. Then $\psi_m \in$* $\mathrm{Hom}_Q(R, N)$*, and the map $m \mapsto \psi_m$ is an R-module homomorphism when* $\mathrm{Hom}_Q(R, N)$ *has the R-module structure described above.*

Proof. It is easy to verify that $\psi_m \in \mathrm{Hom}_Q(R, N)$:

$$\psi_m(a + b) = \psi((a + b)m) = \psi(am) + \psi(bm) = \psi_m(a) + \psi_m(b);$$

$$\psi_m(qa) = \psi(qam) = q\psi(am) = q\psi_m(a).$$

It is equally easy to show that $m \mapsto \psi_m$ is an R-module homomorphism:

$$\psi_{m+m'}(a) = \psi(a(m+m')) = \psi(am) + \psi(am') = \psi_m(a) + \psi_{m'}(a);$$

$$\psi_{rm}(a) = \psi(ram) = \psi_m(ra) = (r\psi_m)(a).$$

□

Lemma C2.4. *If J is an injective Q-module, then* $\operatorname{Hom}_Q(R, J)$ *is an injective R-module.*

Proof. Suppose that M is an R-module, L is a submodule, and $\varphi : L \to \operatorname{Hom}_Q(R, J)$ is an R-module homomorphism. We will extend φ to an R-module homomorphism $\psi : M \to \operatorname{Hom}_Q(R, J)$.

It will work well to write φ_ℓ in place of $\varphi(\ell)$. Let $\tilde{\varphi} : L \to J$ be the map $\ell \mapsto \varphi_\ell(1)$. Then:

$$\tilde{\varphi}(\ell + \ell') = \varphi_{\ell+\ell'}(1) = \varphi_\ell(1) + \varphi_{\ell'}(1) = \tilde{\varphi}(\ell) + \tilde{\varphi}(\ell');$$

$$\tilde{\varphi}(q\ell) = \varphi_{q\ell}(1) = (q\varphi_\ell)(1) = q(\varphi_\ell(1)) = q\tilde{\varphi}(\ell).$$

(The first and last equalities are from the definition of $\tilde{\varphi}$, the second is from the R-module structure of φ, and the third is from the point emphasized above.) Therefore $\tilde{\varphi}$ is an Q-module homomorphism, so it extends to a Q-module homomorphism $\tilde{\psi} : M \to J$. For $m \in M$ let $\psi_m : R \to J$ be the function $\psi_m(a) = \tilde{\psi}(am)$. The last result implies that $m \mapsto \psi_m$ is an R-module homomorphism, and it extends φ because $\tilde{\psi}$ extends $\tilde{\varphi}$:

$$\psi_\ell(a) = \tilde{\psi}(a\ell) = \tilde{\varphi}(a\ell) = \varphi_{a\ell}(1) = (a\varphi_\ell)(1) = \varphi_\ell(a).$$

□

Proof of Theorem C2.1. Let M be an R-module. Lemma C2.2 implies that there is an injective $\mathbb{Z}$-module homomorphism $\varphi : M \to G$, where G is a divisible abelian group. Let α and β be as described above; of course these maps are injective. Lemma C2.3 implies that α is an R-module homomorphism. If $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(R, M)$ and $a, r \in R$, then

$$\beta(r\varphi) : a \mapsto \varphi((r\varphi)(a)) = \varphi(\varphi(ra)) = (r\beta(\varphi))(a).$$

so $\beta(r\varphi) = r\beta(\varphi)$. Therefore β and $\beta \circ \alpha$ are R-module homomorphisms. The last result implies that $\operatorname{Hom}_{\mathbb{Z}}(R, G)$ is an injective R-module. □

We can now develop another useful characterization of injective modules.

Proposition C2.5. *An R-module J is injective if and only if each short exact sequence* $0 \to J \xrightarrow{i} M \xrightarrow{p} N \to 0$ *splits.*

Proof. Suppose that J is injective. Applying (b) of Proposition B8.1 with $f = \mathbf{1}_J$ and $g = i$ gives a homomorphism $j : M \to J$ such that $j \circ i = \mathbf{1}_J$.

Now suppose that each short exact sequence splits. Theorem C2.1 implies that J can be embedded in an injective module, which is to say that there is such a sequence in which M is injective. Since the sequence splits, J is a direct factor of M, and now Lemma B8.2 implies that J is injective. $\square$

C3 Univariate Left Derived Functors

Taking a projective resolution of a module is a form of analysis: we provide a representation of the module in terms of simpler objects, combined in a standard and (in some sense) elementary way. The method described in this section extracts additional information from this decomposition, examining how it is "deformed" by a functor.

Suppose we are given an additive covariant functor T from R-modules to Q-modules, where Q is now a second commutative ring with unit. Let X be a projective resolution of M. We would like to define a sequence of functors $L_0T, L_1T, L_2T, \ldots$ by specifying that $(L_nT)M$ is $H_n(T\mathbf{X})$. Suppose $f : M \to N$ is an R-homomorphism, Y is projective resolution of N, and there is a chain map $f : X \to Y$ extending the given f. Compounding our abuse of notation, let f also denote the truncated chain map from $\mathbf{X}$ to $\mathbf{Y}$. We would like to define

$$(L_nT)f : (L_nT)M \to (L_nT)N$$

to be $H_n(Tf : \mathbf{X} \to \mathbf{Y})$.

One may certainly wonder why one might wish to do this. There is no quick or easy answer to this question; on the contrary, the entire subject can be regarded as a (possibly incomplete) response to this query. But prior to that one must first ask whether the wishes expressed above can be fulfilled at all in any sensible way.

In order for it to be possible to define $(L_nT)f$ there must exist an extension of the given f to a chain map.

Lemma C3.1. *Suppose that*

$$\cdots \to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} M \to 0$$

and

$$\cdots \to Y_n \xrightarrow{\partial_n} Y_{n-1} \to \cdots \to Y_1 \xrightarrow{\partial_1} Y_0 \xrightarrow{\eta} N \to 0$$

are left complexes over M and N respectively, with the first projective and the second exact. Then any homomorphism $f : M \to N$ extends to a chain map given by homomorphisms $f_n : X_n \to Y_n$. Any two extensions are chain homotopic.

Proof. We use homogeneous notation, and in addition set $f_{-1} = f$. For $n \geq -1$ suppose that f_n is given or has already been defined. We have $\partial_n f_n d_{n+1} = f_{n-1} d_n d_{n+1} = 0$, so the image of $f_n d_{n+1}$ is contained in $\operatorname{Ker}(\partial_n) = \operatorname{Im}(\partial_{n+1})$. Since X_{n+1} is projective, there is an $f_{n+1} : X_{n+1} \to Y_{n+1}$ such that $\partial_{n+1} f_{n+1} = f_n d_{n+1}$. Thus the first claim follows by induction.

The second claim is one of the two cases of the following result. □

Lemma C3.2. *Suppose that in the diagram*

$$\begin{tikzcd}
\cdots \arrow[r, "d_3"] & X_2 \arrow[r, "d_2"] \arrow[d, "f_2"', "g_2"] & X_1 \arrow[r, "d_1"] \arrow[d, "f_1"', "g_1"] & X_0 \arrow[r] \arrow[d, "f_0"', "g_0"] & 0 \\
\cdots \arrow[r, "\partial_3"] & Y_2 \arrow[r, "\partial_2"] & Y_1 \arrow[r, "\partial_1"] & Y_0 \arrow[r] & 0
\end{tikzcd}$$

the rows are chain complexes, the second of which is exact, $\{f_n\}$ and $\{g_n\}$ are chain maps, and $X_1, X_2, \ldots$ are projective. If $f_0 = g_0$, or if X_0 is projective, then the two chain maps are homotopic.

Proof. Let $h_n = f_n - g_n$. Let $s_{-1} = 0 : 0 \to Y_0$. For $n \geq 0$ suppose that for $-1 \leq i < n$ we have already defined $s_i : X_i \to Y_{i+1}$ such that $h_i = \partial_{i+1} s_i + s_{i-1} d_i$. The computation

$$\partial_n(h_n - s_{n-1} d_n) = (h_{n-1} - \partial_n s_{n-1}) d_n = s_{n-2} d_{n-1} d_n = 0$$

shows that the image of $h_n - s_{n-1} d_n$ is contained in $\operatorname{Ker}(\partial_n) = \operatorname{Im}(\partial_{n+1})$. Since X_n is projective (or because $h_0 = 0$) there is an R-homomorphism $s_n : X_n \to Y_{n+1}$ such that $\partial_{n+1} s_n = h_n - s_{n-1} d_n$. Thus the existence of a suitable homotopy follows by induction. □

These results at least makes it possible to get our project off the ground. The point of view we adopt is that we have chosen "once and for all" a projective resolution of each R-module, and for each homomorphism $f : M \to N$ we have chosen an extension $f : X \to Y$ where X and Y are the chosen projective resolutions of M and N. The aspirations announced at the beginning of the section are now tangible definitions: (a) if X is the chosen projective resolution of M, then, for each $n = 0, 1, 2, \ldots$, $(L_n T)M$ is defined to be $H_n(T\mathbf{X})$; (b) if in addition $f : M \to N$ is an R-homomorphism, Y is the chosen projective resolution of N, and $f : X \to Y$ is the chosen extension with truncation $f : \mathbf{X} \to \mathbf{Y}$, then $(L_n T)f$ is defined to be $H_n(Tf : \mathbf{X} \to \mathbf{Y})$.

A problem arises when we try to show that each $L_n T$ is a functor. Suppose that $g : N \to P$ is a second homomorphism, Z is the chosen projective resolution of P, and $g : Y \to Z$ is the chosen extension of g. If it happened to be the case that $g \circ f : X \to Z$ was the chosen extension of $g \circ f : M \to P$, we could use the fact that T and H_n are functors to infer that

$$(L_n T)(g \circ f) = (L_n T)g \circ (L_n T)f,$$

but there is no reason to expect that we have been this lucky.

The way to circumvent this problem is to show that our choices don't matter. Specifically, we will show that if we have some second system of choices, say X' for the projective resolution of M, Y' for N, $f' : X' \to Y'$ as the extension of f, and so forth, and we use these choices to define $(L'_n T)M$ for each M and $(L'_n T)f$ for each f, then there is a system of isomorphisms $\iota_n^M : (L_n T)M \to (L'_n T)M$ that are natural in the sense that all diagrams

$$\begin{array}{ccc} (L_nT)M & \xrightarrow{(L_nT)f} & (L_nT)N \\ {\scriptstyle \iota_n^M}\downarrow & & \downarrow{\scriptstyle \iota_n^N} \\ (L'_nT)M & \xrightarrow{(L'_nT)f} & (L'_nT)N \end{array} \qquad (*)$$

commute. Now we can prove that $(L_nT)(g \circ f) = (L_nT)g \circ (L_nT)f$ by appealing to the fact that $(L'_nT)(g \circ f) = (L'_nT)g \circ (L'_nT)f$ if the second system is chosen in such a way that $(g \circ f)' = g' \circ f'$, as is certainly possible. Moreover, in this way we see that, at least up to natural isomorphism, the choices really *don't* matter, which should certainly contribute to our sense that these definitions are well founded and potentially interesting.

This style of reasoning may well strike you as a bit suspicious. Although the choices of projective resolutions and extending chain maps were initially thought of as "fixed forever," ex post we are in effect freely pretending that the choices were whatever we would now happen to find convenient. But there is actually nothing wrong with this, and in fact it seems not at all unnatural provided you understand a fundamental feature of homological algebra: *the industry standard for "sameness" is natural isomorphism,* not literal equality. Put another way, we are *really* dealing with natural isomorphism classes of functors, rather than individual functors, but our language systematically fails to distinguish between an equivalence class and one of its representatives. That is, we speak of "the derived functor L_nT" rather that "one of the many possible naturally isomorphic functors L_nT." Not explicitly acknowledging this rather minor sleight of hand almost never gives rise to problems, so this is a very useful convention, and it quickly becomes second nature, to the point where most authors don't mention it at all.

We now explain the details of this maneuver. Suppose, as above, that we have two systems of choices, with X and X' the two projective resolutions of M, and so forth. For each R-module M we choose a chain map $i^M : X \to X'$ that extends the identity function on M. In connection with a homomorphism $f : M \to N$, it need not be the case that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i^M}\downarrow & & \downarrow{\scriptstyle i^N} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

commutes, but Lemma C3.2 does imply that $f' \circ i^M$ and $i^N \circ f$ are homotopic. However, this turns out to irrelevant. The key point is that Lemma C3.2 implies that the diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \\ {\scriptstyle i^M}\downarrow & & \downarrow{\scriptstyle i^N} \\ \mathbf{X}' & \xrightarrow{f'} & \mathbf{Y}' \end{array}$$

of truncated chain complexes and chain maps commutes up to homotopy, simply because everything in sight is projective. *Since T is additive, it preserves the equation defining chain homotopy,* so the diagram

$$\begin{array}{ccc} T(\mathbf{X}) & \xrightarrow{T(f)} & T(\mathbf{Y}) \\ {\scriptstyle T(i^M)}\downarrow & & \downarrow{\scriptstyle T(i^N)} \\ T(\mathbf{X}') & \xrightarrow{T(f')} & T(\mathbf{Y}') \end{array}$$

also commutes up to homotopy. Applying the homology functor H_n and setting $\iota_n^M = H_n(T(i^M))$, we see that diagram $(*)$ above commutes, as desired. We are now justified in calling (the natural isomorphism class of) L_nT the n^{th} *left derived functor of* T.

There is a second sort of left derived functor that will not play any role in our analysis of $\mathrm{Hom}_R(-,-)$ or $- \otimes_R -$, but which is worth developing, if only because otherwise the subsequent material will be contorted due to its lack of an obvious symmetry. Its development involves the injective analogs of Lemmas C3.1 and C3.2.

Lemma C3.3. *Suppose that*

$$0 \to M \xrightarrow{\eta} I_0 \xrightarrow{d^0} I_1 \to \cdots \to I_n \xrightarrow{d^n} I_{n+1} \to \cdots$$

and

$$0 \to N \xrightarrow{\kappa} J_0 \xrightarrow{\partial^0} J_1 \to \cdots \to J_n \xrightarrow{\partial^n} J_{n+1} \to \cdots$$

are cochain complexes, the first of which is exact, and $J_0, J_1, J_2, \ldots$ are injective. Then any homomorphism $f : M \to N$ extends to a chain map given by homomorphisms $f^n : I_n \to J_n$.

Proof. We use homogeneous notation. Supposing that, for some $n \geq -1$, f^{n-1} and f^n are given or have already been defined, by induction it suffices to find a suitable f^{n+1}. We have $\mathrm{Im}(d^{n-1}) \subset \mathrm{Ker}(\partial^n f_n)$ because $\partial^n f^n d^{n-1} = \partial^n \partial^{n-1} f^{n-1} = 0$, so $\partial^n f^n$ induces a map from $I_n/\mathrm{Im}(d^{n-1}) \cong \mathrm{Im}(d^n)$ to J_{n+1}. Since J_{n+1} is injective this extends to a homomorphism $f^{n+1} : I_{n+1} \to J_{n+1}$ such that $f^{n+1}d^n = \partial^n f^n$. $\square$

Lemma C3.4. *Suppose that in the diagram*

$$\begin{tikzcd} 0 \arrow[r] & I_0 \arrow[r, "d^0"] \arrow[d, shift right, "f^0"'] \arrow[d, shift left, "g^0"] & I_1 \arrow[r, "d^1"] \arrow[d, shift right, "f^1"'] \arrow[d, shift left, "g^1"] & I_2 \arrow[r] \arrow[d, shift right, "f^2"'] \arrow[d, shift left, "g^2"] & 0 \\ 0 \arrow[r] & J_0 \arrow[r, "\partial^0"] & J_1 \arrow[r, "\partial^1"] & J_2 \arrow[r] & 0 \end{tikzcd}$$

the rows are cochain complexes, $\{f^n\}$ *and* $\{g^n\}$ *are chain maps, and* $J_1, J_2, \ldots$ *are injective. If* $f^0 = g^0$*, or if* J_0 *is injective, then the two chain maps are homotopic.*

Proof. Let $h^n = f^n - g^n$. Let $s^{-1} : I_{-1} \to J_{-2}$ and $s^0 : I_0 \to J_{-1}$ both be zero. By induction, it suffices to show that if we have already defined $s^i : I_i \to J_{i-1}$ such that $h^{i-1} = \partial^{i-2}s^{i-1} + s^i d^{i-1}$ for all $i \le n$, then we can also define a satisfactory s^{n+1}. The computation

$$(h^n - \partial^{n-1}s^n)d^{n-1} = \partial^{n-1}(h^{n-1} - s^n d^{n-1}) = \partial^{n-1}\partial^{n-2}s^{n-1} = 0$$

shows that $\mathrm{Ker}(d^n) = \mathrm{Im}(d^{n-1}) \subset \mathrm{Ker}(h^n - \partial^{n-1}s^n)$. Therefore we may regard $h^n - \partial^{n-1}s^n$ as a map defined on $I_n/\mathrm{Ker}(d^n) \cong \mathrm{Im}(d^n)$. Since J_n is injective (or $h^0 = 0$) there is an $s^{n+1} : I_{n+1} \to J_n$ such that $s^{n+1}d^n = h^n - \partial^{n-1}s^n$. $\square$

Let U be an additive contravariant functor from the category of R-modules to the category of Q-modules. Suppose that we have chosen an injective resolution of each R-module, and for each homomorphism $f : M \to N$ we have chosen an extension $f : I \to J$ where I and J are the chosen injective resolutions of M and N. If I is the chosen injective resolution of M, then, for each $n = 0, 1, 2, \ldots$, $(L_n U)M$ is defined to be $H_n(U\mathbf{I})$, and if $f : M \to N$ is an R-homomorphism, J is an injective resolution of N, and $f : I \to J$ is the chosen extension with truncation $f : \mathbf{I} \to \mathbf{J}$, then $(L_n U)f$ is defined to be $H_n(Uf : \mathbf{J} \to \mathbf{I})$.

Suppose, as above, that we have two systems of choices, with I and I' the two projective resolutions of M, and so forth. For each R-module M we choose a chain map $i^M : I \to I'$ that extends the identity function on M. In connection with a homomorhism $f : M \to N$, Lemma C3.4 implies that the diagram

$$\begin{tikzcd} \mathbf{I} \arrow[r, "f"] \arrow[d, "i^M"'] & \mathbf{J} \arrow[d, "i^N"] \\ \mathbf{I}' \arrow[r, "f'"] & \mathbf{J}' \end{tikzcd}$$

of truncated chain complexes and chain maps commutes up to homotopy because all the modules are injective. Since U is additive, it preserves the equation defining chain homotopy, so the diagram

$$\begin{array}{ccc} U(\mathbf{J}') & \xrightarrow{U(f)'} & U(\mathbf{I}') \\ {\scriptstyle U(i^M)}\big\downarrow & & \big\downarrow{\scriptstyle U(i^N)} \\ U(\mathbf{J}) & \xrightarrow{U(f)} & U(\mathbf{I}) \end{array}$$

also commutes up to homotopy. Applying the homology functor H_n and setting $\iota_n^M = H_n(U(i^M))$, we see that for each n the diagram

$$\begin{array}{ccc} (L_n'U)N & \xrightarrow{(L_n'U)f} & (L_n'U)M \\ {\scriptstyle \iota_n^M}\big\downarrow & & \big\downarrow{\scriptstyle \iota_n^N} \\ (L_nU)N & \xrightarrow{(L_nU)f} & (L_nU)M \end{array} \qquad (*)$$

commutes. This implies that $(L_nU)(g\circ f) = (L_nU)f\circ(L_nU)g$ when $g : M \to P$ is a second homomorphism, because we can choose a system of resolutions and extensions with the extension of $g\circ f$ equal to the composition of the extension of f with the extension of g. That is, L_nU is a contravariant functor. And of course it also implies that, up to natural isomorphism, the definition of L_nU does not depend on the chosen resolutions and extensions. We call (the natural isomorphism class of) L_nU the n^{th} *left derived functor of* U.

Let T and U be as above. It turns out that when these are exact, the left derived functors don't give anything new. When they are right exact, L_0T and L_0U are the identity, but it can still be the case that the higher order derived functors are nontrivial.

Proposition C3.5. *If T is exact, then $L_nT = 0$ for all $n \geq 1$. If T is right exact, then T and L_0T are naturally isomorphic. If U is exact, then $L_nU = 0$ for all $n \geq 1$. If U is right exact, then U and L_0U are naturally isomorphic.*

Proof. Suppose

$$\cdots \to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} M \to 0$$

is the chosen projective resolution of M. If T is exact, then the truncated sequence

$$\cdots \to TX_2 \xrightarrow{Td_2} TX_1 \xrightarrow{Td_1} TX_0 \to 0$$

is exact everywhere except at TX_0.

Now suppose that T is only right exact. Then

$$TX_1 \xrightarrow{Td_1} TX_0 \xrightarrow{T\epsilon} TM \to 0$$

is exact, and $T\epsilon$ induces an isomorphism

$$T\epsilon : (L_0T)M = TX_0/\mathrm{Im}(Td_1) \to TM.$$

To establish naturality, suppose that suppose that $f : M \to N$ is a homomorphism, let

$$\cdots \to Y_n \xrightarrow{\partial_n} Y_{n-1} \to \cdots \to Y_1 \xrightarrow{\partial_1} Y_0 \xrightarrow{\eta} N \to 0$$

be the chosen projective resolution of N, and let $\{f_n : X_n \to Y_n\}$ be a chain map extending f, as per Lemma C3.1. Then Tf_1 maps $\mathrm{Im}(Td_1)$ into $\mathrm{Im}(T\partial_1)$ because $T\partial_1 \circ Tf_1 = Tf_0 \circ Td_1$, so we can pass to a commutative diagram

$$\begin{array}{ccc} (L_0T)M = TX_0/\mathrm{Im}(Td_1) & \xrightarrow{T\epsilon} & TM \\ {\scriptstyle (L_0T)f = Tf_0}\downarrow & & \downarrow{\scriptstyle Tf} \\ (L_0T)N = TY_0/\mathrm{Im}(T\partial_1) & \xrightarrow{T\eta} & TN. \end{array}$$

The proofs of the assertions for L_nU follow the same pattern, with injective rather than projective resolutions. □

C4 Univariate Right Derived Functors

The material in this section is dual to what we did in the previous section. We assume that an injective resolution

$$0 \to M \xrightarrow{\eta} I_0 \xrightarrow{d^0} I_1 \to \cdots \to I_n \xrightarrow{d^n} I_{n+1} \to \cdots$$

and a projective resolution

$$\cdots \to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} M \to 0$$

have been assigned to each object M in the category of R-modules. For each homomorphism $f : M \to N$ we assume that a particular chain map $F : X \to Y$ extending f has been chosen, where X and Y are the chosen projective resolutions of M and N. We also assume that a particular cochain map $f : I \to J$ extending f has been chosen, where I and J are the chosen injective resolutions of M and N.

Suppose we are given an additive covariant functor T and an additive contravariant functor U from R-modules to Q-modules. For $n \geq 0$ we define $(R^nT)M$ to be the n^{th} cohomology group of the truncated sequence

$$0 \to TI_0 \xrightarrow{Td^0} TI_1 \to \cdots \to TI_n \xrightarrow{Td^n} TI_{n+1} \to \cdots .$$

and we define $(R^nU)M$ to be the n^{th} cohomology group of the truncated sequence

$$0 \to UX_0 \xrightarrow{Ud_0} UX_1 \to \cdots \to UX_n \xrightarrow{Ud_n} UX_{n+1} \to \cdots .$$

Suppose $f : M \to N$ is an R-homomorphism, and let J and Y be the chosen injective and projective resolutions of N, and let $f : X \to Y$ and $f : I \to J$ be the chosen extensions of f. These can be truncated to a chain map $f : \mathbf{X} \to \mathbf{Y}$ and a cochain map $f : \mathbf{I} \to \mathbf{J}$. For $n = 0, 1, 2, \ldots$ let

$$(R^nT)f = H^n(Tf : T\mathbf{I} \to T\mathbf{J}) : (R^nT)M \to (R^nT)N$$

and

$$(R^nU)f = H^n(Uf : U\mathbf{Y} \to U\mathbf{X}) : (R^nU)N \to (R^nU)M.$$

The methods we used earlier to show that L_nT and L_nU are functors can now be used to show that R_nT and R^nU are functors.

Suppose that there is a second system of choices of injective resolutions, with I' the resolution of M, J' the resolution of N, etc. Suppose also that for each homomorphism $f : M \to N$ we have chosen an extension to a cochain map $f' : I \to J$. Let $(R^{n\prime}T)M = H^n(T\mathbf{I'})$, and let

$$(R^{n\prime}T)f = H^n(Tf' : T\mathbf{I'} \to T\mathbf{J'})) : (R^{n\prime}T)M \to (R^{n\prime}T)N.$$

Also, for each M we choose a cochain map $i^M : I \to I'$ extending the identity. Since all the modules involved are injective, Lemma C3.4 implies that $f' \circ i^M$ and $i^N \circ f$ are homotopic. Since T is additive, it preserves the equations expressing homotopy, so $T(f') \circ T(i^M)$ and $T(i^N) \circ T(f)$ are homotopic. Applying the functor H^n, we find that the diagram

$$\begin{array}{ccc} (R^nT)M & \xrightarrow{(R^nT)f} & (R^nT)N \\ {\scriptstyle H^n(Ti^N)}\downarrow & & \downarrow{\scriptstyle H^n(Ti^M)} \\ (R^{n\prime}T)M & \xrightarrow{(R^{n\prime}T)f} & (R^{n\prime}T)N \end{array}$$

commutes.

This shows that R^nT is, in fact, a covariant functor, because if $g : N \to P$ is a second homomorphism, K is the chosen injective resolution of P, and g is also the extension of g to a cochain map, then it could be the case that $g \circ f : I \to K$ is the chosen extension of $g \circ f : M \to P$. In addition, it shows that, up to natural isomorphism, its definition does not depend on the choices of injective resolutions and extensions to cochain maps. It should be obvious that the methods used to show that L_nT is a functor, and independent of choices, work equally well for R^nU, so we regard these results as established. We are now justified in calling R_nT and R^nU the n^{th} *right derived functors* of T and U.

There is the following analog of Proposition C3.5. The reader is invited to check for herself that the ideas used in the proof of that result, with obvious modifications, work equally well for right derived functors.

Proposition C4.1. *If T is exact, then $R^nT = 0$ for all $n \geq 1$. If T is left exact, then T and R^0T are naturally isomorphic. If U is exact, then $R^nU = 0$ for all $n \geq 1$. If U is left exact, then U and R^0U are naturally isomorphic.*

C5 Short Exact Sequences of Resolutions

In order to derive long exact sequences of derived functors, the settings of Propositions B4.3 and B4.4 must be attained. The results that do this are, unfortunately and unavoidably, technical and computational in character. This section is devoted to them.

Lemma C5.1. *Suppose we are given a diagram*

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & X_1 \arrow[r, "i_1"] \arrow[d, "d_1"'] & Y_1 \arrow[r, "p_1"] \arrow[d, "d_1"'] & Z_1 \arrow[r] \arrow[d, "d_1"'] & 0 \\
0 \arrow[r] & X_0 \arrow[r, "i_0"] \arrow[d, "d_0"'] & Y_0 \arrow[r, "p_0"] \arrow[d, "d_0"'] & Z_0 \arrow[r] \arrow[d, "d_0"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *is a short exact sequence of* R*-modules,* X *and* Z *are left complexes over* A *and* C*, and for all* $n \geq 0$ *we have* $Y_n = X_n \oplus Z_n$ *with* $i_n(x) = (x, 0)$ *and* $p_n(x, z) = z$*. Then (with respect to homogeneous notation):*

(a) For each n *it is the case that* $d_n \circ i_n = i_{n-1} \circ d_n$ *and* $d_n \circ p_n = p_{n-1} \circ d_n$ *if and only if there is a homomorphism* $\eta_n : Z_n \to X_{n-1}$ *such that*

$$d_n(x, z) = (d_n(x) + \eta_n(z), d_n(z)). \qquad (*)$$

(b) If this is the case for all n*, then, for each* n*,* $0 = d_{n-1} \circ d_n : Y_n \to Y_{n-2}$ *if and only if*

$$0 = d_{n-1} \circ \eta_n + \eta_{n-1} \circ d_n : Z_n \to X_{n-2}.$$

Proof. Taking compositions of $d_n : Y_n \to Y_{n-1}$ with the projections gives homomorphisms φ_n and ψ_n such that

$$d_n(x, z) = (\varphi_n(x, z), \psi_n(x, z)).$$

If $p_{n-1} \circ d_n = d_n \circ p_n$, then $\psi_n(x, z) = d_n(z)$, and if $i_{n-1} \circ d_n = d_{n-1} \circ i_n$, then $d_n(x) = \varphi_n(x, 0)$, so setting $\eta_n(z) = \varphi_n(0, z)$ gives $(*)$. Conversely, if, for some homomorphism $\eta_n : Z_n \to X_{n-1}$, we define $d_n : Y_n \to Y_{n-1}$ by setting

$$d_n(x, z) = \big(d_n(x) + \eta_n(z), d_n(z)\big),$$

then $p_{n-1} \circ d_n = d_n \circ p_n$ and $i_{n-1} \circ d_n = d_n \circ i_n$ follow automatically.

Now (b) follows from a simple and straightforward calculation:

$$\begin{aligned} d_{n-1}(d_n(x,z)) &= d_{n-1}\big(d_n(x) + \eta_n(z), d_n(z)\big) \\ &= \big(d_{n-1}(d_n(x) + \eta_n(z)) + \eta_{n-1}(d_n(z)), d_{n-1}(d_n(z))\big) \\ &= \big(d_{n-1}(\eta_n(z)) + \eta_{n-1}(d_n(z)), 0\big). \end{aligned}$$

□

The following is known as the *horseshoe lemma*, in accord with the diagram of its given elements.

Lemma C5.2. *Suppose we are given a short exact sequence*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of R-modules, an acyclic left complex X over A, and a projective left complex Z over C. Then there exists a left complex Y over B and extensions of i and p to chain maps such that the sequence

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

is a short exact sequence of left complexes over $0 \to A \to B \to C \to 0$.

Proof. For each $n \geq 0$, let $Y_n = X_n \oplus Z_n$, and let $i_n : X_n \to Y_n$ and $p_n : Y_n \to Z_n$ be the functions $i_n(x) = (x, 0)$ and $p_n(x, z) = z$. Of course $0 \to X_n \xrightarrow{i_n} Y_n \xrightarrow{p_n} Z_n \to 0$ is exact, so the remaining step is the construction of suitable homomorphisms $d_n : Y_n \to Y_{n-1}$.

In view of the last result, it suffices to construct homomorphisms $\eta_n : Z_n \to X_{n-1}$ such that $d_n \circ \eta_{n+1} + \eta_n \circ d_{n+1} = 0$ for all n, with respect to homogeneous notation. Setting $\eta_n = 0$ for all $n < 0$, it is clear that this equation holds whenever $n < -1$. Fixing $n \geq -1$, suppose that η_j has already been defined for all $j \leq n$ in such a way that this equation holds with n replaced by any $j < n$. In particular, $d_{n-1} \circ \eta_n = -\eta_{n-1} \circ d_n$, so

$$d_{n-1} \circ \eta_n \circ d_{n+1} = -\eta_{n-1} \circ d_n \circ d_{n+1} = 0,$$

and consequently the image of $\eta_n \circ d_{n+1}$ is contained in the kernel of d_{n-1}. Since X is acyclic, the kernel of d_{n-1} is the image of d_n, which is of course also the image of $-d_n$. Since Z_{n+1} is projective, it follows that there is an η_{n+1} such that $-d_n \circ \eta_{n+1} = \eta_n \circ d_{n+1}$. □

In the last result stronger assumptions give stronger conclusions.

Lemma C5.3. *Suppose we are given a short exact sequence*

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

of left complexes over a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of R-modules. If X and Z are acyclic, then so is Y, if X and Z are projective, then so is Y, and if X and Z are free, then so is Y.

Proof. If X and Z are acyclic, then (with respect to homogeneous notation) $H_n(X) = 0$ and $H_n(Z) = 0$ for all n. In this circumstance the long exact sequence for homology implies that $H_n(Y) = 0$ for all n.

Suppose that X and Z are projective. For each $n \geq 0$ the sequence $0 \to X_n \to Y_n \to Z_n \to 0$ is exact. Since Z_n is projective, this sequence splits: there is a $q_n : Z_n \to Y_n$ such that $p_n \circ q_n$ is the identity, so that up to isomorphism we have $Y_n = X_n \oplus Z_n$, $i_n(x) = (x, 0)$, and $p_n(x, z) = z$. Since X_n is also projective, Y_n is a direct sum of two direct summands of free modules, so it is itself a direct summand of a free module. Of course if X and Z are free, this argument shows that Y is free. $\square$

The following result is one of the main points of the various constructions above:

Proposition C5.4. *For any short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ of R-modules there is a short exact sequence $0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$ of left complexes over the given sequence with X, Y, and Z free resolutions of A, B, and C respectively.*

Proof. Lemma C1.1 implies that there are free resolutions X and Z of A and C. Lemma C5.2 implies that there is a left complex Y over B and extensions of i and p to chain maps such that $0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$ is a short exact sequence of left complexes over $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$. The last result implies that Y is acyclic and free, hence a free resolution of Y. $\square$

In the obvious way one can define a category of short exact sequences of left complexes. The next result shows how to complete the construction of a morphism in this category when some of the data is given.

Lemma C5.5. *Let*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of R-modules and homomorphisms, let

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0 \quad \textit{and} \quad 0 \to X' \xrightarrow{i'} Y' \xrightarrow{p'} Z' \to 0$$

be short exact sequences of left complexes over its rows, and let $f : X \to X'$ and $h : Z \to Z'$ be chain maps extending $f : A \to A'$ and $h : C \to C'$. If Z and Z' are projective and X' is acyclic, there is a chain map $g : Y \to Y'$ extending $g : B \to B'$ such that the diagram of chain maps

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "p"] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & X' \arrow[r, "i'"] & Y' \arrow[r, "p'"] & Z' \arrow[r] & 0
\end{tikzcd}$$

commutes.

Proof. We use homogeneous notation, and, to save space, we drop the symbol for composition, writing this operation multiplicatively. Our objective is to construct a sequence of functions $g_n : Y_n \to Y'_n$, where g_{-1} agrees with the given $g : B \to B'$, such that for all n we have:

(a) $i'_n f_n = g_n i_n$;

(b) $p'_n g_n = h_n p_n$;

(c) $d'_n g_n = g_{n-1} d_n$.

If we set $g_{-1} = g$ and $g_n = 0$ for all $n < -1$, then these conditions hold for all $n \leq -1$. By way of induction, suppose that $n \geq -1$, and that we have already constructed satisfactory $g_0, \ldots, g_n$. We need to show that a suitable g_{n+1} exists.

Since Z_n is projective, the sequence $0 \to X_n \xrightarrow{i_n} Y_n \xrightarrow{p_n} Z_n \to 0$ splits, and similarly for Z'_n, so up to isomorphism we have

$$Y_n = X_n \oplus Z_n \quad \text{and} \quad Y'_n = X'_n \oplus Z'_n$$

with

$$i_n(x) = (x, 0), \quad p_n(x, z) = z, \quad i'_n(x') = (x', 0), \quad p'_n(x', z') = z'.$$

We have $p'_n(g_n(0, z)) = h_n(p_n(0, z)) = h_n(z)$, so there is a homomorphism $\theta_n : Z_n \to X'_n$ such that

$$g_n(0, z) = (\theta_n(z), h_n(z)).$$

We also have

$$g_n(x, 0) = g_n(i_n(x)) = i'_n(f_n(x)) = (f_n(x), 0).$$

Conversely, if g_{n+1} is defined by setting

$$g_{n+1}(x,z) = (f_{n+1}(x) + \theta_{n+1}(z), h_{n+1}(z))$$

for some homomorphism $\theta_{n+1} : Z_{n+1} \to X'_{n+1}$, then it is automatically the case that $i'_{n+1}f_{n+1} = g_{n+1}i_{n+1}$ and $p'_{n+1}g_{n+1} = h_{n+1}p_{n+1}$. Our problem is reduced to finding θ_{n+1} such that $d'_{n+1}g_{n+1} = g_n d_{n+1}$.

Now recall that Lemma C5.1 gives homomorphisms $\eta_n : Z_n \to X_{n-1}$ and $\eta'_n : Z'_n \to X'_{n-1}$ for all n such that

$$d_n(x,z) = (d_n(x) + \eta_n(z), d_n(z)), \qquad d'_n(x,z) = (d'_n(x) + \eta'_n(z), d'_n(z)),$$

$$d_n\eta_{n+1} + \eta_n d_{n+1} = 0, \qquad d'_n\eta'_{n+1} + \eta'_n d'_{n+1} = 0.$$

Combining these with the equation above, we find that

$$d'_{n+1}(g_{n+1}(x,z)) = d'_{n+1}(f_{n+1}(x) + \theta_{n+1}(z), h_{n+1}(z))$$
$$= \big(d'_{n+1}(f_{n+1}(x)) + d'_{n+1}(\theta_{n+1}(z)) + \eta'_{n+1}(h_{n+1}(z)), d'_{n+1}(h_{n+1}(z))\big)$$

and

$$g_n(d_{n+1}(x,z)) = g_n(d_{n+1}(x) + \eta_{n+1}(z), d_{n+1}(z))$$
$$= \big(f_n(d_{n+1}(x)) + f_n(\eta_{n+1}(z)) + \theta_n(d_{n+1}(z)), h_n(d_{n+1}(z))\big).$$

Since f and h are chain maps, $d'_{n+1}g_{n+1} = g_n d_{n+1}$ reduces to

$$d'_{n+1}\theta_{n+1} = f_n\eta_{n+1} + \theta_n d_{n+1} - \eta'_{n+1}h_{n+1}.$$

Since Z_{n+1} is projective, a suitable θ_{n+1} exists if the image of the right hand side is contained in the image of d'_{n+1} which is the same (because X' is acyclic) as the kernel of d'_n. The induction hypothesis implies that $d'_n g_n = g_{n-1}d_n$, so this equation holds with n in place of $n+1$, and consequently

$$d'_n\theta_n d_{n+1} = (f_{n-1}\eta_n + \theta_{n-1}d_n - \eta'_n h_n)d_{n+1} = (f_{n-1}\eta_n - \eta'_n h_n)d_{n+1}.$$

Using the fact that f and h are chain maps, we now have

$$d'_n(f_n\eta_{n+1} + \theta_n d_{n+1} - \eta'_{n+1}h_{n+1}) = d'_n(f_n\eta_{n+1} - \eta'_{n+1}h_{n+1}) + (f_{n-1}\eta_n - \eta'_n h_n)d_{n+1}$$
$$= f_{n-1}(d_n\eta_{n+1} + \eta_n d_{n+1}) - (d'_n\eta'_{n+1} + \eta'_n d'_{n+1})h_{n+1} = 0.$$

□

Again, the important point results from combining the particular constructive results.

Proposition C5.6. *If*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of R-modules and homomorphisms, then there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "p"] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & X' \arrow[r, "i'"] & Y' \arrow[r, "p'"] & Z' \arrow[r] & 0 \end{tikzcd}$$

of left complexes in which X, Y, Z, X', Y', and Z' are free resolutions of A, B, C, A', B', and C' and each chain map extends the corresponding given homomorphism.

Proof. Proposition C5.4 gives short exact sequences $0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$ and $0 \to X' \xrightarrow{i'} Y' \xrightarrow{p'} Z' \to 0$ of free resolutions over the two given short exact sequences of R-modules. Lemma C3.1 implies that f and h can be extended to chain maps $X \xrightarrow{f} X'$ and $Z \xrightarrow{h} Z'$, after which Lemma C5.5 implies that there is a chain map $Y \xrightarrow{g} Y'$ such that the diagram commutes. □

Now we are going to repeat everything above for the case of injective resolutions. The reader will quickly realize that everything is dual to what we did above, and was written by copying it, then making required modifications. Most readers will want to pass over it lightly, and will not miss much by doing so.

Lemma C5.7. *Suppose we are given a diagram*

$$\begin{tikzcd} & & & & \\ 0 \arrow[r] & I_1 \arrow[r, "i_1"] \arrow[u] & J_1 \arrow[r, "p_1"] \arrow[u] & K_1 \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & I_0 \arrow[r, "i_0"] \arrow[u, "d_0"] & J_0 \arrow[r, "p_0"] \arrow[u, "d_0"] & K_0 \arrow[r] \arrow[u, "d_0"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] \arrow[u, "d_{-1}"] & B \arrow[r, "p"] \arrow[u, "d_{-1}"] & C \arrow[r] \arrow[u, "d_{-1}"] & 0 \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

in which $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *is a short exact sequence of* R*-modules,* I *and* K *are right complexes over* A *and* C*, and for all* $n \geq 0$ *we have* $J_n = I_n \oplus K_n$ *and* $i_n(x) = (x, 0)$ *and* $p_n(x, z) = z$*. Then (with respect to homogeneous notation):*

(a) *For each* n *it is the case that* $d_n \circ i_n = i_{n+1} \circ d_n$ *and* $d_n \circ p_n = p_{n+1} \circ d_n$ *if and only if there is a homomorphism* $\eta_n : K_n \to I_{n+1}$ *such that*

$$d_n(i, k) = (d_n(i) + \eta_n(k), d_n(k)). \tag{$*$}$$

(b) *If this is the case for all* n*, then, for each* n*,* $0 = d_{n+1} \circ d_n : J_n \to J_{n+2}$ *if and only if*

$$0 = d_{n+1} \circ \eta_n + \eta_{n+1} \circ d_n : K_n \to I_{n+2}.$$

Proof. Taking compositions of $d_n : J_n \to J_{n-1}$ with the projections gives homomorphisms φ_n and ψ_n such that

$$d_n(i, k) = (\varphi_n(i, k), \psi_n(i, k)).$$

If $p_{n+1} \circ d_n = d_n \circ p_n$, then $\psi_n(i, k) = d_n(k)$, and if $i_{n+1} \circ d_n = d_n \circ i_n$, then $\varphi_n(i, 0) = d_n(i)$, so setting $\eta_n(k) = \varphi_n(0, k)$ gives $(*)$. Conversely, if, for some homomorphism $\eta_n : K_n \to I_{n+1}$, we define $d_n : J_n \to J_{n+1}$ by setting

$$d_n(i, k) = \big(d_n(i) + \eta_n(k), d_n(k)\big),$$

then $p_{n+1} \circ d_n = d_n \circ p_n$ and $i_{n+1} \circ d_n = d_n \circ i_n$ follow automatically.

Now (b) follows from a straightforward calculation:

$$\begin{aligned}
d_{n+1}(d_n(i, k)) &= d_{n+1}(d_n(i) + \eta_n(k), d_n(k)) \\
&= (d_{n+1}(d_n(i) + \eta_n(k)) + \eta_{n+1}(d_n(k)), d_{n+1}(d_n(k))) \\
&= (d_{n+1}(\eta_n(k)) + \eta_{n+1}(d_n(k)), 0).
\end{aligned}$$

□

Lemma C5.8. *Suppose we are given a short exact sequence*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of R*-modules, an injective right complex* I *over* A*, and an acyclic left complex* K *over* C*. Then there exists a right complex* J *over* B *and extensions of* i *and* p *to chain maps such that the sequence*

$$0 \to I \xrightarrow{i} J \xrightarrow{p} K \to 0$$

is a short exact sequence of left complexes over $0 \to A \to B \to C \to 0$ *and for each* $n \geq 0$ *the sequence* $0 \to I_n \xrightarrow{i_n} J_n \xrightarrow{p_n} K_n \to 0$ *splits.*

Proof. For each $n \geq 0$, let
$$J_n = I_n \oplus K_n,$$
and let $i_n : I_n \to J_n$ and $p_n : J_n \to K_n$ be the functions
$$i_n(i) = (i, 0) \quad \text{and} \quad p_n(i, k) = k.$$

Then $0 \to I_n \xrightarrow{i_n} J_n \xrightarrow{p_n} K_n \to 0$ is obviously exact, and it splits.

The remaining step is the construction of suitable homomorphisms $d_n : J_n \to J_{n-1}$. In view of the last result, it suffices to construct homomorphisms $\eta_n : K_n \to I_{n+1}$ such that (with respect to homogeneous notation)
$$d_{n+1} \circ \eta_n + \eta_{n+1} \circ d_n = 0$$
for all n. Setting $\eta_n = 0$ for all $n < 0$, it is clear that this equation holds whenever $n < -1$. Fixing $n \geq -1$, suppose that η_j has already been defined for all $j \leq n$ in such a way that this equation holds with n replaced by any $j < n$. Since I_{n+2} is is injective, to show that a suitable η_{n+1} exists it suffices (by (c) of Proposition B8.1) to show that the kernel of $d_{n+1} \circ \eta_n$ contains the kernel of d_n which (since K is acyclic) is the image of d_{n-1}. But since $\eta_n \circ d_{n-1} = -d_n \circ \eta_{n-1}$ we have
$$-d_{n+1} \circ \eta_n \circ d_{n-1} = d_{n+1} \circ d_n \circ \eta_{n-1} = 0.$$
□

Lemma C5.9. *Suppose we are given a short exact sequence*
$$0 \to I \xrightarrow{i} J \xrightarrow{p} K \to 0$$
of right complexes over a short exact sequence
$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$
of R-modules. If I and K are acyclic, then so is J. If I and K are injective, and each sequence $0 \to I_n \xrightarrow{i_n} J_n \xrightarrow{p_n} K_n \to 0$ splits, then J is injective.

Proof. If I and K are acyclic, then (with respect to homogeneous notation) $H^n(I) = 0$ and $H^n(K) = 0$ for all n. In this circumstance the long exact sequence for cohomology implies that $H^n(J) = 0$ for all n.

The second claim follows from the fact that finite direct sums of injective modules are injective, which is easily derived from condition (b) of Proposition B8.1. □

Proposition C5.10. *For any short exact sequence* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *of* R*-modules there is a short exact sequence* $0 \to I \xrightarrow{i} J \xrightarrow{p} K \to 0$ *of right complexes over* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *with* I*,* J*, and* K *injective resolutions of* A*,* B*, and* C *respectively, such that each* $0 \to I_n \xrightarrow{i_n} J_n \xrightarrow{p_n} K_n \to 0$ *splits.*

Proof. Lemma C1.1 implies that there are free resolutions I and K of A and C. Lemma C5.2 implies that there is a left complex J over B and extensions of i and p to chain maps such that $0 \to I \xrightarrow{i} J \xrightarrow{p} K \to 0$ is a short exact sequence of left complexes over $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$, such that each $0 \to I_n \xrightarrow{i_n} J_n \xrightarrow{p_n} K_n \to 0$ splits. The last result implies that J is acyclic and injective, hence an injective resolution of J. $\square$

Lemma C5.11. *Let*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of R*-modules and homomorphisms, let*

$$0 \to I \xrightarrow{i} J \xrightarrow{p} K \to 0 \quad \textit{and} \quad 0 \to I' \xrightarrow{i'} J' \xrightarrow{p'} K' \to 0$$

be short exact sequences of left complexes over its rows such that each $0 \to I_n \xrightarrow{i_n} J_n \xrightarrow{p_n} K_n \to 0$ *and each* $0 \to I'_n \xrightarrow{i'_n} J'_n \xrightarrow{p'_n} K'_n \to 0$ *splits, and let* $f : I \to I'$ *and* $h : K \to K'$ *be chain maps extending* $f : A \to A'$ *and* $h : C \to C'$*. If* K *and* K' *are projective and* I' *is acyclic, there is a chain map* $g : J \to J'$ *extending* $g : B \to B'$ *such that the diagram of chain maps*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"'] & J \arrow[r, "p"] \arrow[d, "g"'] & K \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & I' \arrow[r, "i'"] & J' \arrow[r, "p'"] & K' \arrow[r] & 0 \end{tikzcd}$$

commutes.

Proof. We use homogeneous notation, and, to save space we drop the symbol for composition, writing this operation multiplicatively. Our objective is to construct a sequence of functions $g_n : J_n \to J'_n$, where g_{-1} agrees with the given $g : B \to B'$, such that for all n we have:

(a) $i'_n f_n = g_n i_n$;

(b) $p'_n g_n = h_n p_n$;

(c) $d'_n g_n = g_{n+1} d_n$.

If we set $g_{-1} = g$ and $g_n = 0$ for all $n < -1$, then these conditions hold for all $n \leq -1$. By way of induction, suppose that $n \geq -1$, and that we have already constructed satisfactory $g_0, \dots, g_n$. We need to show that a suitable g_{n+1} exists.

By hypothesis, up to isomorphism we have

$$J_n = I_n \oplus K_n \quad \text{and} \quad J'_n = I'_n \oplus K'_n$$

with

$$i_n(i) = (i, 0), \quad p_n(i, k) = k, \quad i'_n(i') = (i', 0), \quad p'_n(i', k') = k'.$$

We have $p'_n(g_n(0, k)) = h_n(p_n(0, k)) = h_n(k)$, so there is a homomorphism $\theta_n : K_n \to I'_n$ such that

$$g_n(0, k) = (\theta_n(k), h_n(k)).$$

We also have

$$g_n(i, 0) = g_n(i_n(i)) = i'_n(f_n(i)) = (f_n(i), 0).$$

Conversely, if g_{n+1} is defined by setting

$$g_{n+1}(i, k) = (f_{n+1}(i) + \theta_{n+1}(k), h_{n+1}(k))$$

for some homomorphism $\theta_{n+1} : K_{n+1} \to I'_{n+1}$, then it is automatically the case that $i'_{n+1} f_{n+1} = g_{n+1} i_{n+1}$ and $p'_{n+1} g_{n+1} = h_{n+1} p_{n+1}$. Our problem is reduced to finding θ_{n+1} such that $g_{n+1} d_n = d'_n g_n$.

Now recall that Lemma C5.7 gives homomorphisms $\eta_n : K_n \to I_{n+1}$ and $\eta'_n : K'_n \to I'_{n+1}$ for all n such that

$$d_n(i, k) = (d_n(i) + \eta_n(k), d_n(k)), \qquad d'_n(i, k) = (d'_n(i) + \eta'_n(k), d'_n(k)),$$

$$d_{n+1}\eta_n + \eta_{n+1} d_n = 0, \qquad d'_{n+1}\eta'_n + \eta'_{n+1} d'_n = 0.$$

Combining these with the equation above, we find that

$$\begin{aligned} g_{n+1}(d_n(i, k)) &= g_{n+1}(d_n(i) + \eta_n(k), d_n(k)) \\ &= \big(f_{n+1}(d_n(i)) + f_{n+1}(\eta_n(k)) + \theta_{n+1}(d_n(k)), h_{n+1}(d_n(k))\big) \end{aligned}$$

and

$$\begin{aligned} d'_n(g_n(i, k)) &= d'_n(f_n(i) + \theta_n(k), h_n(k)) \\ &= \big(d'_n(f_n(i)) + d'_n(\theta_n(k)) + \eta'_n(h_n(k)), d'_n(h_n(k))\big). \end{aligned}$$

Since f and h are chain maps, $g_{n+1}d_n = d'_n g_n$ reduces to

$$\theta_{n+1}d_n = d'_n\theta_n + \eta'_n h_n - f_{n+1}\eta_n.$$

Since I'_{n+1} is injective, it suffices (by (c) of Proposition B8.1) to show that $\mathrm{Ker}(d_n) \subset \mathrm{Ker}(d'_n\theta_n + \eta'_n h_n - f_{n+1}\eta_n)$, and $\mathrm{Ker}(d_n) = \mathrm{Im}(d_{n-1})$ because K is acyclic.

The induction hypothesis implies that the equation above holds with all subscripts reduced by one, so

$$d'_n\theta_n d_{n-1} = d'_n(\eta'_{n-1}h_{n-1} - f_n\eta_{n-1}).$$

Using the fact that f and h are chain maps, we have

$$\begin{aligned}(d'_n\theta_n + \eta'_n h_n - f_{n+1}\eta_n)d_{n-1} &= d'_n(\eta'_{n-1}h_{n-1} - f_n\eta_{n-1}) + (\eta'_n h_n - f_{n+1}\eta_n)d_{n-1} \\ &= (d'_n\eta'_{n-1} + \eta'_n d'_{n-1})h_{n-1} - f_{n+1}(d_n\eta_{n-1} + \eta_n d_{n-1}) = 0.\end{aligned}$$

□

Proposition C5.12. *If*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of R-modules and homomorphisms, then there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"'] & J \arrow[r, "p"] \arrow[d, "g"'] & K \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & I' \arrow[r, "i'"] & J' \arrow[r, "p'"] & K' \arrow[r] & 0 \end{tikzcd}$$

of left complexes in which I, J, K, I', J', and K' are free resolutions of A, B, C, A', B', and C' and each chain map extends the corresponding given homomorphism.

Proof. Proposition C5.4 gives short exact sequences $0 \to I \xrightarrow{i} J \xrightarrow{p} K \to 0$ and $0 \to I' \xrightarrow{i'} J' \xrightarrow{p'} K' \to 0$ of free resolutions over the two given short exact sequences of R-modules such that each $0 \to I_n \xrightarrow{i_n} J_n \xrightarrow{p_n} K_n \to 0$ and each $0 \to I'_n \xrightarrow{i'_n} J'_n \xrightarrow{p'_n} K'_n \to 0$ splits. Lemma C3.1 implies that f and h can be extended to chain maps $I \xrightarrow{f} I'$ and $K \xrightarrow{h} K'$, after which Lemma C5.5 implies that there is a chain map $J \xrightarrow{g} J'$ such that the diagram commutes. □

C6 The Long Exact Sequences of Derived Functors

Let T be an additive covariant functor from the category of R-modules to the category of Q-modules, and let U be an additive contravariant functor from the category of R-modules to the category of Q-modules.

Proposition C6.1. *For each $n \in \mathbb{Z}$ and each short exact sequence*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of R-modules there are R-module homomorphisms

$$d_n : (L_nT)C \to (L_{n-1}T)A, \qquad \partial_n : (L_nU)A \to (L_{n-1}U)C,$$

$$\partial^n : (R^nT)C \to (R^{n+1}T)A, \qquad d^n : (R^nU)A \to (R^{n+1}U)C,$$

such that the sequences

$$\cdots \longrightarrow (L_nT)A \xrightarrow{i} (L_nT)B \xrightarrow{p} (L_nT)C \xrightarrow{d_n} (L_{n-1}T)A \longrightarrow \cdots$$

$$\cdots \longrightarrow (L_nU)C \xrightarrow{i} (L_nU)B \xrightarrow{p} (L_nU)A \xrightarrow{\partial_n} (L_{n-1}U)C \longrightarrow \cdots$$

$$\cdots \longrightarrow (R^nT)A \xrightarrow{i} (R^nT)B \xrightarrow{p} (R^nT)C \xrightarrow{\partial^n} (R^{n+1}T)A \longrightarrow \cdots$$

$$\cdots \longrightarrow (R^nU)C \xrightarrow{i} (R^nU)B \xrightarrow{p} (R^nU)A \xrightarrow{d^n} (R^{n+1}U)C \longrightarrow \cdots$$

are exact and, for any morphism

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences of R-modules, all diagrams

$$\begin{array}{ccc}
(L_nT)C & \xrightarrow{d_n} & (L_{n-1}T)A \\
{\scriptstyle (L_nT)h}\downarrow & & \downarrow{\scriptstyle (L_{n-1}T)f} \\
(L_nT)C' & \xrightarrow{d_n} & (L_{n-1}T)A'
\end{array}
\qquad
\begin{array}{ccc}
(L_nU)A' & \xrightarrow{\partial_n} & (L_{n-1}U)C' \\
{\scriptstyle (L_nU)f}\downarrow & & \downarrow{\scriptstyle (L_{n-1}U)h} \\
(L_nU)A & \xrightarrow{\partial_n} & (L_{n-1}U)C
\end{array}$$

$$\begin{array}{ccc}
(R^nT)C & \xrightarrow{\partial^n} & (R^{n+1}T)A \\
{\scriptstyle (R^nT)h}\downarrow & & \downarrow{\scriptstyle (R^{n+1}T)f} \\
(R^nT)C' & \xrightarrow{\partial^n} & (R^{n+1}T)A'
\end{array}
\qquad
\begin{array}{ccc}
(R^nU)A' & \xrightarrow{d^n} & (R^{n+1}U)C' \\
{\scriptstyle (R^nU)f}\downarrow & & \downarrow{\scriptstyle (R^{n+1}U)h} \\
(R^nU)A & \xrightarrow{d^n} & (R^{n+1}U)C
\end{array}$$

commute.

Proof. Proposition C5.4 gives a short exact sequence

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

of chain maps, where X, Y, and Z are projective resolutions of A, B, and C respectively. Since each Z_n is projective, (c) of Proposition B7.2 implies that each short exact sequence $0 \to X_n \to Y_n \to Z_n \to 0$ splits, so Proposition B5.2 implies that $0 \to TX_n \to TY_n \to TZ_n \to 0$ is exact (and actually splits, not that it matters here). Applying Proposition B4.3 to the exact sequence $0 \to T\mathbf{X} \xrightarrow{i} T\mathbf{Y} \xrightarrow{p} T\mathbf{Z} \to 0$ gives homomorphisms $d_n : (L_nT)C \to (L_{n-1}T)A$ such that the sequence

$$\cdots \longrightarrow (L_nT)A \xrightarrow{i} (L_nT)B \xrightarrow{p} (L_nT)C \xrightarrow{d_n} (L_{n-1}T)A \longrightarrow \cdots$$

is exact.

If

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

is a morphism of short exact sequences of R-modules, Proposition C5.6 gives a morphism

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences of chain maps, with X, Y, Z, X', Y', and Z' projective resolutions of A, B, C, A', B', and C' respectively. Applying Proposition B4.3 to the morphism

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{X} \arrow[r] \arrow[d, "f"'] & \mathbf{Y} \arrow[r] \arrow[d, "g"'] & \mathbf{Z} \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & \mathbf{X}' \arrow[r] & \mathbf{Y}' \arrow[r] & \mathbf{Z}' \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences of truncated chain complexes shows that the diagram

$$\begin{tikzcd}
(L_nT)C \arrow[r, "d_n"] \arrow[d, "(L_nT)h"'] & (L_{n-1}T)A \arrow[d, "(L_{n-1}T)f"] \\
(L_nT)C' \arrow[r, "d_n"] & (L_{n-1}T)A'
\end{tikzcd}$$

commutes.

The arguments for L_nU, R^nT, and R^nU follow the same general pattern, with obvious modifications (in particular, note the role of Proposition C2.5) so we regard the proof as complete. □

In order to save a bit of space, and to focus on what is new, the statement of the result above has a somewhat narrow and technical flavor, and it is important to understand its real significance. Specifically, there is a functor from the category of short exact sequences $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ of R-modules to the category of exact sequences of Q-modules with the following properties:

(a) The image of the sequence above is the long exact sequence

$$\cdots \longrightarrow (L_nT)A \xrightarrow{i} (L_nT)B \xrightarrow{p} (L_nT)C \xrightarrow{d_n} (L_{n-1}T)A \longrightarrow \cdots.$$

(b) The functor maps a morphism

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences to the chain map

$$\begin{tikzcd}
\to & (L_nT)A \arrow[r] \arrow[d, "f"'] & (L_nT)B \arrow[r] \arrow[d, "g"'] & (L_nT)C \arrow[r, "d_n"] \arrow[d, "h"'] & (L_{n-1}T)A \to \arrow[d, "f"'] \\
\to & (L_nT)A' \arrow[r] & (L_nT)B' \arrow[r] & (L_nT)C' \arrow[r, "d'_n"] & (L_{n-1}T)A' \to
\end{tikzcd}$$

Of course similar statements hold for the left derived functors of U and the right derived functors.

When T is right exact these properties amount to an axiomatic characterization of the left derived functors:

Theorem C6.2. *If T is a right exact functor from the ring of R-modules to the ring of Q-modules, then there are functors L_nT ($n \in \mathbb{Z}$) that take R-modules to Q-modules, and connecting homomorphisms*

$$d_n : (L_nT)M'' \to (L_{n-1}T)M'$$

for short exact sequences $0 \to M' \to M \to M'' \to 0$, with the following properties:

(a) $(L_nT)M = 0$ for all $n < 0$.

(b) L_0T *is naturally isomorphic to* T*.*

(c) If M *is projective, then* $(L_nT)M = 0$ *for all* $n > 0$*.*

(d) For all short exact sequences $0 \to M' \to M \to M'' \to 0$ *the sequence*

$$\cdots \to (L_nT)M' \to (L_nT)M \to (L_nT)M'' \xrightarrow{d_n} (L_{n-1}T)M' \to \cdots$$

is exact.

(e) The connecting homomorphisms are natural: for any morphism

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"'] & M \arrow[r] \arrow[d, "f"'] & M'' \arrow[r] \arrow[d, "f''"'] & 0 \\ 0 \arrow[r] & \tilde{M}' \arrow[r] & \tilde{M} \arrow[r] & \tilde{M}'' \arrow[r] & 0 \end{tikzcd}$$

of short exact sequences the diagram

$$\begin{tikzcd} (L_nT)M'' \arrow[r, "d_n"] \arrow[d, "(L_nT)f''"'] & (L_{n-1}T)M' \arrow[d, "(L_{n-1}T)f'"] \\ (L_nT)\tilde{M}'' \arrow[r, "d_n"] & (L_{n-1}T)\tilde{M}' \end{tikzcd}$$

commutes.

These properties determine the functors L_nT *and the connecting homomorphisms uniquely up to natural isomorphism.*

Proof. Prior to this point we have not defined L_nT for $n < 0$, so we can do so now using (a). That the functors L_nT have the asserted properties is simply a summary of our work up to this point, specifically the definition and Propositions C3.5 and C6.1.

The real work is proving uniqueness. Suppose that there are two different systems of functors and connecting homomorphisms with these properties, say $\{F_n, d_n\}$ and $\{\tilde{F}_n, \tilde{d}_n\}$. We will show that for each n, F_n is naturally isomorphic to $\tilde{F}_n$, and that this system of isomorphisms is natural with respect to the connecting homomorphisms. By symmetry, the same claim holds with the roles of the two variables reversed, and the result follows.

For any R-module M there is a short exact sequence $0 \to K \to P \to M \to 0$ with P projective. (For example we can take P to be the free module on a set of generators of M.) We are given that F_0 and $\tilde{F}_0$ are naturally isomorphic to T, hence to each other, and F_1P and $\tilde{F}_1P$ vanish by (c), so there is a diagram

$$\begin{tikzcd} 0 \to F_1M \arrow[r] & F_0K \arrow[r] \arrow[d, "\cong"'] & F_0P \arrow[d, "\cong"'] \\ 0 \to \tilde{F}_1M \arrow[r] & \tilde{F}_0K \arrow[r] & \tilde{F}_0P \end{tikzcd}$$

with exact rows. Let I_0K and $\tilde{I}_0K$ be the images of $F_1M \to F_0K$ and $\tilde{F}_1M \to \tilde{F}_0K$. Exactness implies that the image of I_0K in $\tilde{F}_0K$ is contained in the kernel of $\tilde{F}_0K \to \tilde{F}_0P$, which is $\tilde{I}_0K$. By symmetry the reverse inclusion also holds, so the isomorphism between F_0K and $\tilde{F}_0K$ restricts to an isomorphism between I_0K and $\tilde{I}_0K$. There is now an isomorphism between F_1M and $\tilde{F}_1M$ induced by requiring that the diagram

$$\begin{array}{ccc} F_1M & \xrightarrow{\cong} & I_0K \\ \big\downarrow & & \big\downarrow\cong \\ \tilde{F}_1M & \xrightarrow{\cong} & \tilde{I}_0K \end{array}$$

commutes. We would like to use this procedure to define a canonical isomorphism between F_1M and $\tilde{F}_1M$, but for this we must show that the definition is independent of choices, and then we will need to show that these isomorphisms constitute a natural isomorphism of F_1 and $\tilde{F}_1$.

Let $f : M \to M'$ be a homomorphism, and let $0 \to K' \to P' \to M' \to 0$ be a short exact sequence with P' projective. There is a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & f\big\downarrow & & \\ 0 & \longrightarrow & K' & \longrightarrow & P' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

of short exact sequences because, since P is projective, we can choose a map $P \to P'$ such that the right hand square commutes, after which exactness implies the existence of a map $K \to K'$ making the left hand square commute. Taking long exact sequences, and applying the naturality of the connecting homomorphism, gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_1M & \longrightarrow & F_0K & \longrightarrow & F_0P \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & F_1M' & \longrightarrow & F_0K' & \longrightarrow & F_0P' \end{array}$$

with exact rows. Since the kernel of $F_0K \to F_0P$ is mapped to the kernel of $F_0K' \to F_0P'$, we find that the diagram

$$\begin{array}{ccc} F_1M & \longrightarrow & I_0K \\ \big\downarrow & & \big\downarrow \\ F_1M' & \longrightarrow & I_0K' \end{array}$$

commutes.

Now consider the diagram

$$
\begin{tikzcd}
I_0K \arrow[rrr, "a"] \arrow[ddd, "e"'] & & & I_0K' \arrow[ddd, "h"] \\
& F_1M \arrow[ul, "b"'] \arrow[r, "d"] \arrow[d, "f"'] & F_1M' \arrow[ur, "c"] \arrow[d, "g"] & \\
& \tilde{F}_1M \arrow[r, "i"] \arrow[dl, "j"'] & \tilde{F}_1M' \arrow[dr, "k"] & \\
\tilde{I}_0K \arrow[rrr, "\ell"] & & & \tilde{I}_0K'
\end{tikzcd}
$$

in which f and g are defined by requiring that the left and right hand quadrilaterals commute. The diagonal maps are isomorphisms. We have just seen that the upper and lower quadrilateral are commutative. The diagram

$$
\begin{tikzcd}
I_0K \arrow[r] \arrow[d] & I_0K' \arrow[d] \\
\tilde{I}_0K \arrow[r] & \tilde{I}_0K'
\end{tikzcd}
\quad \text{commutes because} \quad
\begin{tikzcd}
F_0K \arrow[r] \arrow[d] & F_0K' \arrow[d] \\
\tilde{F}_0K \arrow[r] & \tilde{F}_0K'
\end{tikzcd}
$$

commutes and the kernel of $F_0K \to F_0P$ is mapped to the kernels of $F_0K' \to F_0P'$ and $\tilde{F}_0K \to \tilde{F}_0P$, while the latter kernels are mapped to the kernel of $\tilde{F}_0K' \to \tilde{F}_0P'$. These facts justify the calculation

$$if = k^{-1}\ell jf = k^{-1}\ell eb = k^{-1}hab = k^{-1}hcd = k^{-1}kgd = gd$$

which shows that the inner square commutes.

If $M' = M$ and f is the identity, this amounts to the definition of the isomorphism $F_1M \to \tilde{F}_1M$ being independent of the choice of the short exact sequence $0 \to K \to P \to M \to 0$. The general case amounts to this system of isomorphisms being a natural transformation between F_1 and $\tilde{F}_1$.

To show that this natural isomorphism between F_1 and $\tilde{F}_1$ is also natural with respect to the connecting homomorphisms, consider a short exact sequence $0 \to M' \to M \to M'' \to 0$. Let $0 \to K \to P \to M'' \to 0$ be a short exact sequence with P projective. As before, since P is projective there are maps $P \to M$ and $K \to M'$ such that

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d, "\mathbf{1}_{M''}"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}
$$

commutes. Applying naturality of the connecting homomorphism gives the commutative diagram

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & F_1M'' & \longrightarrow & F_0K & \longrightarrow & F_0P \\ & & & & \downarrow & & \downarrow & & \downarrow \\ F_1M & \longrightarrow & & & F_1M'' & \longrightarrow & F_0M' & \longrightarrow & F_0M. \end{array}$$

Thus $d_1 : F_1M'' \to F_0M'$ is the composition $F_1M'' \to F_0K \to F_0M'$. In the diagram

$$\begin{array}{ccccc} F_1M'' & \longrightarrow & F_0K & \longrightarrow & F_0M' \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{F}_1M'' & \longrightarrow & \tilde{F}_0K & \longrightarrow & \tilde{F}_0M' \end{array}$$

the left hand square commutes because the isomorphism between F_1M'' and $\tilde{F}_1M''$ was defined by requiring that this be the case. The right hand square commutes because F_0 and $\tilde{F}_0$ are naturally isomorphic. Thus the diagram

$$\begin{array}{ccc} F_1M'' & \xrightarrow{d_1} & F_0M' \\ \cong\downarrow & & \cong\downarrow \\ \tilde{F}_1M'' & \xrightarrow{\tilde{d}_1} & \tilde{F}_0M' \end{array}$$

commutes, as desired.

We now proceed by induction. Suppose, for some $n > 1$, that we have already shown that F_{n-1} and $\tilde{F}_{n-1}$ are naturally isomorphic, and that the connecting homomorphisms are natural with respect to this system of isomorphisms. Essentially the same argument as above—simplified somewhat because $F_{n-1}P = 0 = \tilde{F}_{n-1}P$ when P is projective—shows that F_n and $\tilde{F}_n$ are naturally isomorphic, with naturality with respect to d_n and $\tilde{d}_n$. The proof is complete. □

There are similar characterizations of R^nT when T is left exact, and of L_nU and R^nU when U is a contravariant functor, and when this result is cited later it will be understood as encompassing those claims. After all we have done to get here, it is certainly nice to know that this phase of our work is complete, in this sense that we have a set of properties that completely characterizes derived functors. However, subsequent analysis will not be based exclusively on these properties, since it will often be convenient to use the concrete definitions.

Chapter D

Derived Bifunctors

For R-modules M and N, the methods of the last chapter can be applied to $M \otimes_R -$, $- \otimes_R N$, $\mathrm{Hom}_R(M, -)$, and $\mathrm{Hom}_R(-, N)$. Double complexes will be introduced and used to show that the functors derived from $M \otimes_R -$ and $- \otimes_R N$ are the same, as are the functors derived from $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, N)$. In this way we obtain bifunctors $\mathrm{Tor}^R_n(-, -)$ and $\mathrm{Ext}^n_R(-, -)$ for $n \geq 0$. After summarizing the properties of these bifunctors axiomatically, we introduce the technique of "dimension shifting" and apply it to the study of various notions of dimension for R-modules that are defined in terms of the minimal lengths of various types of resolution.

D1 Left Derived Bifunctors

Our agenda now is to apply the method of derived functors to $- \otimes_R -$ in this section and $\mathrm{Hom}_R(-, -)$ in the next. To emphasize the properties of $- \otimes_R -$ that matter, in this section we work with an additive bifunctor $F(-, -)$ from pairs of R-modules to Q-modules that is covariant in both variables. It is interesting to note that the analysis in this section does not require that F be half exact.

For any R-module N, $F(-, N)$ is an additive covariant univariate functor, and has left derived functors. We will show that these combine across the various N to form a system of derived bivariate functors. Symmetrically, we can combine the derived functors of the various $F(M, -)$ to form a second system of derived bifunctors. There is also a third system of derived functors obtained by resolving both variables simultaneously. We will show that if $F(X, -)$ and $F(-, Y)$ are exact functors whenever X and Y is projective, then these three systems of derived functors are naturally isomorphic.

As before, we fix a system of projective resolutions for all R-modules. We also fix a system of chain maps $f : X \to Y$ extending the various homomorphisms $f : M \to N$, where X and Y are the chosen resolutions of M and N.

If X is the chosen resolution of M and N is an R-module, there is a chain complex

$$\cdots \to F(X_2, N) \xrightarrow{F(d_2,N)} F(X_1, N) \xrightarrow{F(d_1,N)} F(X_0, N) \to 0$$

that we denote by $F(\mathbf{X}, N)$. The n^{th} left derived functor of $F(-, N)$ evaluated at M is, by definition, $H_n(F(\mathbf{X}, N))$, and is denoted by $F_n^1(M, N)$. If $f : M \to M'$ is a homomorphism with chosen extension $f : X \to X'$, then $F_n^1(f, N)$ is, by definition, $H_n(F(f, N))$. For each N have defined functors $F_n^1(-, N)$.

For any $g : N \to N'$ there is a chain map

$$F(\mathbf{X}, g) : F(\mathbf{X}, N) \to F(\mathbf{X}, N')$$

whose n^{th} component is $F(X_n, g)$, and it is easy to see that $F(\mathbf{X}, -)$ is a functor from the category of R-modules to the category of chain complexes. For each $n = 0, 1, 2, \ldots$ let

$$F_n^1(M, g) = H_n(F(\mathbf{X}, g)) : F_n^1(M, N) \to F_n^1(M, N').$$

For each M we have defined functors $F_n^1(M, -)$.

We need to check that $F_n^1(-, -)$ is a bifunctor. Let M' be another R-module with chosen resolution X', and let $f : M \to M'$ be a homomorphism. Because F is a bifunctor the diagram of chain complexes

$$\begin{tikzcd}
F(\mathbf{X}, N) \arrow[r, "F(f,N)"] \arrow[d, "F(\mathbf{X},g)"'] & F(\mathbf{X}', N) \arrow[d, "F(\mathbf{X}',g)"] \\
F(\mathbf{X}, N') \arrow[r, "F(f,N')"] & F(\mathbf{X}', N')
\end{tikzcd}$$

commutes, so applying the functor H_n gives the desired commutativity:

$$F_n^1(M', g) \circ F_n^1(f, N) = F_n^1(f, N') \circ F_n^1(M, g).$$

As in our work with univariate derived functors, there is the task of checking that the definition of $F_n^1(-, -)$ does not depend (up to natural isomorphism) on the choices of resolutions and extensions of homomorphisms of chain maps. Consider a second system of choices of resolutions X' for each M and extensions $f' : X' \to Y'$ for each homomorphism $f : M \to N$. Let $F_n^{1\prime}$ denote the bifunctors derived from this system of resolutions and extensions. Fix a system of chain maps $i^M : X \to X'$ where each i^M extends the identity on M. In view of our work with univariate derived functors, we already know that for each $f : M \to M'$ and N, the diagram

$$\begin{tikzcd}
F_n^1(M, N) \arrow[r, "F_n^1(f,N)"] \arrow[d, "H_n(F(i^M,N))"'] & F_n^1(M', N) \arrow[d, "H_n(F(i^{M'},N))"] \\
F_n^{1\prime}(M, N) \arrow[r, "F_n^{1\prime}(f,N)"] & F_n^{1\prime}(M', N)
\end{tikzcd}$$

commutes.

Fix a homomorphism $g : N \to N'$. There is a diagram of chain maps

$$\begin{tikzcd}
F(\mathbf{X}, N) \arrow[r, "F(\mathbf{X},g)"] \arrow[d, "F(i^M,N)"'] & F(\mathbf{X}, N') \arrow[d, "F(i^M,N')"] \\
F(\mathbf{X}', N) \arrow[r, "F(\mathbf{X}',g)"] & F(\mathbf{X}', N')
\end{tikzcd}$$

which is commutative simply because F is a bifunctor. Applying the functor H_n gives the commutative diagram

$$\begin{tikzcd}
F^1_n(\mathbf{X}, N)) \arrow[r, "F^1_n(M,g)"] \arrow[d, "H_n(F(i^M,N))"'] & F^1(M, N') \arrow[d, "H_n(F(i^M,N'))"] \\
F^{1\prime}_n(M, N) \arrow[r, "F^{1\prime}_n(M,g)"] & F^{1\prime}_n(\mathbf{X}', N').
\end{tikzcd}$$

Thus the maps $H_n(F(i^M, N))$ constitute a natural isomorphism between the bifunctor F^1_n and $F^{1\prime}_n$.

Of course we can define a second system of derived bifunctors $F^2_n(-,-)$ by resolving the second variable instead of the first. This gives rise to the problem of showing that the two constructions give the same result, up to natural isomorphism. This aspect requires a rather elaborate construction and some new ideas.

A *double complex* Z is a diagram

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
Z_{02} \arrow[d, "d_{02}"'] & Z_{12} \arrow[l, "\partial_{12}"'] \arrow[d, "d_{12}"'] & Z_{22} \arrow[l, "\partial_{22}"'] \arrow[d, "d_{22}"'] & {} \arrow[l] \\
Z_{01} \arrow[d, "d_{01}"'] & Z_{11} \arrow[l, "\partial_{11}"'] \arrow[d, "d_{11}"'] & Z_{21} \arrow[l, "\partial_{21}"'] \arrow[d, "d_{21}"'] & {} \arrow[l] \\
Z_{00} & Z_{10} \arrow[l, "\partial_{10}"'] & Z_{20} \arrow[l, "\partial_{20}"'] & {} \arrow[l]
\end{tikzcd}$$

that is *anticommutative* in the sense that

$$\partial_{i,j-1} \circ d_{ij} = -d_{i-1,j} \circ \partial_{ij}$$

for all i, j, and that also satisfies

$$\partial_{i-1,j} \circ \partial_{ij} = 0 \quad \text{and} \quad d_{i,j-1} \circ d_{ij} = 0$$

for all i and j. (We extend Z by setting $Z_{ij} = 0$ if $i < 0$ or $j < 0$, so the range of i and j in this and similar conditions below is all of $\mathbb{Z}$.) If Z' is a second double complex defined by modules Z'_{ij} and homomorphisms d'_{ij} and ∂'_{ij}, a *chain map* from Z to Z' is a system of homomorphisms $g_{ij} : Z_{ij} \to Z'_{ij}$ such that $\partial'_{ij} \circ g_{ij} = g_{i-1,j} \circ \partial_{ij}$ and $d'_{ij} \circ g_{ij} = g_{i,j-1} \circ d_{ij}$ for all i and j. Evidently there is a category of double complexes and chain maps.

Now let $\Sigma(Z)$ be the chain complex

$$\cdots \to \Sigma_2(Z) \xrightarrow{\Delta_2} \Sigma_1(Z) \xrightarrow{\Delta_1} \Sigma_0(Z) \to 0$$

where, for each n, $\Sigma_n(Z) = \bigoplus_{i+j=n} Z_{ij}$ and Δ_n (which is sometimes denoted by $\Delta_n(Z)$) is the homomorphism given by

$$\Delta_n(z_{ij}) = \partial_{ij}(z_{ij}) + d_{ij}(z_{ij}).$$

The computation

$$\Delta_{n-1}(\Delta_n(z_{ij})) = \partial_{i,j-1}(\partial_{ij}(z_{ij})) + d_{i,j-1}(\partial_{ij}(z_{ij})) + \partial_{i-1,j}(d_{ij}(z_{ij})) + d_{i-1,j}(d_{ij}(z_{ij}))$$
$$= d_{i,j-1}(\partial_{ij}(z_{ij})) + \partial_{i-1,j}(d_{ij}(z_{ij})) = 0$$

shows that $\Sigma(Z)$ is indeed a chain complex.

If $g : Z \to Z'$ is a chain map of double complexes, then we define homomorphisms $\Sigma_n(g) : \Sigma_n(Z) \to \Sigma_n(Z')$ by setting

$$\Sigma_n(g)(z_{0n} + \cdots + z_{n0}) = g_{0n}(z_{0n}) + \cdots + g_{n0}(z_{n0}).$$

It is simple to verify

$$\Delta_n(Z') \circ \Sigma_n(g) = \Sigma_{n-1}(g) \circ \Delta_n(Z)$$

for all n, so these homomorphisms constitute a chain map

$$\Sigma(g) : \Sigma(Z) \to \Sigma(Z').$$

Clearly Σ is a covariant functor from the category of double complexes to the category of chain complexes.

The algebraic principle underlying our work is:

Proposition D1.1. *Suppose the double complex Z is expanded to the diagram*

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 & W_2 \arrow[l] \arrow[d, "d_2"'] & Z_{02} \arrow[l, "\varepsilon_2"'] \arrow[d, "d_{02}"'] & Z_{12} \arrow[l, "\partial_{12}"'] \arrow[d, "d_{12}"'] & Z_{22} \arrow[l, "\partial_{22}"'] \arrow[d, "d_{22}"'] & \arrow[l] \\
0 & W_1 \arrow[l] \arrow[d, "d_1"'] & Z_{01} \arrow[l, "\varepsilon_1"'] \arrow[d, "d_{01}"'] & Z_{11} \arrow[l, "\partial_{11}"'] \arrow[d, "d_{11}"'] & Z_{21} \arrow[l, "\partial_{21}"'] \arrow[d, "d_{21}"'] & \arrow[l] \\
0 & W_0 \arrow[l] & Z_{00} \arrow[l, "\varepsilon_0"'] & Z_{10} \arrow[l, "\partial_{10}"'] & Z_{20} \arrow[l, "\partial_{20}"'] & \arrow[l]
\end{tikzcd}$$

in which $\cdots \to W_2 \xrightarrow{d_2} W_1 \xrightarrow{d_1} W_0 \to 0$ *is a chain complex* W, $d_n \circ \varepsilon_n = \varepsilon_{n-1} \circ d_{0n}$ *for all* n*, and every row is exact. Then the homomorphisms* $\tilde{\varepsilon}_n : \Sigma_n(Z) \to W_n$ *given by*

$$\tilde{\varepsilon}_n(z_{0n} + \cdots + z_{n0}) = \varepsilon_n(z_{0n})$$

define a chain map $\tilde{\varepsilon} : \Sigma(Z) \to W$*, and each* $H_n(\tilde{\varepsilon}) : H_n(\Sigma(Z)) \to H_n(W)$ *is an isomorphism.*

Proof. The verification that $\tilde{\varepsilon}$ is a chain map is a calculation in which the third equality is from exactness of the rows:

$$\begin{aligned}
&\tilde{\varepsilon}_{n-1}\big(\Delta_n(z_{0n} + \cdots + z_{n0})\big)\\
&\quad = \tilde{\varepsilon}_{n-1}\big(d_{0n}(z_{0n}) + d_{n-1,1}(z_{n-1,1}) + \cdots + \partial_{1,n-1}(z_{1,n-1}) + \partial_{n0}(z_{n0})\big)\\
&\quad = \varepsilon_{n-1}\big(d_{0n}(z_{0n}) + \partial_{1,n-1}(z_{1,n-1})\big)\\
&\quad = \varepsilon_{n-1}\big(d_{0n}(z_{0n})\big) = d_n\big(\varepsilon_n(z_{0n})\big) = d_n\big(\tilde{\varepsilon}_n(z_{0n} + \cdots + z_{n0})\big).
\end{aligned}$$

The calculation above implies that $\tilde{\varepsilon}_n$ maps cycles and boundaries in $\Sigma_n(Z)$ to cycles and boundaries in W_n. We will show that $\tilde{\varepsilon}_n$ maps the cycles in $\Sigma_n(Z)$ onto the cycles in W_n, and that if a preimage of a boundary in W_n is a cycle in $\Sigma_n(Z)$, then it is a boundary. These facts imply that $H_n(\tilde{\varepsilon})$ is surjective and injective, respectively.

Let $w_n \in W_n$ be a cycle. Since row n is exact we can choose $z_{0n} \in Z_{0n}$ with $\varepsilon_n(z_{0n}) = w_n$. Since

$$\varepsilon_{n-1}(d_{0n}(z_{0n})) = d_n(\varepsilon_n(z_{0n})) = d_n(w_n) = 0$$

and row $n-1$ is exact, there is $z_{1,n-1} \in Z_{1,n-1}$ such that $\partial_{1,n-1}(z_{1,n-1}) = -d_{0n}(z_{0n})$. To see that there is a $z_{2,n-2} \in Z_{2,n-2}$ such that $\partial_{2,n-2}(z_{2,n-2}) = -d_{1,n-1}(z_{1,n-1})$ we combine the exactness of row $n-2$ with the computation

$$\partial_{1,n-2}(d_{1,n-1}(z_{1,n-1})) = d_{0,n-1}(\partial_{1,n-1}(z_{1,n-1})) = -d_{0,n-1}(d_{0n}(z_{0n})) = 0.$$

Continuing in this manner eventually produces a cycle $z_{0n} + \cdots + z_{n0} \in \Sigma_n(Z)$ such that $\tilde{\varepsilon}_n(z_{0n} + \cdots + z_{n0}) = w_n$.

Now suppose that $w_n = d_{n+1}(w_{n+1})$ is a boundary and $z_{0n} + \cdots + z_{n0}$ is a cycle in $\Sigma_n(Z)$ with $\tilde{\varepsilon}_n(z_{0n} + \cdots + z_{n0}) = w_n$. Since row $n+1$ is exact there is $z_{0,n+1} \in Z_{0,n+1}$ such that $\varepsilon_{n+1}(z_{0,n+1}) = w_{n+1}$. Now

$$\varepsilon_n(z_{0n} - d_{0,n+1}(z_{0,n+1})) = \varepsilon_n(z_{0n}) - d_{n+1}(\varepsilon_{n+1}(z_{0,n+1})) = 0,$$

so the exactness of row n implies that there is $z_{1n} \in Z_{1n}$ such that $\partial_{1n}(z_{1n}) = z_{0n} - d_{0,n+1}(z_{0,n+1})$. Anticommutativity implies that

$$\begin{aligned}
\partial_{1,n-1}(z_{1,n-1} - d_{1n}(z_{1n})) &= \partial_{1,n-1}(z_{1,n-1}) + d_{0n}(\partial_{1n}(z_{1n}))\\
&= \partial_{1,n-1}(z_{1,n-1}) + d_{0n}(z_{0n} - d_{0,n+1}(z_{0,n+1}))
\end{aligned}$$

$$= \partial_{1,n-1}(z_{1,n-1}) + d_{0n}(z_{0n}) = 0$$

because $z_{0n} + \cdots + z_{n0}$ is a cycle. Therefore the exactness of row $n-1$ implies the existence of $z_{2,n-1} \in Z_{2,n-1}$ such that $\partial_{2,n-1}(2, z_{n-1}) = z_{1,n-1} - d_{1n}(z_{1n})$. Continuing in this manner eventually produces $z_{0,n+1} + \cdots + z_{n+1,0} \in \Sigma_{n+1}(Z)$ such that

$$\Delta_{n+1}(z_{0,n+1} + \cdots + z_{n+1,0}) = z_{0n} + \cdots + z_{n0}.$$

□

Proposition D1.2. *In addition to the hypotheses of the last result, suppose that we have a second diagram of this form, with primes attached to all symbols, and that we have a chain map* $f = (f_n) : W \to W'$ *and a chain map of double complexes* $\varphi = (\varphi_{ij}) : Z \to Z'$ *such that all diagrams*

$$\begin{array}{ccc} Z_{0n} & \xrightarrow{\varphi_{0n}} & Z'_{0n} \\ {\scriptstyle \varepsilon_n}\downarrow & & \downarrow{\scriptstyle \varepsilon'_n} \\ W_n & \xrightarrow{f_n} & W'_n \end{array}$$

commute. Then for each n *we have* $f_n \circ \tilde{\varepsilon}_n = \tilde{\varepsilon}'_n \circ \Sigma_n(\varphi)$*, so that the following diagram is commutative:*

$$\begin{array}{ccc} H_n(\Sigma(Z)) & \xrightarrow{H_n(\Sigma(\varphi))} & H_n(\Sigma(Z')) \\ {\scriptstyle H_n(\tilde{\varepsilon})}\downarrow & & \downarrow{\scriptstyle H_n(\tilde{\varepsilon}')} \\ H_n(W) & \xrightarrow{H_n(f)} & H_n(W'). \end{array}$$

Proof. This is a straightforward calculation:

$$f_n(\tilde{\varepsilon}_n(z_{0n} + \cdots + z_{n0})) = f_n(\varepsilon_n(z_{0n})) = \varepsilon'_n(\varphi_{0n}(z_{0n}))$$
$$= \tilde{\varepsilon}'_n(\varphi_{0n}(z_{0n}) + \cdots + \varphi_{n0}(z_{n0})) = \tilde{\varepsilon}'_n(\Sigma_n(\varphi)(z_{0n} + \cdots + z_{n0})).$$

□

We can now proceed as follows. Suppose that X is the chosen projective resolution of M and Y is the chosen projective resolution of N. Let $F(\mathbf{X}, \mathbf{Y})$ be the double complex defined by setting:

$$F_{ij}(\mathbf{X}, \mathbf{Y}) = F(X_i, Y_j), \quad d_{ij} = F(X_i, d_j), \quad \partial_{ij} = (-1)^i F(d_i, Y_j).$$

Define

$$F^*_n(M, N) = H_n(\Sigma(F(\mathbf{X}, \mathbf{Y}))).$$

If $f : M \to M'$ and $g : N \to N'$ are homomorphisms, X' and Y' are the chosen resolutions of M' and N', and as usual f and g also denote the extensions to chain maps $f : X \to X'$ and $g : Y \to Y'$, we let $F(f, g)$ denote the map of double complexes whose ij-component is $F(f_i, g_j) : F(X_i, Y_j) \to F(X'_i, Y'_j)$, and we set

$$F_n^*(f, g) = H_n(\Sigma(F(f, g))) : F_n^*(M, N) \to F_n^*(M', N').$$

A natural approach at this point would be to show that the F_n^* are bifunctors that are naturally isomorphic to the F_n^1 and (by symmetry) the F_n^2. It turns out that showing that the F_n^* are bifunctors is unnecessary, since the following result is enough to show that the F_n^1 and F_n^2 are naturally isomorphic, which is our ultimate concern. (As it happens, that F^* is in fact a bifunctor also follows directly from the next result.)

Proposition D1.3. *If $F(X, -)$ and $F(-, Y)$ are exact whenever X and Y are projective, then there is a system of homomorphisms $\iota_{(M,N)} : F_n^*(M, N) \to F_n^1(M, N)$ such that the diagram*

$$\begin{tikzcd} F_n^*(M,N) \arrow[r, "F_n^*(f,g)"] \arrow[d, "\iota_{(M,N)}"'] & F_n^*(M',N') \arrow[d, "\iota_{(M',N')}"] \\ F_n^1(M,N) \arrow[r, "F_n^1(f,g)"] & F_n^1(M',N') \end{tikzcd}$$

commutes.

Proof. Consider the diagram

$$\begin{tikzcd} & \arrow[d] & \arrow[d] & \arrow[d] & \\ 0 & F(X_1, N) \arrow[l] \arrow[d, "d_1"'] & F(X_1, Y_0) \arrow[l, "\varepsilon_1"'] \arrow[d, "\partial^{10}"'] & F(X_1, Y_1) \arrow[l, "d^{11}"'] \arrow[d, "\partial^{11}"'] & \arrow[l, "d^{12}"'] \\ 0 & F(X_0, N) \arrow[l] & F(X_0, Y_0) \arrow[l, "\varepsilon_0"'] & F(X_0, Y_1) \arrow[l, "d^{01}"'] & \arrow[l, dotted, "d^{02}"'] \end{tikzcd}$$

The rows are exact because each $F(X_j, -)$ is an exact functor. Therefore Proposition D1.1 gives isomorphisms $\iota_{(M,N)} : F_n^*(M, N) \to F_n^1(M, N)$, and, by virtue of two applications of Proposition D1.2, the diagrams

$$\begin{tikzcd} F_n^*(M,N) \arrow[r, "F_n^*(f,N)"] \arrow[d, "\iota_{(M,N)}"'] & F_n^*(M',N) \arrow[d, "\iota_{(M',N)}"] \\ F_n^1(M,N) \arrow[r, "F_n^1(f,N)"] & F_n^1(M',N) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F_n^*(M',N) \arrow[r, "F_n^*(M',g)"] \arrow[d, "\iota_{(M',N)}"'] & F_n^*(M',N') \arrow[d, "\iota_{(M',N')}"] \\ F_n^1(M',N) \arrow[r, "F_n^1(M',g)"] & F_n^1(M',N') \end{tikzcd}$$

commute. The claim is obtained by combining these. □

Summarizing, we started with an additive bifunctor F that is covariant in each variable, and we defined three sequences of derived bifunctors F_n^1, F_n^2, and F_n^*. The last result implies that these are naturally isomorphic if $F(X, -)$ and

$F(-,Y)$ are exact whenever X and Y are projective. Now recall (Proposition B7.4) that a projective module is flat, so $X \otimes_R -$ and $- \otimes_R Y$ are exact whenever X and Y are projective. Therefore we can apply the method described above to define a sequence of derived bifunctors, which are denoted by $\mathrm{Tor}_n^R(-,-)$. These may be computed from $- \otimes_R -$ by resolving either variable.

D2 Right Derived Bifunctors

Our task in this section is to define bifunctors derived from $\mathrm{Hom}_R(-,-)$. Let M and N be R-modules. Recall that $\mathrm{Hom}_R(M,-)$ is covariant and left exact, so if J is an injective resolution of N, then we can take the cohomology of the sequence $\mathrm{Hom}_R(M, \mathbf{J})$. On the other hand $\mathrm{Hom}_R(-, N)$ is contravariant and left exact, so if X is a projective resolution of M, then we can take the cohomology of $\mathrm{Hom}_R(\mathbf{X}, N)$. The overall plan is to do both, then show that the resulting derived bifunctors are naturally isomorphic. To a large extent the work parallels what we did in the last section, but, because of the mixed variances, things are in some ways a bit different.

Let F be an additive bifunctor taking pairs of R-modules to Q-modules that (like $\mathrm{Hom}_R(-,-)$) is contravariant in its first variable and covariant in its second variable. For the time being there is no need to assume that F is half exact. Also, note that our terminology of "left" and "right" derived bifunctors is at least a bit misleading, because there are various possibilities in addition to those considered here and in the last section.

We retain the system of projective resolutions and chain maps between them that we fixed in the last section, and we now suppose that a system of injective resolutions and cochain maps have been selected.

We first define bifunctors F_1^n for $n = 0, 1, 2, \ldots$. For each N the univariate functor $F_1^n(-, N)$ is given by the version of $R^n F(-, N)$ resulting from the chosen projective resolutions. Concretely, $F_1^n(M, N) = H^n(F(\mathbf{X}, N))$ and if $f : M \to M'$, then

$$F_1^n(f, N) = H^n(F(f, N)) : F_1^n(M', N) \to F_1^n(M, N).$$

If X is the chosen projective resolution of M and $g : N \to N'$ is a homomorphism, there is a chain map

$$F(\mathbf{X}, g) : F(\mathbf{X}, N) \to F(\mathbf{X}, N')$$

whose n^{th} component is $F(X_n, g)$, and we set

$$F_1^n(M, g) = H^n(F(\mathbf{X}, g)) : F_1^n(M, N) \to F_1^n(M, N').$$

As before we must show that the we have defined bifunctors insofar as

$$F_1^n(f, N') \circ F_1^n(M', g) = F_1^n(M, g) \circ F_1^n(f, N)$$

for all homomorphisms $f : M \to M'$ and $g : N \to N'$, and we must show that (up to natural isomorphism) nothing depends on the chosen resolutions. The arguments follow the patterns laid out in our discussion of left derived bifunctors, with slight and obvious modifications, so we will not write them out here.

We now define a second system of bifunctors F_2^n. For each M the univariate functor $F_2^n(M,-)$ is given by the version of $R^nF(M,-)$ resulting from the chosen injective resolutions. If J is the chosen injective resolution of N, then $F_2^n(M,N) = H^n(F(M,\mathbf{J}))$, and if $g : N \to N'$ is a homomorphism, then

$$F_2^n(M,g) = H^n(F(M,g)) : F_2^n(M,N) \to F_2^n(M,N').$$

If $f : M \to M'$, then

$$F_2^n(f,N) = H^n(F(f,\mathbf{J})) : F_2^n(M',N) \to F_2^n(M,N).$$

Again, we omit the verification that nothing depends on the choices of resolutions, and that

$$F_2^n(f,N') \circ F_2^n(M',g) = F_2^n(M,g) \circ F_2^n(f,N)$$

for all homomorphisms $f : M \to M$ and $g : N \to N'$.

We need to show that if F shares certain properties of $\mathrm{Hom}_R(-,-)$, then F_1^n and F_2^n are naturally isomorphic; we will prove this by constructing an intermediate bifunctor that is naturally isomorphic to each.

A *codouble complex* A is a diagram

$$\begin{tikzcd}
 & & & & & \\
A^{02} \arrow[u] \arrow[rr, "\partial^{02}"] & & A^{12} \arrow[u] \arrow[rr, "\partial^{12}"] & & A^{22} \arrow[u] \arrow[r] & \\
A^{01} \arrow[u, "d^{01}"] \arrow[rr, "\partial^{01}"] & & A^{11} \arrow[u, "d^{11}"] \arrow[rr, "\partial^{11}"] & & A^{21} \arrow[u, "d^{21}"] \arrow[r] & \\
A^{00} \arrow[u, "d^{00}"] \arrow[rr, "\partial^{00}"] & & A^{10} \arrow[u, "d^{10}"] \arrow[rr, "\partial^{10}"] & & A^{20} \arrow[u, "d^{20}"] \arrow[r] &
\end{tikzcd}$$

that is *anticommutative* in the sense that

$$d^{i+1,j} \circ \partial^{ij} = -\partial^{i,j+1} \circ d^{ij}$$

for all i, j, and that also satisfies

$$\partial^{i+1,j} \circ \partial^{ij} = 0 \quad \text{and} \quad d^{i,j+1} \circ d^{ij} = 0$$

for all i and j. (We extend A by setting $A^{ij} = 0$ if $i < 0$ or $j < 0$.) If A' is a second codouble complex defined by modules A'^{ij} and homomorphisms d'^{ij} and ∂'^{ij}, a *chain map* from A to A' is a system of homomorphisms $g^{ij} : A^{ij} \to A'^{ij}$ such that $\partial'^{ij} \circ g^{ij} = g^{i+1,j} \circ \partial^{ij}$ and $d'^{ij} \circ g^{ij} = g^{i,j+1} \circ d^{ij}$ for all i and j. Evidently there is a category of codouble complexes and chain maps.

Now let $\Sigma(A)$ be the cochain complex

$$0 \to \Sigma^0(A) \xrightarrow{\Delta^0} \Sigma^1(A) \xrightarrow{\Delta^1} \Sigma^2(A) \to$$

where, for each n, $\Sigma^n(A) = \bigoplus_{i+j=n} A^{ij}$ and Δ^n (which is sometimes denoted by $\Delta^n(A)$) is the homomorphism given by

$$\Delta^n(a^{ij}) = \partial^{ij}(a^{ij}) + d^{ij}(a^{ij}).$$

The computation

$$\begin{aligned}\Delta^{n+1}(\Delta^n(a^{ij})) &= \partial^{i+1,j}(\partial^{ij}(a^{ij}))+d^{i+1,j}(\partial^{ij}(a^{ij}))+\partial^{i,j+1}(d^{ij}(a^{ij}))+d^{i,j+1}(d^{ij}(a^{ij})) \\ &= d^{i+1,j}(\partial^{ij}(a^{ij})) + \partial^{i,j+1}(d^{ij}(a^{ij})) = 0\end{aligned}$$

shows that $\Sigma(A)$ is indeed a cochain complex.

If $g : A \to A'$ is a chain map of codouble complexes, then we define homomorphisms $\Sigma^n(g) : \Sigma^n(A) \to \Sigma^n(A')$ by setting

$$\Sigma^n(g)(a^{0n} + \cdots + a^{n0}) = g^{0n}(a^{0n}) + \cdots + g^{n0}(a^{n0}).$$

It is simple to verify

$$\Delta^n(A') \circ \Sigma^n(g) = \Sigma^{n+1}(g) \circ \Delta^n(A)$$

for all n, so these homomorphisms constitute a chain map

$$\Sigma(g) : \Sigma(A) \to \Sigma(A').$$

Clearly Σ is a covariant functor from the category of codouble complexes to the category of cochain complexes.

As in the last section, we now need to attend to the underlying algebra.

Proposition D2.1. *Suppose the codouble complex A is expanded to the diagram*

$$\begin{tikzcd}
 & {} & {} & {} & {} & \\
0 \arrow[r] & C^2 \arrow[u] \arrow[r, "\eta^2"] & A^{02} \arrow[u] \arrow[r, "\partial^{02}"] & A^{12} \arrow[u] \arrow[r, "\partial^{12}"] & A^{22} \arrow[u] \arrow[r] & {} \\
0 \arrow[r] & C^1 \arrow[u, "d^1"] \arrow[r, "\eta^1"] & A^{01} \arrow[u, "d^{01}"] \arrow[r, "\partial^{01}"] & A^{11} \arrow[u, "d^{11}"] \arrow[r, "\partial^{11}"] & A^{21} \arrow[u, "d^{21}"] \arrow[r] & {} \\
0 \arrow[r] & C^0 \arrow[u, "d^0"] \arrow[r, "\eta^0"] & A^{00} \arrow[u, "d^{00}"] \arrow[r, "\partial^{00}"] & A^{10} \arrow[u, "d^{10}"] \arrow[r, "\partial^{10}"] & A^{20} \arrow[u, "d^{20}"] \arrow[r] & {}
\end{tikzcd}$$

in which $0 \to C^1 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^0 \to \cdots$ *is a chain complex* C, $d^{n0} \circ \eta^n = \eta^{n+1} \circ d^n$ *for all* n, *and every row is exact. When each* η^n *is regarded as a homomorphism from* C^n *to* $\Sigma^n(A)$, *these homomorphisms constitute a cochain map, and each* $H^n(\eta) : H^n(C) \to H^n(\Sigma(A))$ *is an isomorphism.*

Proof. That η is a cochain map follows easily from the exactness of the rows:

$$\Delta^n(\eta^n(c^n)) = d^{0n}(\eta^n(c^n)) + \partial^{0n}(\eta^n(c^n)) = d^{0n}(\eta^n(c^n)) = \eta^{n+1}(d^n(c^n)).$$

This calculation also shows that η^n takes cocycles to cocycles and coboundaries to coboundaries. We will show if the image of a cocycle $c^n \in C^n$ is a coboundary, then c^n is itself a coboundary, and that any cocycle in $\Sigma_n(A)$ is the sum of the image of a cocycle C_n and a coboundary in $\Sigma_n(C)$. These facts imply that $H^n(\eta)$ is injective and surjective respectively.

Suppose that $c^n \in C^n$ is a cocycle, and that $a^{0n} = \eta^n(c^n) \in A^{0n}$ is a coboundary. When $n = 0$ this means that $a^{0n} = 0$, in which case we much have $c^n = 0$ because η^0 is injective. Otherwise a^{0n} is a coboundary when regarded as an element of $\Sigma^n(A)$. Of the various preimages $a^{0,n-1} + \cdots + a^{n-1,0}$ of a^{0n} in $\Sigma^{n-1}(A)$, choose one that is minimal for the k such that $a^{k,n-k-1} = \cdots = a^{n-1,0} = 0$. Aiming at a contradiction, suppose that $k > 1$. Since $\partial^{k-1,n-k}(a^{k-1,n-k}) = 0$ and row $n-k$ is exact, there is some $a^{k-2,n-k}$ such that $a^{k-1,n-k} = \partial^{k-2,n-k}(a^{k-2,n-k})$. Evidently

$$a^{0,n-1} + \cdots + a^{k-1,n-k} - \Delta^{n-2}(a^{k-2,n-k})$$

is also mapped to a^{n0} by Δ^{n-1}, which contradicts the minimality of k. Therefore $k = 1$, which is to say that there is some $a^{0,n-1} \in A^{0,n-1}$ with $d^{0,n-1}(a^{0,n-1}) = a^{0n}$ and $\partial^{0,n-1}(a^{0,n-1}) = 0$. The latter fact implies that it has a preimage $c^{n-1} \in C^{n-1}$. Since

$$\eta^n(c^n) = a^n = d^{0,n-1}(\eta_{n-1}(c^{n-1})) = \eta^n(d^{n-1}(c^{n-1})),$$

the injectivity of η^n implies that $d^{n-1}(c^{n-1}) = c^n$, so c^n is a coboundary.

Now suppose that $a^{0n} + \cdots + a^{k,n-k}$ is a cocycle. We may suppose that, among the various cocycles that represent the same element of $H^n(\eta)$, this one minimizes k. If $k > 0$, then, since $\partial^{k,n-k}(a^{k,n-k}) = 0$ and row $n-k$ is exact, there is some $a^{k-1,n-k}$ such that $a^{k,n-k} = \partial^{k-1,n-k}(a^{k-1,n-k})$. Evidently

$$a^{0n} + \cdots + a^{k,n-k} - \Delta^{n-1}(a^{k-1,n-k})$$

is another representative of the cohomology class that contradicts the minimality of k. Therefore $k = 0$, which means that the cohomology class is represented by some $a^{0n} \in A^{0n}$. Since $\partial^{0n}(a^{0n}) = 0$ and row n is exact, a^{0n} has a preimage $c^n \in C^n$. Since a^{0n} is in the kernel of d^{0n}, $\eta^{n+1}(d^n(c^n)) = d^{0n}(\eta^n(c^n)) = 0$, and since η^{n+1} is injective, it follows that c^n is a cocycle. □

Proposition D2.2. *In addition to the hypotheses of the last result, suppose that we have a second diagram of this form, with primes attached to all symbols, and that we have a chain map $f = (f^n) : C \to C'$ and a chain map of double complexes $\varphi = (\varphi^{ij}) : A \to A'$ such that all diagrams*

$$\begin{tikzcd} C^n \arrow[r, "f^n"] \arrow[d, "\eta^n"'] & C'^n \arrow[d, "\eta'^n"] \\ A^{0n} \arrow[r, "\varphi^{0n}"] & A'^{0n} \end{tikzcd}$$

commute. Then for each n we have $\eta'^n \circ f^n = \Sigma^n(\varphi) \circ \eta^n$, so that the following diagram is commutative:

$$\begin{tikzcd} H^n(C) \arrow[r, "H^n(f)"] \arrow[d, "H^n(\eta)"'] & H^n(C') \arrow[d, "H^n(\eta')"] \\ H^n(\Sigma(A)) \arrow[r, "H^n(\Sigma(\varphi))"] & H^n(\Sigma(A')). \end{tikzcd}$$

Proof. This is a straightforward consequence of the fact that $\varphi^{0n} \circ \eta^n = \eta'^n \circ f^n$. ☐

We apply these concepts to F as follows. Suppose that X is the chosen projective resolution of M and J is the chosen projective resolution of N. Let $F(\mathbf{X}, \mathbf{J})$ be the codouble complex defined by setting:

$$F^{ij}(\mathbf{X}, \mathbf{J}) = F(X_i, J_j), \quad \partial^{ij} = F(d^{i+1}, J_j), \quad d^{ij} = (-1)^i F(X_i, d^j).$$

If $f : M' \to M$ and $g : N \to N'$ are homomorphisms, X' and J' are the chosen resolutions of M' and N', and as usual f and g also denote the extensions to chain maps $f : X' \to X$ and $g : J \to J'$, we let $F(f, g)$ denote the map of codouble complexes whose ij-component is $F(f_i, g_j) : F(X_i, J_j) \to F(X'_i, J'_j)$.

Define

$$F^n_*(M, N) = H^n(\Sigma(F(\mathbf{X}, \mathbf{J}))),$$

and set

$$F^n_*(f, g) = H^n(\Sigma(F(f, g))) : F^n_*(M, N) \to F^n_*(M', N').$$

Proposition D2.3. *If $F(X, -)$ is exact whenever X is projective and $F(-, I)$ is exact whenever I is injective, then there is a system of homomorphisms $\iota^{(M,N)} : F^n_*(M, N) \to F^n_1(M, N)$ such that the diagram*

$$\begin{tikzcd} F^n_1(M, N) \arrow[r, "F^n_1(f,g)"] \arrow[d, "\iota^{(M,N)}"'] & F^n_1(M', N') \arrow[d, "\iota^{(M',N')}"] \\ F^n_*(M, N) \arrow[r, "F^n_*(f,g)"] & F^n_*(M', N') \end{tikzcd}$$

commutes.

Proof. We apply Proposition D2.1 to the diagram

$$
\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & F(X_1,N) \arrow[u] \arrow[r, "\eta^1"] & F(X_1,J_0) \arrow[u] \arrow[r, "\partial^{10}"] & F(X_1,J_1) \arrow[u] \arrow[r, "\partial^{11}"] & {} \\
0 \arrow[r] & F(X_0,N) \arrow[u, "d_1"] \arrow[r, "\eta^0"] & F(X_0,J_0) \arrow[u, "d^{00}"] \arrow[r, "\partial^{00}"] & F(X_0,J_1) \arrow[u, "d^{01}"] \arrow[r, "\partial^{01}"] & {}
\end{tikzcd}
$$

to obtain isomorphisms $\iota^{(M,n)} : F_1^n(M,N) \to F_*^n(M,N)$. Two applications of the last result imply that the diagrams

$$
\begin{tikzcd}
F_1^n(M',N) \arrow[r, "F_1^n(f,N)"] \arrow[d, "\iota^{(M,N)}"'] & F_1^n(M,N) \arrow[d, "\iota^{(M,N)}"] \\
F_*^n(M',N) \arrow[r, "F_*^n(f,N)"] & F_*^n(M,N)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F_1^n(M,N) \arrow[r, "F_1^n(M,g)"] \arrow[d, "\iota^{(M,N)}"'] & F_1^n(M,N') \arrow[d, "\iota^{(M,N')}"] \\
F_*^n(M,N) \arrow[r, "F_*^n(M,g)"] & F_*^n(M,N')
\end{tikzcd}
$$

commute, and the claim is obtained by combining these. □

In the last section the situation was entirely symmetric, so that the analogue of the last result could be taken also as a proof that the functors F_n^2 were naturally isomorphic to the F_n^*. In this case we do not have complete symmetry, but in fact one can prove that the functors F_2^n and F_*^n are naturally isomorphic in the same way. Instead of belaboring the details we simply mention that in this case the critical diagram is

$$
\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & F(M,J_1) \arrow[u] \arrow[r, "\eta^1"] & F(X_0,J_1) \arrow[u] \arrow[r, "d^{01}"] & F(X_1,J_1) \arrow[u] \arrow[r, "d^{11}"] & {} \\
0 \arrow[r] & F(M,J_0) \arrow[u, "d_1"] \arrow[r, "\eta^0"] & F(X_0,J_0) \arrow[u, "\partial^{00}"] \arrow[r, "d^{00}"] & F(X_1,J_0) \arrow[u, "\partial^{10}"] \arrow[r, "d^{01}"] & {}
\end{tikzcd}
$$

We originally defined projective and injective modules by specifying that X is projective if $\mathrm{Hom}_R(X,-)$ is exact and J is injective if $\mathrm{Hom}_R(-,J)$ is exact, so we can apply the method described above to define a sequence of derived bifunctors, which are denoted by $\mathrm{Ext}_R^n(-,-)$. These may be computed from $\mathrm{Hom}_R(-,-)$ by resolving the first variable projectively or the second variable injectively.

D3 *Axiomatic Characterizations of* Tor *and* Ext

With the spade work complete, we can now "officially" define the bifunctors $\mathrm{Tor}_n^R(-,-)$ and $\mathrm{Ext}_R^n(-,-)$ to be the derived bifunctors of $-\otimes_R-$ and $\mathrm{Hom}_R(-,-)$ respectively. Suppose that X and Y are projective resolutions of R-modules M and N. Then $\mathrm{Tor}_n^R(M,N)$ is the homology in dimension n of $\mathbf{X} \otimes_R N$, and also the homology in dimension n of $M \otimes_R \mathbf{Y}$, and $\mathrm{Ext}^n_R(M, N)$ is the cohomology in dimension n of $\mathrm{Hom}_R(\mathbf{X}, N)$. If J is an injective resultion of Y, then $\mathrm{Ext}^n_R(M, N)$ is also the cohomology in dimension n of $\mathrm{Hom}_R(M, \mathbf{J})$.

The next result summarizes the properties of $\mathrm{Tor}^R_n(-,-)$ that were established during its construction. It turns out that these properties suffice to characterize this system of bifunctors completely.

Theorem D3.1. $\mathrm{Tor}^R_n(-,-)$ *($n \in \mathbb{Z}$) that take pairs of R-modules to R-modules, with* $\mathrm{Tor}^R_n(M,-)$ *and* $\mathrm{Tor}^R_n(-,N)$ *covariant, and connecting homomorphisms*

$$d_n : \mathrm{Tor}^R_n(M'', N) \to \mathrm{Tor}^R_{n-1}(M', N) \text{ and } d_n : \mathrm{Tor}^R_n(M, N'') \to \mathrm{Tor}^R_{n-1}(M, N')$$

for short exact sequences $0 \to M' \to M \to M'' \to 0$ *and* $0 \to N' \to N \to N'' \to 0$*, with the following properties:*

(a) $\mathrm{Tor}^R_n(M, N) = 0$ *for all* $n < 0$*.*

(b) $\mathrm{Tor}^R_0(-,-)$ *is naturally isomorphic to* $- \otimes_R -$*.*

(c) If either M *or* N *is projective, then* $\mathrm{Tor}^R_n(M, N) = 0$ *for all* $n > 0$*.*

(d) For all short exact sequences $0 \to M' \to M \to M'' \to 0$ *and* $0 \to N' \to N \to N'' \to 0$ *the sequences*

$$\cdots \to \mathrm{Tor}^R_n(M', N) \to \mathrm{Tor}^R_n(M, N) \to \mathrm{Tor}^R_n(M'', N) \to \mathrm{Tor}^R_{n-1}(M', N) \to \cdots$$

and

$$\cdots \to \mathrm{Tor}^R_n(M, N') \to \mathrm{Tor}^R_n(M, N) \to \mathrm{Tor}^R_n(M, N'') \to \mathrm{Tor}^R_{n-1}(M, N') \to \cdots$$

are exact.

(e) The connecting homomorphisms are natural: for any morphism

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"'] & M \arrow[r] \arrow[d, "f"'] & M'' \arrow[r] \arrow[d, "f''"'] & 0 \\ 0 \arrow[r] & \tilde{M}' \arrow[r] & \tilde{M} \arrow[r] & \tilde{M}'' \arrow[r] & 0 \end{tikzcd}$$

of short exact sequences the diagram

$$\begin{tikzcd} \mathrm{Tor}^R_n(M'', N) \arrow[r, "d_n"] \arrow[d, "\mathrm{Tor}^R_n(f'', N)"'] & \mathrm{Tor}^R_{n-1}(M', N) \arrow[d, "\mathrm{Tor}^R_{n-1}(f', N)"] \\ \mathrm{Tor}^R_n(\tilde{M}'', N) \arrow[r, "d_n"] & \mathrm{Tor}^R_{n-1}(\tilde{M}', N) \end{tikzcd}$$

commutes, and similarly for the second variable.

These properties determine the functors $\mathrm{Tor}_n^R(-,-)$ *and the connecting homomorphisms uniquely up to natural isomorphism.*

Proof. Prior to this point we have not defined $\mathrm{Tor}_n^R(-,-)$ for $n < 0$, so we can do so now using (a). In view of our definition of $\mathrm{Tor}_n^R(M,-)$ and $\mathrm{Tor}_n^R(-,N)$ as right derived functors of $M \otimes_R -$ and $- \otimes_R N$, properties (d) and (e) follow immediately from the long exact sequence for right derived functors. Since $- \otimes_R -$ is right exact, and $M \otimes_R -$ and $- \otimes_R N$ are exact when M and N are projective, (b) and (c) follow from Proposition C3.5. The uniqueness assertion of Theorem C6.2 implies the uniqueness asserted here. □

There is a similar characterization of $\mathrm{Ext}_R^n(-,-)$. As in the last result, the proof that these functors have the indicated properties is just a review our work to this point. We use (a) to define $\mathrm{Ext}_R^n(-,-)$ for $n < 0$. Properties (d) and (e) follow immediately from the long exact sequence for left derived functors. Since $\mathrm{Hom}_R(-,-)$ is left exact, and $\mathrm{Hom}_R(M,-)$ is exact when M is projective while $\mathrm{Hom}_R(-,N)$ is exact when N is injective, (b) and (c) follow from Proposition C4.1. The uniqueness asserted in the next result follows from the uniqueness assertion of Theorem C6.2, when it is construed as applying to both variances and both left and right derived functors. Thus:

Theorem D3.2. *There are bifunctors* $\mathrm{Ext}_R^n(-,-)$ *($n \in \mathbb{Z}$) that take pairs of R-modules to R-modules, with* $\mathrm{Ext}_R^n(M,-)$ *covariant and* $\mathrm{Ext}_R^n(-,N)$ *contravariant, and connecting homomorphisms*

$$d_n : \mathrm{Ext}_R^n(M',N) \to \mathrm{Ext}_R^{n+1}(M'',N) \text{ and } d_n : \mathrm{Ext}_R^n(M,N'') \to \mathrm{Ext}_R^{n+1}(M,N')$$

for short exact sequences $0 \to M' \to M \to M'' \to 0$ *and* $0 \to N' \to N \to N'' \to 0$*, with the following properties:*

(a) $\mathrm{Ext}_R^n(M,N) = 0$ *for all* $n < 0$.

(b) $\mathrm{Ext}_R^0(-,-)$ *is naturally isomorphic to* $\mathrm{Hom}_R(-,-)$.

(c) If M is projective, or if N is injective, then $\mathrm{Ext}_R^n(M,N) = 0$ *for all* $n > 0$.

(d) For all short exact sequences $0 \to M' \to M \to M'' \to 0$ *and* $0 \to N' \to N \to N'' \to 0$ *the sequences*

$$\cdots \to \mathrm{Ext}_R^n(M'',N) \to \mathrm{Ext}_R^n(M,N) \to \mathrm{Ext}_R^n(M',N) \to \mathrm{Ext}_R^{n+1}(M'',N) \to \cdots$$

and

$$\cdots \to \mathrm{Ext}_R^n(M,N') \to \mathrm{Ext}_R^n(M,N) \to \mathrm{Ext}_R^n(M,N'') \to \mathrm{Ext}_R^{n+1}(M,N') \to \cdots$$

are exact.

(e) *The connecting homomorphisms are natural: for any morphism*

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"'] & M \arrow[r] \arrow[d, "f"'] & M'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & \tilde{M}' \arrow[r] & \tilde{M} \arrow[r] & \tilde{M}'' \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences the diagram

$$\begin{tikzcd}
\mathrm{Ext}_R^n(\tilde{M}', N) \arrow[r, "d_n"] \arrow[d, "\mathrm{Ext}_R^n(f'',N)"'] & \mathrm{Ext}_R^{n+1}(\tilde{M}'', N) \arrow[d, "\mathrm{Ext}_R^{n-1}(f',N)"] \\
\mathrm{Ext}_R^n(M', N) \arrow[r, "d_n"] & \mathrm{Ext}_R^{n+1}(M'', N)
\end{tikzcd}$$

commutes, and similarly for the second variable.

These properties determine the functors $\mathrm{Ext}_R^n(-,-)$ *and the connecting homomorphisms uniquely up to natural isomorphism.*

D4 The Iterated Connecting Homomorphism

There is a computational device that is used in the application of derived functors to exact sequences that are not short. The starting point of the discussion is a rather cumbersome definition that abstracts the properties of derived functors that make it work. A *connected sequence* of covariant functors is a system

$$T = \{\dots, T^{-2}, T^{-1}, T^0, T^1, T^2, \dots\}$$

of covariant functors from R-modules to Q-modules, together with connecting homomorphisms

$$T^n(C) \to T^{n-1}(A)$$

for all short exact sequences $0 \to A \to B \to C \to 0$ and all n, such that:

(a) The composition of any two maps in the sequence

$$\cdots \to T^{n+1}(C) \to T^n(A) \to T^n(B) \to T^n(C) \to T^{n-1}(A) \to \cdots \qquad (*)$$

is zero.

(b) For any morphism

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences all of the diagrams

$$\begin{array}{ccc} T^n(C) & \longrightarrow & T^{n-1}(A) \\ \downarrow & & \downarrow \\ T^n(C') & \longrightarrow & T^{n-1}(A') \end{array}$$

commute.

We say that T is *exact* if the long sequence $(*)$ is always exact.

Intuitively, one may think of a connected sequence of functors as a functor from the category of short exact sequences to the category of "generalized cochain complexes" of the sort appearing in long exact sequences. Homological algebra doesn't have standard terminology for the latter category, which is why the definition above is a bit bulky.

Let

$$0 \to A_p \to A_{p-1} \cdots \to A_0 \to A_{-1} \to 0$$

be an exact sequence. For $i = -1, 0, \ldots, p$ let Z_i be the kernel of $A_i \to A_{i-1}$. Then $Z_{-1} = A_{-1}$ and we can identify Z_{p-1} with A_p.

For each $i = 0, \ldots, p-1$ there is a short exact sequence $0 \to Z_i \to A_i \to Z_{i-1} \to 0$ that gives rise to connecting homomorphisms $T^{n+p-i}(Z_{i-1}) \to T^{n+p-i-1}(Z_i)$. The *iterated connecting homomorphism*

$$c_T : T^{n+p}(A_{-1}) \to T^n(A_p)$$

is the composition

$$T^{n+p}(Z_{-1}) \to T^{n+p-1}(Z_0) \to \cdots \to T^{n+1}(Z_{p-2}) \to T^n(Z_{p-1}).$$

Now let $\tilde{T} = \{\tilde{T}^n\}$ be a second connected sequence of functors. A *natural transformation* from T to $\tilde{T}$ is a collection of homomorphisms $\eta_A^n : T^n(A) \to \tilde{T}^n(A)$ that are natural, in the sense that for any homomorphism $A \to B$ the diagrams

$$\begin{array}{ccc} T^n(A) & \longrightarrow & T^n(B) \\ {\scriptstyle \eta_A^n}\downarrow & & \downarrow{\scriptstyle \eta_B^n} \\ \tilde{T}^n(A) & \longrightarrow & \tilde{T}^n(B) \end{array}$$

commute, and are also natural with respect to the connecting homomorphisms, so that for every short sequence $0 \to A \to B \to C \to 0$ the diagram

$$\begin{array}{ccc} T^n(C) & \longrightarrow & T^{n-1}(A) \\ {\scriptstyle \eta_C^n}\downarrow & & \downarrow{\scriptstyle \eta_A^{n-1}} \\ \tilde{T}^n(C) & \longrightarrow & \tilde{T}^{n-1}(A) \end{array}$$

commutes.

Lemma D4.1. *If $\{\eta_A^n\}$ is a natural transformation from T to $\tilde{T}$, then the iterated connecting homomorphism is natural in the sense that for any exact sequence $0 \to A_p \to \cdots \to A_0 \to 0$ all diagrams*

$$\begin{array}{ccc} T^{n+p}(A_0) & \xrightarrow{c_T} & T^{n+1}(A_p) \\ \eta_{A_0}^{n+p}\Big\downarrow & & \Big\downarrow \eta_{A_p}^{n+1} \\ \tilde{T}^{n+p}(A_0) & \xrightarrow{c_{\tilde{T}}} & \tilde{T}^{n+1}(A_p) \end{array}$$

commute.

Proof. By assumption every square in the diagram

$$\begin{array}{ccccccccc} T^{n+p+1}(Z_{-1}) & \longrightarrow & T^{n+p}(Z_0) & \longrightarrow & \cdots & \longrightarrow & T^{n+2}(Z_{p-2}) & \longrightarrow & T^{n+1}(Z_{p-1}) \\ \Big\downarrow & & \Big\downarrow & & & & \Big\downarrow & & \Big\downarrow \\ \tilde{T}^{n+p+1}(Z_{-1}) & \longrightarrow & \tilde{T}^{n+p}(Z_0) & \longrightarrow & \cdots & \longrightarrow & \tilde{T}^{n+2}(Z_{p-2}) & \longrightarrow & \tilde{T}^{n+1}(Z_{p-1}) \end{array}$$

commutes. □

We are particularly concerned with a situation that arises in connection with resolutions that are "almost projective" or "almost flat." An R-module X is a *T-module* if $T_n(X) = 0$ for all $n > 0$. The isomorphisms given by the following result are often described as *dimension shifting.*

Proposition D4.2. *Suppose that T is exact. If*

$$0 \to M \to X_{p-1} \to \cdots \to X_0 \to A \to 0$$

exact and $X_{p-1}, \ldots, X_0$ are T-modules, then there is an exact sequence

$$0 \to T^p(A) \xrightarrow{c_T} T^0(M) \to T^0(X_{p-1}),$$

and for each $n > 0$ the connecting homomorphism $c_T : T^{p+n}(A) \to T^n(M)$ is an isomorphism.

Proof. For each $n \geq 0$ and $i = 0, \ldots, p-1$ the exact sequence

$$T^{n+p-i}(X_i) \to T^{n+p-i}(Z_{i-1}) \to T^{n+p-i-1}(Z_i) \to T^{n+p-i-1}(X_i)$$

is a portion of $(*)$. When $n > 0$ or $i < p-1$ the hypotheses imply that the outer modules vanish, so that the inner map is an isomorphism. When $n = 0$ and $i = p-1$ the sequence is

$$0 \to T^1(Z_{p-2}) \to T^0(Z_{p-1}) \to T^0(X_{p-1}).$$

Thus the iterated connecting homomorphism is an isomorphism when $n > 0$, and when $n = 0$ it is the isomorphism $T^p(Z_{-1}) \to T^1(Z_{p-2})$ composed with $T^1(Z_{p-2}) \to T^0(Z_{p-1})$. Of course $Z_{-1} = A$ and $Z_{p-1} \cong M$. □

The technique developed in this section can also be applied when the connected sequence of functors is contravariant and/or the connecting homomorphisms increase dimension. Thus there are four cases, and we regard the discussion to this point as establishing the relevant result in the other three as well.

For ease of reference we now describe one other particular situation that will arise later. Consider an exact connected sequence $T = \{\dots, T_{-1}, T_0, T_1, \dots\}$ that is covariant and whose boundary homomorphisms go from T_n to T_{n+1}, with $T_n(I) = 0$ whenever I is injective and $n > 0$. Suppose we are given an exact sequence

$$0 \to N \to I_0 \to \cdots \to I_{p-1} \to X \to 0$$

with $I_0, \dots, I_{p-1}$ injective. For $i = 0, \dots, p-2$ let Z_i be the kernel of $X_i \to X_{i+1}$ (so $N \cong Z_0$) let Z_{p-1} be the kernel of $I_{p-1} \to X$, and let $Z_p = X$. For each integer n there is an iterated connecting homomorphism $T_n(X) \to T_{n+p}(N)$ given by the composition

$$T_n(Z_p) \to T_{n+1}(Z_{p-1}) \to \cdots \to T_{n+p-1}(Z_1) \to T_{n+p}(Z_0)$$

where each $T_{n+i}(Z_{p-i}) \to T_{n+i+1}(Z_{p-i-1})$ is the central homomorphism of the exact sequence

$$T_{n+i}(I_{p-i-1}) \to T_{n+i}(Z_{p-i}) \to T_{n+i+1}(Z_{p-i-1}) \to T_{n+i+1}(I_{p-i-1})$$

obtained by applying T to the short exact sequence $0 \to Z_{p-i-1} \to I_{p-i-1} \to Z_{p-i} \to 0$. When $n > 0$ the outer modules vanish for all $i = 0, \dots, p-1$, so the iterated connecting homomorphism is an isomorphism. When $n = 0$ and $i = 0$ the sequence is $T_0(I_{p-1}) \to T_0(Z_p) \to T_1(Z_{p-1}) \to 0$, so there is an exact sequence

$$T_0(I_{p-1}) \to T_0(X) \xrightarrow{c_T} T_p(N) \to 0.$$

D5 Projective and Weak Dimension

Let M be an R-module. The *projective dimension* of M, denoted by $\mathrm{pd}_R\, M$, is the smallest n such that there is a projective resolution

$$\cdots \to 0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

with $X_{n+1} = 0$. If, in the definition above, we replaced "projective" with "flat" or "free," we arrive at the notions of *flat dimension* and *free dimension*. Since free modules are projective and projective modules are flat, the flat dimension is not larger than the projective dimension, which is in turn not larger than the free dimension. The *injective dimension* of M is defined in the same way using injective resolutions.

If $\mathrm{pd}_R\, M = 0$, then there is a projective resolution X of M such that $X_1 = 0$, so that $0 \to X_0 \to M \to 0$ is exact and M is projective because it is isomorphic to X_0. On the other hand, if M is projective, then $0 \to M \to M \to 0$ is a projective resolution, so $\operatorname{pd}_R M = 0$. Therefore if $M \neq 0$, then M is projective if and only if $\operatorname{pd}_R M = 0$. With suitable adjustments, this argument shows that M is injective (free, flat) if and only if its injective (free, flat) dimension is zero.

The *weak dimension* of M, denoted by $\operatorname{wd}_R M$, is the largest n for which there is some R-module N such that $\operatorname{Tor}_n^R(M, N) \neq 0$. Due to the asymmetric nature of Ext_R^n, there are also two notions of dimension than can be defined using these bifunctors, but these receive less attention. Since a projective resolution of M can be used to compute $\operatorname{Tor}_n^R(M, N)$, the weak dimension of M is never greater than the projective dimension. Below we will show that flat resolutions of M can be used to compute $\operatorname{Tor}_n^R(M, N)$, so the weak dimension is also not greater than the flat dimension. One of the main results of this section is that if R is Noetherian and M is finitely generated, then $\operatorname{wd}_R M = \operatorname{pd}_R M$. Under a wide variety of conditions the free dimension is not greater than the projective dimension.

We begin by showing that Tor^R gives a characterization of flatness. Later we will see that, similarly, Ext_R characterizes projectivity and injectivity.

Proposition D5.1. *The following are equivalent:*

(a) M *is flat.*

(b) $\operatorname{Tor}_n^R(M, N) = 0$ *for all* R*-modules* N *and all* $n > 0$.

(c) $\operatorname{Tor}_1^R(M, N) = 0$ *for all* R*-modules* N.

Proof. If M is flat and $X : \cdots \to X_1 \to X_0 \to N \to 0$ is a projective resolution of N, then $\operatorname{Tor}_n^R(M, N) = H_n(M \otimes_R \mathbf{X}) = 0$ for all $n > 0$ because $M \otimes_R -$ is an exact functor. Thus (a) implies (b), which automatically implies (c). Suppose (c) holds. If $0 \to N' \to N \to N'' \to 0$ is a short exact sequence, then

$$0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$

is exact because it is

$$\operatorname{Tor}_1^R(M, N'') \to \operatorname{Tor}_0^R(M, N') \to \operatorname{Tor}_0^R(M, N) \to \operatorname{Tor}_0^R(M, N'') \to 0.$$

Thus $M \otimes_R -$ is exact, so we have shown that (c) implies (a). $\square$

Proposition D5.2. *If* R *is a local ring and* M *is a finitely generated flat* R*-module, then every finite set of generators has a subset that generates* M *freely, so* M *is free.*

Proof. The images of a finite set of generators of M map to a set that spans $M/\mathfrak{m}$ and consequently has a subset that is a basis. Therefore it suffices to show that $x_1, \dots, x_n$ generate M freely whenever their images in $M/\mathfrak{m}M$ are a basis of this vector space. By Nakayama's lemma, $x_1, \ldots, x_n$ is a set of generators for M, so we need to show that they are R-linearly independent.

We argue by induction on n. If $rx_1 = 0$ for some nonzero r, then the euational criterion for flatness gives $y_j \in M$ and $a_j \in R$ such that $x = \sum_j a_j y_j$ and $ra_j = 0$ for all j. But $x \notin \mathfrak{m}M$, so at least one a_j must be a unit, which implies that $r = 0$.

Now suppose that $n > 1$, and let $\sum_i r_i x_i = 0$. Since $x_1, \ldots, x_n$ goes to a basis of $M/\mathfrak{m}M$, each r_i goes to zero in k, i.e., $r_i \in \mathfrak{m}$. The equational criterion for flatness gives y_j and a_{ij} such that $x_i = \sum_j a_{ij} y_j$ for all i and $\sum_j r_j a_{ij} = 0$ for all i. There is some h such that a_{nh} that is not in $\mathfrak{m}$ and is consequently a unit, so $r_n = -\sum_{i=0}^{n-1} r_i (a_{ih}/a_{nh})$. The relation can be written as $\sum_{i=1}^{n-1} r_i (x_i - (a_{ih}/a_{nh}) x_n) = 0$. Since the $x_i - (a_{ih}/a_{nh}) x_n$ map to $n-1$ linearly independent elements of $M/\mathfrak{m}M$, $r_1 = \cdots = r_{n-1} = 0$, and it follows that $r_n = 0$. □

The following is Theorem VIII.6.1′ of CE.

Theorem D5.3. *If R is a Noetherian local ring, M is finitely generated, and $\mathrm{Tor}_1^R(M, k) = 0$, then M is free, and every finite set of generators contains a base.*

Proof. Since Proposition D5.1 implies that M is flat, this follows from the last result. □

Flat resolutions can be used to compute Tor^R.

Theorem D5.4. *If $F : \cdots \to F_1 \to F_0 \to M \to 0$ is a flat resolution of M, then $\mathrm{Tor}_n^R(M, N) = H_n(\mathbf{F} \otimes_R N)$ for all R-modules N and all n.*

Proof. Since $- \otimes_R N$ is right exact, the sequence

$$F_1 \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0$$

is exact (recall Lemma B5.3) which gives

$$\mathrm{Tor}_0^R(M, N) = M \otimes_R N = H_0(\mathbf{F} \otimes_R N).$$

Let K be the kernel of $F_0 \to M$, and let $F' : \cdots \to F_2 \to F_1 \to K \to 0$ be the derived flat resolution of K. In view of the last result, the long exact sequence for $0 \to K \to F_0 \to M \to 0$ ends with

$$0 \to \mathrm{Tor}_1^R(M, N) \to K \otimes_R N \to F_0 \otimes_R N \to M \otimes_R N \to 0.$$

Right exactness of $- \otimes_R N$ applied to $F_2 \to F_1 \to K \to 0$ gives the second equality in the computation

$$\mathrm{Tor}_1^R(M, N) = \mathrm{Ker}\Big(\frac{F_1}{\mathrm{Im}(F_2 \to F_1)} \otimes_R N \to F_0 \otimes_R N\Big)$$

$$= \operatorname{Ker}\Big(\frac{F_1 \otimes_R N}{\operatorname{Im}(F_2 \otimes_R N \to F_1 \otimes_R N)} \to F_0 \otimes_R N\Big)$$

$$= \frac{\operatorname{Ker}(F_1 \otimes_R N \to F_0 \otimes_R N)}{\operatorname{Im}(F_2 \otimes_R N \to F_1 \otimes_R N)} = H_1(\mathbf{F} \otimes_R N).$$

Now induction on $n \geq 2$ gives

$$\operatorname{Tor}_n^R(M,N) = \operatorname{Tor}_{n-1}^R(K,N) = H_{n-1}(\mathbf{F}' \otimes_R N) = H_n(\mathbf{F} \otimes_R N),$$

where the first equality comes from the last result applied to the earlier portions of the long exact sequence above. □

Generalizing Proposition D5.1, the next result gives a characterization of flat dimension in terms of Tor^R. Later we will see analogous characterizations of projective and injective dimension in terms of Ext_R.

Proposition D5.5. *For any integer n the following are equivalent:*

(a) $\operatorname{wd}_R M \leq n$.

(b) $\operatorname{Tor}_i^R(M,N) = 0$ *for all R-modules N and all $i > n$.*

(c) $\operatorname{Tor}_{n+1}^R(M,N) = 0$ *for all R-modules N.*

(d) Whenever

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

is exact and $F_{n-1}, \ldots, F_0$ are flat, F_n is also flat.

Proof. Recall (c.f. the discussion at the beginning of chapter C) that M has a flat resolution. Suppose $F_{n-1} \to \cdots \to F_0 \to M \to 0$ is part of a flat resolution of M and F_n is the kernel of $F_{n-1} \to F_{n-2}$. If (d) holds, then the last result implies that $0 \to F_n \to \cdots \to F_0 \to 0$ may be used to compute $\operatorname{Tor}^R(M,\cdot)$. Thus (d) implies (a). Since M has a flat resolution, (a) implies (b), from which (c) follows automatically. Let an exact sequence as in (d) be given. For any N we apply the theory of the iterated connecting homomorphism to the connected sequence $\operatorname{Tor}^R(\cdot,N) = \{\operatorname{Tor}_n^R(\cdot,N)\}$. Flat modules are $\operatorname{Tor}^R(\cdot,N)$-modules, so Proposition D4.2 ("dimension shifting") gives $\operatorname{Tor}_1^R(F_n,N) = \operatorname{Tor}_{n+1}^R(M,N) = 0$, and the last result implies that F_n is flat. □

Much of our work up to this point has been aimed at the following result.

Theorem D5.6. *If R is Noetherian and M is finitely generated, then*

$$\operatorname{wd}_R M = \operatorname{pd}_R M.$$

Proof. Let $d = \operatorname{wd}_R M$. We already know that $d \le \operatorname{pd}_R M$. Proposition C1.2 gives a projective resolution of M whose modules are all finitely generated, so there is an exact sequence

$$0 \to X_d \to X_{d-1} \to \cdots \to X_0 \to M \to 0$$

where $X_{d-1}, \ldots, X_0$ are projective and finitely generated, and X_d is the kernel of $X_{d-1} \to X_{d-2}$. Projective modules are flat, so if $d = \operatorname{wd}_R M$, then the last result implies that X_d is flat. Since finitely generated R-modules are Noetherian (Proposition A4.6) X_d is finitely generated, hence Noetherian, hence finitely presented. Now Theorem B9.11 implies that X_d is projective, so $\operatorname{pd}_R M \le d$. □

Proposition D5.7. *If S is a multiplicatively closed subset of R, then for any R-modules M and N,*

$$\operatorname{Tor}_n^{S^{-1}R}(S^{-1}M, S^{-1}N) = S^{-1}(\operatorname{Tor}_n^R(M, N)).$$

Proof. Let X be a projective resolution of M. Each module in $S^{-1}R \otimes_R \mathbf{X} = S^{-1}\mathbf{X}$ (recall Lemma A6.6) is projective (Proposition B7.3) and $S^{-1}R$ is a flat R-module (Proposition A6.8) so $S^{-1}\mathbf{X}$ is a projective resolution of $S^{-1}M$. We have

$$\begin{aligned} S^{-1}(\operatorname{Tor}_n^R(M, N)) &= S^{-1}(H_n(\mathbf{X} \otimes_R N)) = H_n(S^{-1}(\mathbf{X} \otimes_R N)) \\ &= H_n(S^{-1}\mathbf{X} \otimes_{S^{-1}R} S^{-1}N) = \operatorname{Tor}_n^{S^{-1}R}(S^{-1}M, S^{-1}N). \end{aligned}$$

The second equality is the fact that localization commutes with homology (Proposition A5.5) and the third is the fact that localization commutes with tensor products (Proposition A6.5). □

The following is Exercise VII.11 in CE.

Theorem D5.8. *For any R-module M,*

$$\operatorname{wd}_R M = \sup \operatorname{wd}_{R_\mathfrak{m}} M_\mathfrak{m}$$

where the supremum is over all maximal ideals $\mathfrak{m} \subset R$. If R is Noetherian and M is finitely generated, then

$$\operatorname{pd}_R M = \sup \operatorname{pd}_{R_\mathfrak{m}} M_\mathfrak{m}.$$

Proof. For any maximal ideal $\mathfrak{m}$ and any $R_\mathfrak{m}$-module N we have $N_\mathfrak{m} = N \otimes_{R_\mathfrak{m}} R_\mathfrak{m} = N$, so for every n Proposition D5.7 gives

$$\operatorname{Tor}_n^{R_\mathfrak{m}}(M_\mathfrak{m}, N) = \operatorname{Tor}_n^{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) = (\operatorname{Tor}_n^R(M, N))_\mathfrak{m}.$$

It follows that

$$\operatorname{wd}_R M \ge \sup \operatorname{wd}_{R_\mathfrak{m}} M_\mathfrak{m}.$$

On the other hand the last result implies that if $\operatorname{Tor}_n^R(M, N) \neq 0$, then $(\operatorname{Tor}_n^R(M, N))_{\mathfrak{m}}$ is nonzero for some $\mathfrak{m}$. Therefore

$$\operatorname{wd}_R M \leq \sup \operatorname{wd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}.$$

Any system of generators for M is a system of generators of each $M_{\mathfrak{m}}$, so if M is finitely generated, then so is each $M_{\mathfrak{m}}$. If, in addition, R is Noetherian, then so is each $R_{\mathfrak{m}}$, and Theorem D5.6 gives $\operatorname{wd}_R M = \operatorname{pd}_R M$ and $\operatorname{wd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} = \operatorname{pd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ for all $\mathfrak{m}$. □

D6 Ext *and Dimension*

We now provide analogues of Propositions D5.1 and D5.5 for projective and injective modules. Their proofs follow the same pattern as above.

Proposition D6.1. *The following are equivalent:*

(a) M *is projective.*

(b) $\operatorname{Ext}_R^n(M, N) = 0$ *for all R-modules N and all $n > 0$.*

(c) $\operatorname{Ext}_R^1(M, N) = 0$ *for all R-modules N.*

Proof. If M is projective and $X : \cdots \to X_1 \to X_0 \to N \to 0$ is a projective resolution of N, then $\operatorname{Ext}_R^n(M, N) = H_n(\operatorname{Hom}_R(M, \mathbf{X})) = 0$ for all $n > 0$ because $\operatorname{Hom}_R(M, \cdot)$ is an exact functor. Thus (a) implies (b), which automatically implies (c). Suppose (c) holds. If $0 \to N' \to N \to N'' \to 0$ is a short exact sequence, then

$$0 \to \operatorname{Hom}_R(M, N') \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, N'') \to 0$$

is exact because it is

$$0 \to \operatorname{Ext}_R^0(M, N') \to \operatorname{Ext}_R^0(M, N) \to \operatorname{Ext}_R^0(M, N'') \to \operatorname{Ext}_R^1(M, N').$$

Thus $\operatorname{Hom}_R(M, \cdot)$ is exact, so we have shown that (c) implies (a). □

Proposition D6.2. *For any integer n the following are equivalent:*

(a) $\operatorname{pd}_R M \leq n$.

(b) $\operatorname{Ext}_R^i(M, N) = 0$ *for all R-modules N and all $i > n$.*

(c) $\operatorname{Ext}_R^{n+1}(M, N) = 0$ *for all R-modules N.*

(d) Whenever

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

is exact and $X_{n-1}, \ldots, X_0$ are projective, X_n is also projective.

Proof. Let $\cdots \to X_1 \to X_0 \to M \to 0$ be a projective resolution of M, and let X_n be the kernel of $X_{n-1} \to X_{n-2}$. If (d) holds, then $0 \to X_n \to \cdots \to X_0 \to M \to 0$ is a projective resolution, so (d) implies (a). Of course (a) implies (b) (logically the only issue is the existence of projective resolutions) from which (c) follows automatically. Let an exact sequence as in (d) be given. For any N the connected sequence $\mathrm{Ext}_R(\cdot, N) = \{\mathrm{Ext}^n_R(\cdot, N)\}$ is exact, and projective modules are $\mathrm{Ext}_R(\cdot, N)$-modules, so Proposition D4.2 gives $\mathrm{Ext}^1_R(X_n, N) = \mathrm{Ext}^{n+1}_R(M, N) = 0$. Now the last result implies that X_n is projective. □

The last result has the following consequence.

Corollary D6.3. *Suppose that* $0 \to M' \to M \to M'' \to 0$ *is exact and* M *is projective. If* $\mathrm{pd}_R\, M' > 0$*, then* $\mathrm{pd}_R\, M'' = \mathrm{pd}_R\, M' + 1$*, and if* $\mathrm{pd}_R\, M' = 0$*, then* $\mathrm{pd}_R\, M'' \le 1$*.*

Proof. For any R module N the long exact sequence

$$\cdots \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Ext}^n_R(M', N) \to \mathrm{Ext}^{n+1}_R(M'', N) \to \mathrm{Ext}^{n+1}_R(M, N) \to \cdots$$

implies that $\mathrm{Ext}^n_R(M', N)$ and $\mathrm{Ext}^{n+1}_R(M'', N)$ are isomorphic for all $n > 0$, so this follows from the equivalence of (a) and (b) in the last result. □

Serre applies the following variant of Proposition D6.2.

Proposition D6.4. *If* R *is Noetherian and* M *is finitely generated, then for any integer* n *the following are equivalent:*

(a) $\mathrm{pd}_R\, M \le n$.

(b) $\mathrm{Ext}^{n+1}_R(M, N) = 0$ *for all finitely generated* R*-modules* N.

(c) $\mathrm{Ext}^n_R(M, N') \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Ext}^n_R(M, N'') \to 0$ *is exact whenever* $0 \to N' \to N \to N'' \to 0$ *is a short exact sequence of finitely generated* R*-modules.*

(d) Whenever

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

is exact and $X_{n-1}, \dots, X_0$ *are projective and finitely generated,* X_n *is also projective.*

Proof. Proposition C1.2 implies that there is a sequence with the properties required in (d), so (d) implies (a). We already know that (a) implies (more than) (b). In view of the exact sequence

$$\mathrm{Ext}^n_R(M, N') \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Ext}^n_R(M, N'') \to \mathrm{Ext}^{n+1}_R(M, N')$$

(b) and (c) are equivalent.

Suppose (b) holds, and let an exact sequence as in (d) be given. The argument in the last proof shows that for any finitely generated N, $\mathrm{Ext}^1_R(X_n, N) = 0$. If $0 \to N' \to N \to N'' \to 0$ is exact and N', N'', and N'' are finitely generated, then

$$0 \to \mathrm{Hom}_R(X_n, N') \to \mathrm{Hom}_R(X_n, N) \to \mathrm{Hom}_R(X_n, N'') \to 0 \qquad (*)$$

is exact because it is

$$0 \to \mathrm{Ext}^0_R(X_n, N') \to \mathrm{Ext}^0_R(X_n, N) \to \mathrm{Ext}^0_R(X_n, N'') \to \mathrm{Ext}^1_R(X_n, N').$$

We need to show that $\mathrm{Hom}(X_n, \cdot)$ is exact. Since it is left exact, this boils down to $\mathrm{Hom}_R(X_n, N) \to \mathrm{Hom}_R(X_n, N'')$ being surjective even when N', N, and N'' need not be finitely generated. Aiming at a contradiction, suppose that $\alpha : X_n \to N''$ is not in the image. Insofar as X_n is isomorphic to a submodule of the finitely generated module X_{n-1}, and R is Noetherian, Proposition A4.7 implies that it is finitely generated. Let $\tilde{N}''$ be the image of α. Since it is finitely generated, it is the image of a finitely generated submodule $\tilde{N} \subset N$. Proposition A4.7 implies that the kernel of $\tilde{N} \to \tilde{N}''$ is finitely generated, and of course it is isomorphic to its preimage $\tilde{N}' \subset N'$. Now $0 \to \tilde{N}' \to \tilde{N} \to \tilde{N}'' \to 0$ is a short exact sequence of finitely generated R-modules, but α is not in the image of $\mathrm{Hom}_R(X_n, \tilde{N}) \to \mathrm{Hom}_R(X_n, \tilde{N}'')$, contrary to what has already been shown. □

Finally we handle injective modules. There are no surprises.

Proposition D6.5. *The following are equivalent:*

(a) M *is injective.*

(b) $\mathrm{Ext}^n_R(N, M) = 0$ *for all* R*-modules* N *and all* $n > 0$*.*

(c) $\mathrm{Ext}^1_R(M, N) = 0$ *for all* R*-modules* N*.*

Proof. If M is injective and $X : 0 \to N \to I_0 \to I_1 \to \cdots$ is an injective resolution of N, then $\mathrm{Ext}^n_R(N, M) = H^n(\mathrm{Hom}_R(\mathbf{I}, M)) = 0$ for all $n > 0$ because $\mathrm{Hom}_R(\cdot, M)$ is an exact functor. Thus (a) implies (b), which automatically implies (c). Suppose (c) holds. If $0 \to N' \to N \to N'' \to 0$ is a short exact sequence, then

$$0 \to \mathrm{Hom}_R(N'', M) \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N', M) \to 0$$

is exact because it is

$$0 \to \mathrm{Ext}^0_R(N'', M) \to \mathrm{Ext}^0_R(N, M) \to \mathrm{Ext}^0_R(N', M) \to \mathrm{Ext}^1_R(N'', M).$$

Thus $\mathrm{Hom}_R(\cdot, M)$ is exact, so we have shown that (c) implies (a). □

Proposition D6.6. *For any integer n the following are equivalent:*

(a) the injective dimension of M is $\leq n$.

(b) $\mathrm{Ext}^i_R(N, M) = 0$ *for all R-modules N and all $i > n$.*

(c) $\mathrm{Ext}^{n+1}_R(N, M) = 0$ *for all R-modules N.*

(d) Whenever

$$0 \to M \to I_0 \to \cdots \to I_{n-1} \to I_n \to 0$$

is exact and $I_0, \ldots, I_{n-1}$ are injective, I_n is also injective.

Proof. Let $0 \to M \to I_0 \to I_1 \to \cdots$ be an injective resolution of M, and let I_n be the cokernel of $I_{n-1} \to I_{n-2}$. If (d) holds, then the sequence $0 \to M \to I_0 \to \cdots \to I_{n-1} \to I_n \to 0$ (where $I_{n-1} \to I_n$ is the quotient map) is an injective resolution which may be used to compute $\mathrm{Ext}_R(\cdot, M)$, so (d) implies (a). Of course (a) implies (b) (since we know that M has injective resolutions) from which (c) follows automatically. Let an exact sequence as in (d) be given. For any N the connected sequence $\mathrm{Ext}_R(N, \cdot) = \{\mathrm{Ext}^n_R(N, \cdot)\}$ is exact, and injective modules are $\mathrm{Ext}_R(\cdot, N)$-modules, so the contravariant variant of Proposition D4.2 gives $\mathrm{Ext}^1_R(N, I_n) = \mathrm{Ext}^{n+1}_R(N, M) = 0$. Now the last result implies that I_n is injective. $\square$

Chapter E

Derived Rings and Modules, and Completions

Let G be an abelian topological group, whose group operation is written additively. That is, G is an abelian group endowed with a topology such that addition and negation are continuous. We will always assume that G is first countable, i.e., 0 has a countable neighborhood basis. For much of what we do this is not strictly necessary, insofar as the concepts and results can be suitably generalized, but it avoids considerable complications, and we have no interest in groups that are not first countable. A *Cauchy sequence* is a sequence $g_1, g_2, \ldots$ in G such that for every neighborhood U of 0, $g_i - g_{i'} \in U$ for all sufficiently large i and i'. Two sequences $g_1, g_2, \ldots$ and $g'_1, g'_2, \ldots$ are *equivalent* if, for every neighborhood U of 0, $g_i - g'_i \in U$ for sufficiently large i. (Since the group operation is continuous, for any such U there is a neighborhood V of 0 with $V + V \subset U$. Using this fact, it is easy to see that equivalence is transitive, and it is obviously reflexive and symmetric, so it is indeed an equivalence relation.) The set of equivalence classes of Cauchy sequences can be endowed with an abelian group operation and a topology in a natural manner that generalizes the passage from $\mathbb{Q}$ to $\mathbb{R}$, and parallels the completion of a metric space. We will begin by studying the basic properties of this construction.

A descending chain

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots$$

of subgroups of G is a *filtration* of G. We denote such a filtration by (G_n). We may define a topology on G by taking the various $g + G_n$ to be a base of the collection of open sets. (To show that finite intersections of open sets are open observe that if $g'' \in (g + G_n) \cap (g' + G_{n'})$, and $n' \geq n$, then $g'' + G_{n'} \subset (g + G_n) \cap (g' + G_{n'})$.) This is the *Krull topology* defined by (G_n). Since $G_n + G_n \subset G_n$ and $-G_n = G_n$, addition and negation are continuous.

In general an *inverse system* of abelian groups is a diagram

$$H : H_0 \leftarrow H_1 \leftarrow H_2 \leftarrow \cdots$$

of abelian groups and homomorphisms. For example, for the filtration (G_n) there is an inverse system given by the natural homomorphisms $G/G_{n+1} \to G/G_n$. The *inverse limit* $\varprojlim H_n$ is the set of sequences $\{h_n\}$, where each h_n is an element of H_n, such that for all $m \leq n$, the image of h_n in H_m is h_m.

Among the numerous possibilities for deriving new groups from (G_n), we will be particularly interested in

$$\oplus_{n\geq 0} G_n, \quad \oplus_{n\geq 0} G_n/G_{n+1}, \quad \text{and} \quad \varprojlim G/G_n.$$

In particular, let I be an ideal of R. Then there are the derived rings

$$R^* = \oplus_{n\geq 0} I^n, \quad G_I(R) = \oplus_{n\geq 0} I^n/I^{n+1}, \quad \text{and} \quad \hat{R} = \varprojlim R/I^n.$$

The ring $\hat{R}$ is the *I-adic completion* of R. A filtration (M_n) of an R-module M is an *I-filtration* if $IM_n \subset M_{n+1}$ for all n. We will study the modules

$$M^* = \oplus_{n\geq 0} M_n, \quad G(M) = \oplus_{n\geq 0} M_n/M_{n+1}, \quad \text{and} \quad \hat{M} = \varprojlim M/I^n M.$$

The work of this chapter has a somewhat miscellaneous character, insofar as it studies only a few of the many possibilities presented by these constructions, and it is further restricted to the development of the results used to prove that the I-adic completion of a Noetherian ring is Noetherian, and other results that will be needed later.

E1 Completing a Topological Group

Let G be an abelian topological group. Since the topology is invariant under translation, it is determined by the neighborhoods of 0. A set $S \subset G$ is *balanced* if $S = -S$. If U is a neighborhood of 0, then so is $U \cap -U$, so every neighborhood of 0 contains a balanced neighborhood. Let K_G be the intersection of the neighborhoods of 0.

Lemma E1.1. *K_G is a balanced subgroup of G that is the closure of $\{0\}$, and G is Hausdorff if and only if $K_G = \{0\}$.*

Proof. For any neighborhood U of 0 the continuity of addition gives neighborhoods V and W of 0 such that $K_G + K_G \subset V + W \subset U$, so $K_G + K_G \subset K_G$. Since K_G is the intersection of all balanced neighborhoods of 0, $K_G = -K_G$. Thus K_G is a balanced subgroup.

We have $g \in K_G$ if and only if g is an element of every neighborhood of 0, and this is true if and only if 0 is in every neighborhood of g, which is to say that g is an element of the closure of $\{0\}$.

Distinct elements of K_G do not have disjoint neighborhoods, so G is not Hausdorff if K_G has multiple elements. On the other hand suppose that $K_G = \{0\}$. Let g and g' be distinct elements of G, and let U be a neighborhood of 0 that does not contain $g' - g$. Since addition is continuous, there are neighborhoods V, W of 0 such that $V + W \subset U$. It is easily checked that $g' - V$ and $g + W$ are disjoint neighborhoods of g' and g. □

We endow G/K_G with the quotient topology induced by the quotient homomorphism $\kappa_G : G \to G/K_G$.

Lemma E1.2. *κ_G is an open map.*

Proof. We need to show that if $U \subset G$ is open, then $\kappa_G(U)$ is open, which is the same as $\kappa_G^{-1}(\kappa_G(U))$ being open. But from the definition of K_G we have $U + K_G = U$, so $\kappa_G^{-1}(\kappa_G(U)) = U$. □

The *completion* of G, denoted by $\hat{G}$, is the set of equivalence classes of Cauchy sequences in G. The equivalence class of the componentwise sum of two sequences is easily shown to depend only on the equivalence classes of the sequences, so this operation induces a binary operation on $\hat{G}$, and the axioms for an abelian group are easily verified.

There is a map $\iota_G : G \to \hat{G}$ that takes each g to the equivalence class of the constant sequence with value g.

Lemma E1.3. *If $g \in G$, $\hat{g} \in \hat{G}$, and $\{g_i\}$ is a representative of $\hat{g}$, then $\hat{g} = \iota_G(g)$ if and only if $g_i \to g$. Therefore K_G is the kernel of ι_G.*

Proof. By definition $\hat{g} = \iota_G(g)$ if and only if $\{g_i\}$ is equivalent to the constant sequence with value g, and this is the case if and only $\{g_i - g\}$ is eventually insider each neighborhood of 0, which is in turn the case if and only if $\{g_i\}$ is eventually inside each neighborhood of g. □

In view of this result there is a homomorphism $\lambda_G : G/K_G \to \hat{G}$ that is defined implicitly by requiring that $\lambda_G \circ \kappa_G = \iota_G$.

For an open $U \subset G$ let $\hat{U}$ be the set of $\hat{g} \in \hat{G}$ for which there is a representative $\{g_i\}$ and an neighborhood V of the origin such that $g_i + V \subset U$ for all sufficiently large i.

Lemma E1.4. *The collection of sets $\hat{U}$ is a base of a topology for $\hat{G}$ with respect to which $\hat{G}$ is an abelian topological group, and ι_G and λ_G are continuous.*

Proof. In order to be a base of a topology, a collection of sets needs to cover the space (here $\hat{G}$ is in the base because G is open) and the intersection of any two base elements must be a union of base elements. We will show that if U and U' are open, then $\hat{U} \cap \hat{U}' = \widehat{U \cap U'}$. Obviously $\widehat{U \cap U'} \subset \hat{U} \cap \hat{U}'$. To verify the reverse inclusion suppose that $\hat{g} \in \hat{U} \cap \hat{U}'$. Take representatives $\{g_i\}$ and $\{g_i'\}$ of $\hat{g}$ and neighborhoods V and V' of the origin such that $g_i + V \subset U$ and $g_i' + V' \subset U'$ for large i. Let W be a neighborhood of the origin such that $W + W \subset V \cap V'$. Of course $g_i + W \subset U$ for large i. In addition, for large i we have $g_i - g_i' \in W$ and thus $g_i + W \subset g_i' + W + W \subset U'$. Thus $\hat{g} \in \widehat{U \cap U'}$.

For any open U we have $\widehat{-U} = -\hat{U}$, so negation is continuous. If $\hat{g} + \hat{g}' \in \hat{U}$, then the techniques illustrated in the argument above can easily be used to construct open V and V' with $\hat{g} \in \hat{V}$, $\hat{g}' \in \hat{V}'$, and $\hat{V} + \hat{V}' \subset \hat{U}$. Thus $\hat{G}$ is an abelian topological group.

Evidently $\iota_G^{-1}(\hat{U}) = U$, so ι_G is continuous, and $\lambda_G^{-1}(\hat{U}) = \kappa_G(U)$ is open because κ_G is an open map, so λ_G is continuous. □

Whenever V is open, $\hat{V}$ contains $\iota_G(g)$ for all $g \in V$, so:

Lemma E1.5. *$\iota_G(G)$ is dense in $\hat{G}$.*

Lemma E1.6. *$\hat{G}$ is Hausdorff.*

Proof. If $\{g_i\}$ is a representative of $\hat{g} \in K_{\hat{G}}$, then g_i is eventually in every open U containing the origin, which means that $\{g_i\}$ is equivalent to the constant sequence with value 0, and thus $\hat{g} = 0 \in \hat{G}$. □

If $f : G \to G'$ is a continuous homomorphism, f maps Cauchy sequences to Cauchy sequences, and it maps equivalent sequences to equivalent sequences. Therefore f induces a map

$$\hat{f} : \hat{G} \to \hat{G}'$$

taking each equivalence class of Cauchy sequences to the equivalence class of any of its elements. This is easily shown to be a continuous homomorphism. Because G is dense in $\hat{G}$, it is in fact the unique continuous extension of f. If $f' : G' \to G''$ is a second continuous homomorphism, then $\widehat{f' \circ f} = \hat{f}' \circ \hat{f}$. Thus completion is a functor from the category of abelian topological groups to itself.

It is an immediate consequence of the definitions that if $f : G \to G'$ is a continuous homomorphism, then the diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\iota_G"'] & G' \arrow[d, "\iota_{G'}"] \\ \hat{G} \arrow[r, "\hat{f}"] & \hat{G}' \end{tikzcd}$$

commutes. This simple fact can be dressed up as abstract nonsense:

Lemma E1.7. *The system of maps ι_G constitute a natural transformation from the identity functor, for the category of abelian topological groups, to the completion functor.*

If ι_G is surjective we say that G is *complete*. In view of Lemma E1.3, G is complete if and only if each Cauchy sequence has a limit in G. This section's most intricate result provides some justification of our terminology.

Proposition E1.8. *$\hat{G}$ is complete.*

Proof. At this point we use the assumption that there is a countable neighborhood basis $U_1, U_2, U_3, \ldots$ at 0. We can replace U_n with $U_1 \cap \ldots \cap U_n$, so we may assume that $U_1 \supset U_2 \supset U_3 \supset \cdots$. For each n choose an open V_n containing 0 such that $V_n + V_n + V_n + V_n + V_n \subset U_n$. Again, we may assume that $V_1 \supset V_2 \supset \cdots$.

We need to show that a given Cauchy sequence $\{\hat{g}^h\}$ in $\hat{G}$ has a limit in $\hat{G}$. For each h choose a representative $\{g_i^h\}$ of $\hat{g}^h$.

For each n choose an h_n such that $\hat{g}^h - \hat{g}^{h'} \in \hat{V}_n$ for all $h, h' \geq h_n$. This property is preserved if we replace each h_n with the maximum of $h_1, \ldots, h_n$, so we may assume that $h_1 \leq h_2 \leq \cdots$. For each h and n choose an i_{hn} such that $g_i^h - g_{i'}^h \in V_n$ for all $i, i' \geq i_{hn}$. As above, we may assume that i_{hn} is weakly increasing in h and n.

For each $i = 1, 2, \ldots$ let $g_i = g_{i_{h_i i}}^{h_i}$. If $i, i' \geq n$, then for all sufficiently large i'' we have

$$g_i - g_{i'} = g_{i_{h_i i}}^{h_i} - g_{i_{h_{i'} i'}}^{h_{i'}} = (g_{i_{h_i i}}^{h_i} - g_{i''}^{h_i}) + (g_{i''}^{h_i} - g_{i''}^{h_{i'}}) + (g_{i''}^{h_{i'}} - g_{i_{h_{i'} i'}}^{h_{i'}})$$

$$\in V_i + V_n + V_{i'} \subset U_n.$$

Therefore $\{g_i\}$ is Cauchy. Let $\hat{g}$ be its equivalence class.

For each n, if $h \geq h_n$, $i > n$, and i and i' are sufficiently large, then

$$g_i^h - g_i = (g_i^h - g_{i'}^h) + (g_{i'}^h - g_{i'}^{h_n}) + (g_{i'}^{h_n} - g_{i'}^{h_i}) + (g_{i'}^{h_i} - g_{i_{h_i i}}^{h_i})$$

$$\in V_n + V_n + V_n + V_i \subset V_n + V_n + V_n + V_n,$$

so $g_i^h - g_i + V_n \subset U_n$. Therefore $\hat{g}^h - \hat{g} \in \hat{U}_n$ for sufficiently large h. This is true for all n, so $\hat{g}^h \to \hat{g}$. □

Now suppose that R is a topological ring, so R is a ring with a topology with respect to which addition, negation, and multiplication are continuous, and 0 has a countable neighborhood base. In the following discussion does *not* need to be commutative and need not have a unit. The completion $\hat{R}$ of R is its completion as a topological group. We check that $\hat{R}$ has a well defined and well behaved multiplication.

Lemma E1.9. *If $\{r_i\}$ and $\{s_i\}$ are Cauchy sequences, then $\{r_i s_i\}$ is Cauchy. It $\{r_i'\}$ and $\{s_i'\}$ are Cauchy sequences that are equivalent to $\{r_i\}$ and $\{s_i\}$ respectively, then $\{r_i' s_i'\}$ is equivalent to $\{r_i s_i\}$.*

Proof. Fix a neighborhood U of 0. Choose a neighborhood V of 0 such that $V+V+V+V \subset U$, and choose a neighborhood W of 0 such that $W \cdot W \subset V$. There is an i such that $r_j - r_k, s_j - s_k \in W$ for all $j, k \geq i$. The continuity of multiplication implies that for sufficiently large j and k we have $r_i(s_j - s_k), (r_j - r_k)s_i \in V$, so that

$$r_j s_j - r_k s_k = r_j(s_j - s_k) + (r_j - r_k)s_k$$

$$= r_i(s_j - s_k) + (r_j - r_i)(s_j - s_k) + (r_j - r_k)(s_k - s_i) + (r_j - r_k)s_i \in U.$$

Thus $\{r_i s_i\}$ is Cauchy.

Now choose an i such that $r'_j - r_j, r_j - r_k, s'_j - s_j, s'_j - s'_k \in W$ whenever $j, k \geq i$. If j is sufficiently large, then $(r'_j - r_j)s'_i, r_i(s'_j - s_j) \in V$, so that

$$r'_j s'_j - r_j s_j = (r'_j - r_j)s'_j + r_j(s'_j - s_j)$$

$$= (r'_j - r_j)s'_i + (r'_j - r_j)(s'_j - s'_i) + (r_j - r_i)(s'_j - s_j) + r_i(s'_j - s_j) \in U.$$

Thus $\{r'_i s'_i\}$ and $\{r_i s_i\}$ are equivalent. □

We may now define multiplication in $\hat{R}$ by specifying that if $\{r_i\}$ and $\{s_i\}$ are representatives of $\hat{r}$ and $\hat{s}$, then $\{r_i s_i\}$ is a representative of $\hat{r}\hat{s}$. Of course $\hat{R}$ is commutative if R is commutative, and if R has a unit, its image is a unit for $\hat{R}$.

Lemma E1.10. *Multiplication in $\hat{R}$ is continuous.*

Proof. Suppose that $\hat{r}\hat{s} \in \hat{U}$, where $U \subset R$ is open. Choose a neighborhood V of $0 \in R$ such that $r_i s_i + V + V \subset U$ for all large i. Let W be a neighborhood of 0 such that $W + W + W + W \subset V$, and let Z be a neighborhood of 0 such that $Z \cdot Z \subset W$. Suppose that $\hat{r}' \in \hat{r} + \hat{Z}$ and $\hat{s}' \in \hat{s} + \hat{Z}$, and $\{r'_i\}$ and $\{s'_i\}$ are representatives. Choose i such the $r_j - r_k, s'_j - s'_k \in Z$ for all $j, k \geq i$. Then for all sufficiently large j we have $(r'_j - r_j)s'_i, r_i(s'_j - s_j) \in Z$, because multiplication is continuous, and $r'_j - r_j, s'_j - s_j \in Z$ because the sequences are Cauchy. The second calculation of the last proof now gives $r'_j s'_j - r_j s_j \in V$ for large j, which implies that $r'_j s'_j + V \subset U$, so $\hat{r}'\hat{s}' \in \hat{U}$. □

E2 The Completion of a Filtered Group

Let G be an abelian group, and let (G_n) be a given filtration. We endow G with the Krull topology induced by this filtration, and we let $\hat{G}$ be the completion of G with respect to this topology. Lemma E1.4 implies that:

Lemma E2.1. *If G has the Krull topology, then the topology of $\hat{G}$ coincides with the Krull topology induced by the filtration $\hat{G} = \hat{G}_0 \supset \hat{G}_1 \supset \hat{G}_2 \supset \cdots$.*

We adopt the following notation. For $m \leq n$ let $\theta^n_m : G/G_n \to G/G_m$ be the natural map. An element of $\varprojlim G/G_n$ is a sequence $\{\tilde{g}_n\}$ with $\tilde{g}_n \in G/G_n\}$ such that $\theta^n_m(\tilde{g}_n) = \tilde{g}_m$ for all $m \leq n$. Let

$$\theta_m : \varprojlim G/G_n \to G/G_m$$

be the projection $\{\tilde{g}_n\} \mapsto \tilde{g}_m$.

Suppose $\{g_i\}$ is a Cauchy sequence in G. For each n, $g_i - g_{i'}$ for all sufficiently large i and i', which is to say that the sequence $\{g_i + G_n\}$ in G/G_n stabilizes; let $\tilde{g}_n$ be its terminal value. Clearly $\theta^n_m(\tilde{g}_n) = \tilde{g}_m$ for all $m \leq n$. Moreover, equivalent Cauchy sequences induce the same terminal elements in each G/G_n and thus the same element of $\varprojlim G/G_n$, so there is a map from

$\hat{G}$ to $\varprojlim G/G_n$. This map is obviously a homomorphism, and it is easily seen to be injective. It is in fact an isomorphism because for any $\{\tilde{g}^n\} \in \varprojlim G/G_n$ there is a preimage, namely the equivalence class of any sequence $\{g_n\}$ in which each g_n is an element of the coset $\tilde{g}_n$. Thus we can identify $\hat{G}$ with $\varprojlim G/G_n$.

The concrete representation of $\hat{G}$ given by the inverse limit is very useful. First of all, there is the flexibility provided by the following result.

Lemma E2.2. *Suppose that $G = G'_0 \supset G'_1 \supset G'_2 \supset \cdots$ is a second filtration of G such that for each m there is some n with $G'_n \subset G_m$, and for each n there is some m such that $G_m \subset G'_n$. Then*

$$\varprojlim G/G_m \cong \varprojlim G/G'_n.$$

Proof. Choose strictly increasing functions $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ such that $G_{\beta(m)} \subset G'_{\alpha(m)} \subset G_m$ for all m. The natural projections induce homomorphisms

$$\varprojlim G/G_{\beta(m)} \to \varprojlim G/G'_{\alpha(m)} \to \varprojlim G/G_m.$$

It is easy to see that this composition is an isomorphism, and for the same reason $\varprojlim G/G'_{\alpha(n)} \cong \varprojlim G/G'_n$. $\square$

We now apply homological methods to the study of inverse systems. Let $A : A_0 \leftarrow A_1 \leftarrow \cdots$ and $B : B_0 \leftarrow B_1 \leftarrow \cdots$ be inverse systems. A *homomorphism* $\phi : A \to B$ of inverse systems is a collection of homomorphisms $\phi_n : A_n \to B_n$ such that each diagram

$$\begin{array}{ccc} A_{n-1} & \xrightarrow{\phi_{n-1}} & B_{n-1} \\ \uparrow & & \uparrow \\ A_n & \xrightarrow{\phi_n} & B_n \end{array} \tag{$*$}$$

commutes. In the obvious way such a homomorphism induces a homomorphism $\hat{\phi} : \varprojlim A_n \to \varprojlim B_n$, and it is easy to see that passage to the inverse limit is a covariant functor from the category of inverse systems to the category of groups.

The inverse system A is *surjective* if each homomorphism $A_n \to A_{n-1}$ is surjective, in which case the natural homomorphism $\varprojlim A_n \to A_m$ is surjective for each m.

Proposition E2.3. *Passage to inverse limits is a left exact functor: if $0 \to A \to B \to C \to 0$ is an exact sequence of inverse systems, then*

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n$$

is exact. If A is surjective, then

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$

is exact.

Proof. Let $\tilde{A} = \prod_n A_n$. Denote the homomorphism $A_n \to A_{n-1}$ by θ^n_{n-1}, and let $d^A : \tilde{A} \to \tilde{A}$ be the homomorphism with n^{th} component $d^A_n(a) = a_n - \theta^{n+1}_n(a_{n+1})$. Define $\tilde{B}$, d^B, $\tilde{C}$, and d^C similarly. There is a diagram

$$\begin{array}{ccccccccc} 0 & \to & \tilde{A} & \to & \tilde{B} & \to & \tilde{C} & \to & 0 \\ & & {\scriptstyle d^A}\big\downarrow & & {\scriptstyle d^B}\big\downarrow & & {\scriptstyle d^C}\big\downarrow & & \\ 0 & \to & \tilde{A} & \to & \tilde{B} & \to & \tilde{C} & \to & 0 \end{array}$$

that commutes because each diagram $(*)$ commutes. The snake lemma gives an exact sequence

$$0 \to \operatorname{Ker} d^A \to \operatorname{Ker} d^B \to \operatorname{Ker} d^C \to \operatorname{Coker} d^A \to \operatorname{Coker} d^B \to \operatorname{Coker} d^C.$$

(The sequence given by Lemma B1.3 is supplemented here with the observation that $\operatorname{Ker} d^A \to \operatorname{Ker} d^B$ is injective because it is the restriction of $\tilde{A} \to \tilde{B}$ to $\operatorname{Ker} d^A$.) Since $\operatorname{Ker} d^A = \varprojlim A_n$, and similarly for B and C, the first assertion follows.

Supposing now that A is surjective, we need to show that $\operatorname{Coker} d^A = 0$, i.e., d^A is surjective. Concretely, for a given $x \in \tilde{A}$ we need to find $a \in \tilde{A}$ such that $x_n - a_n = \theta^{n+1}_n(a_{n+1})$ for all n. Since A is surjective, for any choice of a_0 a solution of this system of equations can be constructed inductively. □

Corollary E2.4. *Suppose that $0 \to G' \to G \to G'' \to 0$ is exact. For each n let G'_n be the preimage of G_n in G', and let G''_n be the image of G_n in G''. Let $\hat{G}' = \varprojlim G'/G'_n$ and $\hat{G}'' = \varprojlim G''/G''_n$. Then the sequence*

$$0 \to \hat{G}' \to \hat{G} \to \hat{G}'' \to 0$$

is exact.

Proof. We first show that for each n, the sequence $0 \to G'/G'_n \to G/G_n \to G''/G''_n \to 0$ is exact. Clearly $G'/G'_n \to G/G_n$ is injective, $G/G_n \to G''/G''_n$ is surjective, and their composition is zero. If an element $g + G_n$ of G/G_n goes to zero in G''/G''_n, then the image of g is in G''_n, and is consequently equal to the image of some $\tilde{g} \in G_n$. Since $g - \tilde{g}$ maps to zero, it is the image of some $g' \in G'$, and the image of $g' + G'_n$ in G/G_n is $g + G_n$.

For each n the diagram

$$\begin{array}{ccccccccc} 0 & \to & G'/G'_{n+1} & \longrightarrow & G/G_{n+1} & \longrightarrow & G''/G''_{n+1} & \to & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \to & G'/G'_n & \longrightarrow & G/G_n & \longrightarrow & G''/G''_n & \to & 0 \end{array}$$

commutes, obviously. The inverse system $0 \leftarrow G'/G'_1 \leftarrow G'/G'_2 \leftarrow \cdots$ is surjective, so the claim follows from the last result. □

Corollary E2.5. *If $G_n \subset G'$ for some n, then $\hat{G}/\hat{G}' \cong G/G'$.*

Proof. The last result gives $\hat{G}/\hat{G}' \cong \hat{G}''$, and $\hat{G}'' = \varprojlim (G/G_n)/(G'/G_n) = G/G'$. □

E3 The Associated Graded Group

Let (G_n) be a filtration of G. We now turn our attention to the associated graded group

$$G(G) = \oplus_{n=0}^{\infty} G_n/G_{n+1}.$$

Suppose H is a second group with filtration (H_n). A *filtered homomorphism* from G to H is a homomorphism $\phi : G \to H$ such that $\phi(G_n) \subset H_n$ for all n. Let $G(\phi) : G(G) \to G(H)$ be the graded homomorphism with component homomorphisms $G_n(\phi) : G_n/G_{n+1} \to H_n/H_{n+1}$ given by $g_n + G_{n+1} \mapsto \phi(g_n) + H_{n+1}$. Simple verifications demonstrate that we can understand $G(\cdot)$ as a covariant functor from the category of filtered groups and filtered homomorphisms to the category of graded groups and graded homomorphisms.

There is no clear or obvious relationship between $G(G)$ and $\hat{G}$. Nevertheless there is two interesting implications.

Lemma E3.1. *If G and H have filtrations (G_n) and (H_n) and $\phi : G \to H$ is a filtered homomorphism, then:*

(a) if $G(\phi)$ is injective, then $\hat{\phi}$ is injective;

(b) if $G(\phi)$ is surjective, then $\hat{\phi}$ is surjective.

Proof. For each n there is diagram

$$\begin{array}{ccccccc} 0 \to & G_n/G_{n+1} & \longrightarrow & G/G_{n+1} & \longrightarrow & G/G_n & \to 0 \\ & \Big\downarrow{\scriptstyle G_n(\phi)} & & \Big\downarrow{\scriptstyle \phi_{n+1}} & & \Big\downarrow{\scriptstyle \phi_n} & \\ 0 \to & H_n/H_{n+1} & \longrightarrow & H/H_{n+1} & \longrightarrow & H/H_n & \to 0 \end{array}$$

which commutes and has exact rows. Applying the snake lemma (Lemma B1.3) gives an exact sequence

$$\operatorname{Ker} G_n(\phi) \to \operatorname{Ker} \phi_{n+1} \to \operatorname{Ker} \phi_n \to \operatorname{Coker} G_n(\phi) \to \operatorname{Coker} \phi_{n+1} \to \operatorname{Coker} \phi_n.$$

Since $G/G_0 = 0 = H/H_0$, ϕ_0 has null kernel and cokernel. If $G(\phi)$ is injective, then the first half of this sequence implies (by induction) that $\operatorname{Ker} \phi_n = 0$ for all n. Since $\hat{G}$ is a subgroup of $\prod_{n=0}^{\infty} G/G_n$ it follows that $\hat{\phi}$ is injective.

If $G(\phi)$ is surjective, then the second half of the sequence implies that $\operatorname{Coker} \phi_n = 0$ for all n. Thus the rows of the diagram

$$\begin{array}{ccccccc} 0 \to & \operatorname{Ker} \phi_{n+1} & \longrightarrow & G/G_{n+1} & \xrightarrow{\phi_{n+1}} & H/H_{n+1} & \to 0 \\ & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & \\ 0 \to & \operatorname{Ker} \phi_n & \longrightarrow & G/G_n & \xrightarrow{\phi_n} & H/H_n & \to 0 \end{array}$$

are exact. This diagram commutes because the diagram above commutes and the left hand vertical map is the restriction of the central vertical map. In addition, since $\operatorname{Coker}\phi_n = 0$, the sequence above implies that $\operatorname{Ker}\phi_{n+1} \to \operatorname{Ker}\phi_n$ is surjective. Therefore (b) follows from Proposition E2.3. $\square$

E4 Derived Graded Rings

A *graded ring* is a graded group $S = \oplus_{n\geq 0} S_n$ that is also a ring, with the multiplication satisfying $S_m S_n \subset S_{m+n}$ for all $m, n \geq 0$. Elements of S_n are said to be *homogeneous of degree* n. Since S_0 is closed under addition and multiplication, it is a subring. Let $S_+ = \oplus_{n\geq 1} S_n$. Evidently S_+ is an ideal, and $S_0 \cong S/S_+$.

Let R be an ungraded ring, and let I be an ideal. Then (I^n) is a filtration, so there is a derived graded ring

$$G_I(R) = \oplus_{n\geq 0} I^n/I^{n+1},$$

which is a specific instance of the derived group $G(G)$ seen in the last section. Another derived graded ring is

$$R^* = \oplus_{n\geq 0} I^n.$$

Our agenda in this section is to study when these derived rings are Noetherian. We begin with a generalization of the Hilbert basis theorem.

Proposition E4.1. *A graded ring $S = \oplus_{n\geq 0} S_n$ is Noetherian if and only S_0 is Noetherian and S is finitely generated as an S_0-algebra.*

Proof. First suppose that S_0 is Noetherian and S is finitely generated as an S_0-algebra, say $S = S_0[x_1, \ldots, x_r]$. Of course $S_0[X_1, \ldots, X_r]$ is Noetherian (Hilbert basis theorem) and the image of a surjective ring homomorphism is Noetherian if the domain is Noetherian. There is an obvious such surjection from $S_0[X_1, \ldots, X_n]$ to S, so S is Noetherian.

Now suppose that S is Noetherian. Then $S_0 \cong S/S_+$ is Noetherian.

Let $x_1, \ldots, x_r$ be a system of generators for the ideal S_+, and let $S' = S_0[x_1, \ldots, x_r]$. We will show, by induction, that $S_n \subset S'$ for all n. This is obviously the case for $n = 0$. Each x_i is a sum of homogeneous elements of degree ≥ 1, and may be replaced by them in the list of generators, so we may assume that each x_i is homogeneous, say of degree $d_i \geq 1$. Then any element of S_n has the form $a_1x_1 + \cdots + a_rx_r$ where each a_i is homogeneous of degree $n - d_i$. By induction, each a_i is in S', so $a_1x_1 + \cdots + a_rx_r \in S'$. $\square$

There are now two results concerning $G_I(R)$ and R^* respectively.

Lemma E4.2. *If R is Noetherian, then $G_I(R)$ is Noetherian.*

Proof. Since R is Noetherian, I is finitely generated, say by $x_1, \ldots, x_r$, and R/I is Noetherian. Now $(R/I)[X_1, \ldots, X_r]$ is Noetherian by the Hilbert basis theorem, and there is a surjective homomorphism from this ring to $G_I(R)$ that takes each $X_1^{\alpha_1} \cdots X_r^{\alpha_r}$ to the image of $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ in $I^\alpha / I^{\alpha+1}$, where $\alpha = \alpha_1 + \cdots + \alpha_r$. □

Lemma E4.3. *R^* is Noetherian if and only if R is Noetherian.*

Proof. There is a bijection between the ideals J of R and the ideals of R^* of the form $J \oplus I \oplus I^2 \oplus \cdots$, so R is Noetherian if R^* is Noetherian. If R is Noetherian and $x_1, \ldots, x_n$ generate I, then they also generate R^* as an R-algebra, so R^* is Noetherian by Proposition E4.1. □

E5 Filtered Modules and the Artin-Rees Lemma

Let M be an R-module, and let (M_n) and (M'_n) be filtrations. If, for each n', there is some n such that $M_n \subset M'_{n'}$, and, for each n, there is some n' such that $M'_{n'} \subset M_n$, then the two filtrations induce the same Krull topology on M, and Lemma E2.2 implies that the two completions are isomorphic. We will be interested in a stronger condition: the filtrations *have bounded difference* if there is a nonnegative integer n_0 such that $M_{n+n_0} \subset M'_n$ and $M'_{n+n_0} \subset M_n$ for all n.

In general, if $S = \oplus_{n \geq 0} S_n$ is a graded ring, a *graded S-module* is an S module $N = \oplus_{n \geq 0} N_n$ such that $S_m N_n \subset N_{m+n}$ for all m and n. Fix an ideal I. The filtration (M_n) is an *I-filtration* if $IM_n \subset M_{n+1}$ for all n. If this is the case, then $M^* = \oplus_{n \geq 0} M_n$ is a graded R^*-module with scalar multiplication

$$(r_0, r_1, \ldots)(m_0, m_1, \ldots) = (r_0 m_0, r_0 m_1 + r_1 m_0, \ldots).$$

An I-filtration (M_n) is *stable* if $IM_n = M_{n+1}$ for all sufficiently large n.

Lemma E5.1. *If R is Noetherian, I is an ideal, M is a finitely generated R-module, and (M_n) is an I-filtration, then M^* is finitely generated as an R^*-module if and only if (M_n) is stable.*

Proof. For each $n \geq 0$ let

$$M_n^* = M_0 \otimes \cdots \otimes M_n \otimes IM_n \otimes I^2 M_n \otimes \cdots .$$

This is a submodule of M^* that is finitely generated because $M_0 \otimes \cdots \otimes M_n$ is a finitely generated R-module. The M_n^* are an increasing sequence whose union is M^*, and M^* is finitely generated as an R^*-module if and only if the sequence stabilizes, i.e., there is some n_0 such that $M_{n_0}^* = M^*$, which is to say that $M_{n_0+i} = I^i M_{n_0}$ for all $i \geq 0$. □

Lemma E5.2. *If (M_n) and (M'_n) are both stable I-filtrations of M, then they have bounded difference.*

Proof. Since having bounded difference is an equivalence relation, it suffices to prove the claim when $M'_n = I^n M$. We have $I^n M \subset M_n$ by induction: $I^{n+1}M = I(I^n M) \subset IM_n \subset M_{n+1}$. If $IM_n = M_{n+1}$ for all $n \geq n_0$, then $M_{n+n_0} \subset I^n M$ for all n. □

We now have some rather famous results.

Theorem E5.3 (Artin-Rees Lemma). *If R is Noetherian, M is a finitely generated R-module, (M_n) is a stable I-filtration of M, and M' is a submodule of M, then $(M' \cap M_n)$ is a stable I-filtration of M'.*

Proof. Since $I(M' \cap M_n) = IM' \cap IM_n \subset M' \cap M_{n+1}$ for each n, $(M' \cap M_n)$ is an I-filtration of M'. Therefore $M'^* = \oplus_{n \geq 0} M' \cap M_n$ is a submodule of M^*. Since (M_n) is stable, Lemma E5.1 implies that M^* is finitely generated, so M'^* is finitely generated, and a second application of Lemma E5.1 gives the desired conclusion. □

Since $(I^n M')$ is automatically a stable I-filtration of M', the last result and Lemma E5.2 give:

Corollary E5.4. *If R is Noetherian, M is a finitely generated R-module, (M_n) is a stable I-filtration of M, and M' is a submodule of M, then $(I^n M')$ and $(M' \cap M_n)$ have bounded difference.*

Theorem E5.5 (Krull Intersection Theorem). *If R is Noetherian and M is a finitely generated R-module, then there is $r \in I$ such that $(1-r)\bigcap_{n=0}^{\infty} I^n M = 0$. If R is either a local ring or an integral domain, then $\bigcap_{n=0}^{\infty} I^n = 0$.*

Proof. We apply the Artin-Rees lemma with $M_n = I^n M$ and $M' = \bigcap_{n=0}^{\infty} I^n M$, finding that $(M' \cap M_n)$ is stable, so $M' = M' \cap I^{n+1}M = I(M' \cap I^n M) = IM'$, for large n. Now Corollary A3.2 gives the desired r.

For the second assertion we apply the first with $M = R$. Since I is proper, $r \neq 1$, and the claim follows immediately if R is an integral domain. If R is a local ring, then I is contained in the maximal ideal, so $1 - r$ is not in the maximal ideal and is consequently a unit. □

Corollary E5.6. *If R is Noetherian and local and $G_I(R)$ is an integral domain, then R is an integral domain.*

Proof. For $r \in R$ let $\iota(r) \in G_I(R)$ be the image of r in $G_n = I^n/I^{n+1}$ if n is the largest integer such that $r \in I^n$, and let $\iota(r) = 0$ if there is no such n. Proposition E5.5 implies that $\iota^{-1}(0) = 0$. Suppose that $0 \neq r, s \in R$. Then $0 \neq \iota(r) \in G_m$ and $0 \neq \iota(s) \in G_n$ for some m and n. Since $G_I(R)$ is an integral domain, $\iota(r)\iota(s) \neq 0$, but $\iota(r)\iota(s)$ is the image of rs in $I^{m+n} \setminus I^{m+n+1}$, so $rs \neq 0$. □

Let (M_n) be an I-filtration of M. Recall that $G(M) = \oplus_{n\geq 0} G_n(M)$ where $G_n(M) = M_n/M_{n+1}$. The scalar multiplication

$$(i_\ell + I^{\ell+1})(m_n + M_{n+1}) = i_\ell m_n + M_{\ell+n+1}$$

(where $i_\ell \in I^\ell$ and $m_n \in M_n$) is easily seen to be independent of the choice of representatives, and makes $G(M)$ into a graded $G(R)$-module.

Lemma E5.7. *If R is Noetherian, M is a finitely generated R-module, and (M_n) is a stable I-filtration, then $G(M)$ is a finitely generated $G(R)$-module.*

Proof. Each $G_n(M) = M_n/M_{n+1}$ is finitely generated as an R module, and it is annihilated by I, so it may be understood as a finitely generated R/I-module. There is an n_0 such that $M_{n_0+r} = I^r M_{n_0}$ for all $r \geq 0$, so $G(M)$ is generated as a $G(R)$-module by $\oplus_{n=0}^{n_0} G_n(M)$. Now $\oplus_{n=0}^{n_0} G_n(M)$ is finitely generated as an R/I-module, and $R/I = G_0(R)$, so $G(M)$ is finitely generated as a $G(R)$-module. $\square$

E6 Completions of Rings and Modules

Now that some required tools have been developed, we turn to the more specific types of completions we are interested in. Let I be an ideal of R. The Krull topology induced by the filtration $R \subset I \subset I^2 \supset \cdots$ is called the *I-adic topology*. We have seen the completion of R with respect to this topology is a ring $\hat{R}$ that is isomorphic to $\varprojlim R/I^n$ and whose multiplication is continuous (Lemma E1.10).

More generally, recall that if M is an R-module, then the completion $\hat{M}$ of M is an $\hat{R}$-module that is isomorphic to $\varprojlim M/I^n M$. If R is Noetherian, then for finitely generated R-modules the I-adic completion functor is exact:

Lemma E6.1. *If $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ is an exact sequence of R-modules, then $\hat{M} \to \hat{M}'' \to 0$ is an exact sequence of $\hat{R}$-modules. If, in addition, R is Noetherian and M is finitely generated, then $0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$ is exact.*

Proof. Let $A_n = M'/i^{-1}(I^n M)$, $B_n = M/I^n M$, and $C_n = M''/p(I^n M)$, regarded as inverse systems. Since the first of these is surjective, Proposition E2.3 implies that

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$

is exact. We have $\varprojlim B_n = \hat{M}$ and $\varprojlim C_n = \hat{M}''$ because $p(I^n M) = I^n p(M) = I^n M''$, which establishes the first assertion.

Now suppose that R is Noetherian. Regarding M' as a submodule of M, Theorem E5.3 implies that $(M' \cap I^n M)$ is a stable I-filtration of M', after which Lemma E5.2 implies that the filtrations $(M' \cap I^n M)$ and $(I^n M')$ have bounded difference. In view of Lemma E2.2, $\varprojlim A_n = \hat{M}'$. $\square$

Let $\iota_{R,I} : R \to \hat{R}$ and $\iota_{M,I} : M \to \hat{M}$ be the natural homomorphisms. This first of these makes $\hat{R}$ into an R-algebra, so there is an $\hat{R}$-module $\hat{R} \otimes_R M$, and $\iota_{M,I}$ induces a map $\hat{R} \otimes_R M \to \hat{R} \otimes_R \hat{M}$. In turn $\iota_{R,I}$ induces a map $\hat{R} \otimes_R \hat{M} \to \hat{R} \otimes_{\hat{R}} \hat{M} = \hat{M}$.

Proposition E6.2. *If M is finitely generated, the composition*

$$\hat{R} \otimes_R M \to \hat{R} \otimes_R \hat{M} \to \hat{R} \otimes_{\hat{R}} \hat{M} = \hat{M}$$

is surjective. If, in addition, R is Noetherian, then this composition is an isomorphism.

Proof. First consider an R-module homomorphism $M \to N$. The first square in the diagram

$$\begin{tikzcd}
\hat{R} \otimes_R M \arrow[r] \arrow[d] & \hat{R} \otimes_R \hat{M} \arrow[r] \arrow[d] & \hat{R} \otimes_{\hat{R}} \hat{M} \arrow[d] \\
\hat{R} \otimes_R N \arrow[r] & \hat{R} \otimes_R \hat{N} \arrow[r] & \hat{R} \otimes_{\hat{R}} \hat{N}.
\end{tikzcd}$$

commutes because the maps $\iota_{M,I}$ constitute a natural transformation (Lemma E1.7). To see that the second square commutes consider that for $\hat{r} \otimes_R \hat{m}$ the two paths around the square are, by definition,

$$\hat{r} \otimes_R \hat{m} \to \hat{r} \otimes_{\hat{R}} \hat{m} \to \hat{r} \otimes_{\hat{R}} \hat{f}(\hat{m}) \text{ and } \hat{r} \otimes_R \hat{m} \to \hat{r} \otimes_R \hat{f}(\hat{m}) \to \hat{r} \otimes_{\hat{R}} \hat{f}(\hat{m}).$$

Insofar as M is finitely generated, there is an exact sequence $0 \to K \to F \to M \to 0$ where $F = R^n$ for some n. There is a diagram

$$\begin{tikzcd}
 & \hat{R} \otimes_R K \arrow[r] \arrow[d, "\alpha"'] & \hat{R} \otimes_R F \arrow[r] \arrow[d, "\beta"'] & \hat{R} \otimes_R M \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \hat{K} \arrow[r] & \hat{F} \arrow[r] & \hat{M} \arrow[r] & 0
\end{tikzcd}$$

that commutes by virtue of the argument above. The top row is exact because $\hat{R} \otimes_R -$ is a right exact functor, and the last result implies that $\hat{F} \to \hat{M} \to 0$ is exact. The composition $\hat{R} \otimes_R R \to \hat{R} \otimes_R \hat{R} \to \hat{R} \otimes_{\hat{R}} \hat{R} = \hat{R}$ takes $\hat{r} \otimes_R r = \hat{r}r \otimes_R 1$ to $\hat{r}r$. Since it is the n-fold cartesian product of this, β is an isomorphism. Consequently γ is surjective.

If R is Noetherian, then the last result implies that the bottom row is exact. In addition K is finitely generated, so α is surjective by virtue of what we have established so far. To show that γ is injective, suppose that $\gamma(\tilde{m}) = 0$, choose a preimage $\tilde{f}$ in $\hat{R} \otimes_R F$, choose a preimage $\hat{k}$ of $\beta(\tilde{f})$ in $\hat{K}$, and choose a preimage $\tilde{k}$ of $\hat{k}$ in $\hat{R} \otimes_R K$. Since the diagram commutes and β is an isomorphism, $\tilde{f}$ must be the image of $\tilde{k}$, and consequently $\tilde{m} = 0$. $\square$

We are particularly interested in applying the results above to ideals of R, and to I itself.

Proposition E6.3. *If R is Noetherian and J is an ideal, then:*

(a) $\hat{J} \cong \hat{R} \otimes_R J = J\hat{R}$*;*

(b) $\widehat{J^n} = \hat{J}^n$*;*

(c) if $I^n \subset J$ *for some* n*, then* $\hat{J}^n/\hat{J}^{n+1} \cong J^n/J^{n+1}$*;*

(d) $\hat{I}$ *is contained in the Jacobson radical of* $\hat{R}$*.*

Proof. Since R is Noetherian, J is finitely generated, so the last result implies $\hat{J} \cong \hat{R} \otimes_R J$, and the definition of the tensor product gives $\hat{R} \otimes_R J = J\hat{R}$, so (a) holds. To prove (b) note that (a) allows the computation

$$\widehat{J^n} = J^n\hat{R} = J^n\hat{R}^n = (J\hat{R})^n = \hat{J}^n.$$

For (c) there is now $\hat{J}^n/\hat{J}^{n+1} = \widehat{J^n}/\widehat{J^{n+1}} = J^n/J^{n+1}$, where the second equality is from Corollary E2.5.

For $\hat{x} \in \hat{I}$ consider the sequence of sums $1 + \hat{x} + \cdots + \hat{x}^n$. This is a Cauchy sequence, and (Proposition E1.8 and Lemma E2.1) $\hat{R}$ is complete, so it has a limit $\hat{z}$. Because multiplication is continuous we have $(1 - \hat{x})\hat{z} = 1$. More generally, for any $\hat{y} \in \hat{R}$ we have $\hat{x}\hat{y} \in \hat{I}$, so $1 - \hat{x}\hat{y}$ is a unit. Therefore $\hat{x}$ is in the Jacobson radical (Proposition A2.11). $\square$

Theorem E6.4. *If R is Noetherian and local, and $\hat{R}$ is its $\mathfrak{m}$-adic completion, then $\hat{R}$ is local with maximal ideal $\hat{\mathfrak{m}}$.*

Proof. By (c) above we have $\hat{R}/\hat{\mathfrak{m}} \cong R/\mathfrak{m}$, so $\hat{R}/\hat{\mathfrak{m}}$ is a field and $\hat{\mathfrak{m}}$ is maximal. Now (d) implies that $\hat{\mathfrak{m}}$ is contained in the Jacobson radical, and thus in every maximal ideal, so it is the unique maximal ideal. $\square$

E7 Noetherian Completions

Fix an ideal I. The main result in this section is that if R is Noetherian, then so is its I-adic completion $\hat{R}$. Due to its implications for complex analysis, functions of several complex variables, complex algebraic geometry, and p-adic analysis, this is one of the major contributions of commutative algebra to mathematics as a whole.

The argument is based on a careful analysis of the derived ring $G_I(R) = \oplus_{n=0}^{\infty} I^n/I^{n+1}$. We have already seen (Lemma E4.2) that this ring is Noetherian whenever R is Noetherian. The following basic fact is an immediate consequence of Proposition E6.3(c).

Lemma E7.1. *If R is Noetherian, then $G_I(R)$ and $G_{\hat{I}}(\hat{R})$ are isomorphic as graded rings.*

The bulk of the section's work goes into proving the following.

Proposition E7.2. *Suppose that R is complete in the I-adic topology and M is an R-module with I-filtration (M_n) such that $\bigcap_n M_n = \{0\}$. (That is, the Krull topology of the filtration is Hausdorff.)*

(a) *If $G(M)$ is a finitely generated $G_I(R)$-module, then M is a finitely generated R-module.*

(b) *If $G(M)$ is a Noetherian $G_I(R)$-module, then M is a Noetherian R-module.*

We first explain how this implies the desired result.

Theorem E7.3. *If R is Noetherian, then $\hat{R}$ is Noetherian.*

Proof. By Lemma E4.2 $G_I(R)$ is Noetherian, so $G_{\hat{I}}(\hat{R})$ is Noetherian because it is isomorphic to $G_I(R)$. Since $\hat{R}$ is Hausdorff (Lemma E1.6) $\bigcap \hat{I}^n = 0$. The desired conclusion is obtained from Proposition E7.2(b) with $\hat{R}$ in place of R and M. $\square$

Proof of Proposition E7.2. Let $\xi_1, \ldots, \xi_r$ be a system of generators of $G(M)$. We can decompose these into their homogeneous components, so we may assume that each ξ_i is homogeneous, say of degree $n(i)$, and is consequently the image in $M_{n(i)}/M_{n(i)+1}$ of some $x_i \in M_{n(i)}$. Let $F = \oplus_{i=1}^r F^i$ where each F_i is R. Mapping the generator 1 of each F_i to x_i induces a homomorphism $\phi : F \to M$. There is a filtration (F_n) of F with $F_n = \oplus_{i=1}^r F_n^i$, where $F_n^i = R$ if $i \leq n(i)$ and $F_n^i = I^{n-n(i)}$ if $n > n(i)$. Since (M_n) is an I-filtration, ϕ is a filtered homomorphism. If $(0, \ldots, 1, \ldots, 0)$ has its 1 in the i^{th} component, then it is contained in $F_{n(i)}$, and $(0, \ldots, 1, \ldots, 0) + F_{n(i)+1}$ is mapped to ξ_i by $G(\phi)$. Thus $G(\phi)$ is surjective.

That the diagram

$$\begin{array}{ccc} F & \xrightarrow{\phi} & M \\ {\scriptstyle \iota_{F,I}} \downarrow & & \downarrow {\scriptstyle \iota_M} \\ \hat{F} & \xrightarrow{\hat{\phi}} & \hat{M} \end{array}$$

commutes is a straigthforward consequence of the definitions. The map $\iota_{F,I}$ is the r-fold cartesian product of $\iota_{R,I}$, which is surjective because R is complete. Thus $\iota_{F,I}$ is surjective. Since $G(\phi)$ is surjective, Lemma E3.1(b) implies that $\hat{\phi}$ is surjective. The assumption that $\bigcap_n M_n = \{0\}$ implies (Lemma E1.3) that ι_M is injective. We can now conclude that ϕ is a surjection, so $x_1, \ldots, x_r$ generate M as an R-module.

To prove (b) it suffices (Lemma A4.1) to show that a given submodule M' is finitely generated. We accomplish this by verifying that M' satisfies the hypotheses of part (a). Setting $M'_n = M' \cap M_n$ defines an I-filtration (M'_n) of M', and $\bigcap M'_n \subset \bigcap M_n = 0$. The inclusions $M'_n \subset M_n$ give rise to injections $M'_n/M'_{n+1} \to M_n/M_{n+1}$, and thus to an injection $G(M') \to G(M)$. Since $G(M)$ is Noetherian, $G(M')$ is finitely generated. $\square$

Chapter F

Initial Perspectives on Dimension

Once one advances beyond linear algebra, for many fields of mathematics dimension is a concept that is simultaneously fundamental and complex. For example, it is possible to understand the advances in topology related to the Brouwer fixed point theorem, invariance of domain, the Jordan curve theorem, and so on, as coming from a conceptual breakthrough that was ultimately a matter of finding the correct machinery for expressing dimension in the context of the concepts that eventually evolved into algebraic topology.

Algebraic geometry studies objects that are less general, and more structured, than those studied in topology or differential geometry. For this reason one should expect that the notions related to dimension appearing in related fields will still be relevant in algebraic geometry, and that entirely new perspectives may also become available. Relatedly, we should emphasize at the outset that the study of dimension in commutative algebra and algebraic geometry is not a matter of finding the one true and correct definition, then pursuing its properties and consequences. Instead, there are many definitions that express intutions concerning aspects of dimension, and the analytic substance of the theory is in large part a matter of showing that they relate to each other as expected in the settings of interest.

This chapter begins our study of dimension by laying out that portion of the theory that does not depend on “heavy” homological methods. This distinction is not entirely precise—some elementary aspects do creep into the discussion—but in the subsequent chapters the deeper aspects of homological algebra will be applied frequently.

F1 Two Basic Definitions

Dimension is a local property. That is, an affine variety has a dimension at each of its points, which is the dimension of arbitrarily small neighborhoods of the point. The dimension of the variety is then the maximum of these local dimensions.

As before, and throughout the remainder, we work with a fixed commutative ring with unit R. In principle the dimension of a variety at a point should be a property of the local ring of the variety at that point, and consequently many of the definitions and results in dimension theory concern local rings. We remind the reader of our convention concerning local rings: whenever R

is local, it is automatically the case that $\mathfrak{m}$ is its maximal ideal and $k = R/\mathfrak{m}$ is its residue field.

There is a basic point that will come up repeatedly. Suppose that R is local and Q is an ideal. If $\mathfrak{m}$ is minimal over Q, then $\operatorname{rad}(Q) = \mathfrak{m}$, because $\operatorname{rad}(Q)$ is the intersection of the ideals that are minimal over Q. In turn Proposition A12.5 implies that any ideal Q such that $\operatorname{rad}(Q) = \mathfrak{m}$ is $\mathfrak{m}$-primary, and of course if Q is $\mathfrak{m}$-primary, then $\operatorname{rad}(Q) = \mathfrak{m}$, so that $\mathfrak{m}$ is minimal over Q. In short, $\mathfrak{m}$ is a minimal prime over Q if and only if $\operatorname{rad}(Q) = \mathfrak{m}$ if and only if Q is $\mathfrak{m}$-primary.

For any R, the *Krull dimension* of R, denoted by $\dim R$, is the maximal length n of a chain of prime ideals $P_0 \supset \cdots \supset P_n$. Insofar as the algebraic variety associated with a prime ideal of $K[X_1, \ldots, X_n]$ is irreducible, this definition is based on the idea that passing to a proper subvariety of an irreducible variety reduces dimension by at least one, and that it is always possible to find a subvariety of codimension one. The *codimension* $\operatorname{codim} P$ of a prime ideal P is the supremum of the lengths of chains of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n = P$. In view of the bijection between prime ideals of R_P and prime ideals of R contained in P (Proposition A5.6) $\operatorname{codim} P = \dim R_P$. In particular, if R is local, then the codimension of $\mathfrak{m}$ is $\dim R$. The *codimension* of an arbitrary ideal I is the minimum of the codimensions of the primes containing I.

A different perspective on dimension comes from the intuition that the dimension of a space can be understood in terms of the number of "coordinates" required to specify a point. If the ideal $(x_1, \ldots, x_s)$ generated by $x_1, \ldots, x_s \in R$ is $\mathfrak{m}$-primary, then $x_1, \ldots, x_s$ is said to be a *system of parameters* for R. Let $\delta(R)$ be the minimal number of elements of such a system. We will now show that $\delta(R) \le \dim R$.

The induction step in the proof of our target result is perhaps best viewed in isolation.

Lemma F1.1. *Suppose R is Noetherian and local, and I is an ideal that is contained in a prime other than $\mathfrak{m}$. Then there is x such that the ideal J generated by I and x is proper, and the least codimension of a prime that is minimal over J is greater than the least codimension of a prime that is minimal over I.*

Proof. By Propositions A2.5 and A4.10 the set $\{P_1, \ldots, P_r\}$ of primes that are minimal over I is nonempty and finite. By hypothesis $\mathfrak{m}$ is not one of them, so prime avoidance gives an $x \in \mathfrak{m} \setminus \bigcup_{j=1}^r P_j$. The ideal J generated by I and x is proper because it is contained in $\mathfrak{m}$. Any prime that is minimal over J properly contains a prime P that is minimal over I, and its codimension is greater than the codimension of P. $\square$

Proposition F1.2. *If R is Noetherian and local, then $\delta(R) \le \dim R$.*

Proof. Applying the last result inductively, there are $x_1, \ldots, x_s$ such that $\mathfrak{m}$ is minimal over $(x_1, \ldots, x_s)$ and $s \leq \operatorname{codim} \mathfrak{m} = \dim R$. □

Actually $\delta(R) = \dim R$, but proving that $\delta(R) \geq \dim R$ will be much harder. There is a third number measuring the dimension of R, the degree $d(R)$ of the Hilbert polynomial, that will be defined and analyzed over the course of the next several sections, and we will see that $\delta(R) \geq d(R)$ and $d(R) \geq \dim R$. If we view $k[X_1, \ldots, X_r]$ as a graded ring, then the dimension of the vector space of homogeneous polynomials of degree n is a polyomial function of n of degree $r-1$, and we may think of such a polynomial as determining a hypersurface in $(r-1)$-dimensional projective space. The coordinate ring of a projective variety is $k[X_1, \ldots, X_r]/I$, where I is a homogeneous ideal. It is also a graded ring $S = \oplus_n S_n$, and each S_n is a vector space. One may suspect that the rate of growth of the dimension of S_n measures the dimension of the variety, and the theory of the Hilbert polynomial validates this guess. In this case, and much more generally, the dimension of S_n agrees with a polynomial function of n when n is large, and the degree of this polynomial is the measure of dimension, which becomes $d(R)$ in the context of a local ring R.

If the minimal number $\delta(R)$ of generators of an $\mathfrak{m}$-primary ideal is a good measure of the dimension of a local ring R with maximal ideal $\mathfrak{m}$, what about the minimal number of elements required to generate $\mathfrak{m}$ itself? A local ring R is *regular* if $\mathfrak{m}$ can be generated by $\dim R$ elements. This turns out to be an extremely important concept because regularity of the local ring of a point in a variety is the correct indicator, in the widest range of settings, of whether the point is nonsingular. (The terms "smooth," "simple," and "regular" are also used to describe such points.) Regular local rings will be one of our main concerns throughout the remainder. Later in this chapter we describe certain aspects of the theory that do not require heavy homological algebra.

The equation $x^2 + y^2 = z$ defines a 2-dimensional subset of $\mathbb{R}^3$, and when we add the additional equation $z = 0$ the dimension collapses to zero. Such perversities should not arise when working over an algebraically complete field: adding one more equation should reduce the dimension by at most one. Perhaps the main result fulfilling this intuition is the Krull principle ideal theorem, which asserts that if R is Noetherian, $x_1, \ldots, x_s \in R$, and P is minimal over $(x_1, \ldots, x_s)$, then $\operatorname{codim} P \leq s$. Geometrically, any of the irreducible components of the algebraic variety defined by s equations has codimension at most s. In this chapter's final section we present this result and a few of its simpler consequences.

F2 The Hilbert-Poincaré Series

The Hilbert-Poincaré series is a technical device which provides quite a bit of important information. Its main properties are its primary motivation, and before these have been established there is not much to say.

Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian graded ring with R_0 Artinian. Let $\mathcal{C}$ be a class of R_0-modules that contains every submodule of any of its elements, and every quotient of any of its elements by a submodule. An *additive function* for $\mathcal{C}$ is a function λ from $\mathcal{C}$ to the nonnegative integers that satisfies:

(a) $\lambda(M) = 0$ if and only if $M \cong 0$, and

(b) $\lambda(M') - \lambda(M) + \lambda(M'') = 0$ whenever $0 \to M' \to M \to M'' \to 0$ is a short exact sequence.

Note that setting $M'' = 0$ in (b) shows that $\lambda(M) = \lambda(M')$ whenever M and M' are isomorphic. That is, λ is really a function defined on a class of isomorphism classes of R_0-modules. Also, condition (b) extends to arbitrary finite exact sequences:

Lemma F2.1. *If $0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence of R_0-modules, then $\sum_{i=1}^n (-1)^i \lambda(M_i) = 0$.*

Proof. The cases $n = 1$ and $n = 3$ are (a) and (b), and $n = 2$ was just noted. Proceeding inductively, let I_{n-1} be the image of $M_{n-2} \to M_{n-1}$. Applying the induction hypothesis to $0 \to M_1 \to \cdots \to M_{n-2} \to I_{n-1} \to 0$ and (b) to $0 \to I_{n-1} \to M_{n-1} \to M_n \to 0$ gives

$$\sum_{i=1}^{n-2} (-1)^i \lambda(M_i) + (-1)^{n-1} \lambda(I_{n-1}) = 0 \text{ and } \lambda(I_{n-1}) - \lambda(M_{n-1}) + \lambda(M_n) = 0,$$

from which the result follows easily. □

Henceforth $\mathcal{C}$ will be the class of finitely generated R_0-modules. Since R_0 is Artinian, Corollary A14.9 implies that each element of $\mathcal{C}$ has finite length. Of course the length of an R_0-module M is zero if and only if $M = 0$, so Proposition A13.6 implies that the length is an additive function on $\mathcal{C}$.

From Theorem A14.7 R_0 is Noetherian, so Proposition E4.1 implies that R is generated as an R_0-algebra by finitely many generators, and is a Noetherian R_0-module. Each generator is a sum of homogeneous elements, and can be replaced by these, so the generators may be assumed to be homogeneous. Replacing any generators in R_0 by 1, we find that R is generated by $1 \in R_0$ and homogeneous generators $x_1, \ldots, x_s$ of positive degrees, say $k_1, \ldots, k_s$.

Let $M = \oplus_{n \geq 0} M_n$ be a finitely generated graded R-module. Fix a finite system of generators, which may be assumed to be homogeneous, as above. Each R_i is a submodule of the Noetherian R_0-module R, so it is finitely generated. For any specification of a finite system of generators for each R_i, we may think of an element of M_n as an R_0-linear combination of the products of the generators of M and the generators of the R_i for $i \leq n$, so

M_n is a finitely generated R_0-module. The *Hilbert-Poincaré series* of M (with respect to λ) is the formal power series

$$P_M(t) = \sum_{n=1}^{\infty} \lambda(M_n)t^n.$$

Theorem F2.2 (Hilbert, Serre). $P_M(t) = f(t)/\prod_{i=1}^{s}(1-t^{k_i})$ *for some polynomial* $f \in \mathbb{Z}[t]$.

Proof. We argue by induction on s. If $s = 0$, then $R = R_0$, and since M is finitely generated we have $M_n = 0$ for large n, so $P_M(t) \in \mathbb{Z}[t]$. Therefore we may assume the claim holds when R has $s-1$ generators.

Multiplication by x_s is an R_0-module homomorphism from M_n to M_{n+k_s}. Letting K_s and L_{n+k_s} be the kernel and cokernel of this map gives an exact sequence

$$0 \to K_n \to M_n \xrightarrow{x_s} M_{n+k_s} \to L_{n+k_s} \to 0,$$

and the last result gives

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0.$$

Let $K = \oplus_{n\geq 0} K_n$ and $L = M/x_sM$. Then $L = \oplus_{n\geq 0} L_n$ where $L_i = M_i$ for $i = 0, \dots, k_s - 1$. Since K is a submodule of M while L is a quotient of M, both are finitely generated R-modules, so $P_K(t)$ and $P_L(t)$ are well defined. Multiplying the equation above by t^{n+k_s}, summing over n, then adding the equation $\sum_{i=0}^{k_s-1}(\lambda(M_i) - \lambda(L_i))t^i = 0$, we find that

$$t^{k_s}P_K(t) + (1-t^{k_s})P_M(t) - P_L(t) = 0.$$

Since K and L are annihilated by x_s, any system of generators for one of these R-modules also generates it as a $R_0[x_1, \dots, x_{s-1}]$-module. Therefore both are finitely generated $R_0[x_1, \dots, x_{s-1}]$-modules, and the claim follows from the induction hypothesis. $\square$

We now extract some information from this construction. Let $d(M)$ be the order of the pole of $P_M(t)$ at $t = 1$. (Although $P_M(t)$ began life as a purely formal power series, and remained so in the result above and its proof if we regard $1/(1-t^{k_s})$ as a shorthand for $1 + t^{k_s} + t^{2k_s} + \cdots$, this result implies that it converges absolutely and uniformly on compact subdomains of $\mathbb{C} \setminus \{1\}$, so it is legitimate to treat it analytically.) Insofar as $1 - t^{k_i} = (1-t)(1 + \cdots + t^{k_i - 1})$, $d(M)$ is s less the order of 1 as a root of f. Therefore $d(M) \leq s$.

We would like to show that $d(M) \geq 0$ when $M \neq 0$. Indeed, all the $\lambda(M_n)$ are nonnegative, so from the defining formula for $P_M(t)$ we see that $P_M(1)$ is defined only when all but finitely many of them vanish, and $P_M(1) = 0$ occurs only when they are all zero, which is the case if and only if $M = 0$. Thus:

Lemma F2.3. *If $M \neq 0$, then $0 \leq d(M) \leq s$.*

In the case of greatest interest we can say more:

Proposition F2.4. *If $M \neq 0$ and $k_1 = \cdots = k_s = 1$, then there is a $g \in \mathbb{Q}[t]$ of degree $d(M) - 1$ such that $\lambda(M_n) = g(n)$ for sufficiently large n.*

Proof. The last result implies that $\lambda(M_n)$ is the coefficient of t^n in the power series expansion of $f(t)/(1-t)^s$. We have $f(t)/(1-t)^s = f_0(t)/(1-t)^d$ where $d = d(M)$ and $f_0 \in \mathbb{Z}[t]$ is not divisible by $1-t$. Let $f_0(t) = \sum_{k=0}^{N} a_k t^k$.

We are done if $d = 0$, and for $d > 0$ there is the formula

$$(1-t)^{-d} = (1 + t + t^2 + \cdots)^d = \sum_{h=0}^{\infty} \binom{d+h-1}{d-1} t^h. \qquad (*)$$

To see this observe that for each h

$$(i_1, \ldots, i_d) \leftrightarrow \{i_1 + 1, i_1 + i_2 + 2, \ldots, i_1 + \cdots + i_{d-1} + d - 1\}$$

is a bijection between the set of d-tuples of nonnegative integers that sum to h and the set of $(d-1)$-element subsets of $\{1, \ldots, h + d - 1\}$.

For $n \geq N$ we have

$$\lambda(M_n) = g(n) = \sum_{k=0}^{N} \alpha_k \binom{d+n-k-1}{d-1}.$$

Finally observe that each binomial coefficient

$$\binom{d+n-k-1}{d-1} = \frac{(d+n-k-1)(d+n-k-2)\cdots(n-k+1)}{(d-1)!}$$

is a polynomial of degree $d-1$ as a function of n. Therefore g is a polynomial function of n, and its degree is $d-1$ because its leading term is $(\sum_k \alpha_k) n^{d-1}/(d-1)!$. (Since 1 is not a root of f_0, $\sum_k \alpha_k \neq 0$.) □

The next result will support the induction step in one of the arguments later on. Its proof applies the techniques developed above in a slightly different direction.

Proposition F2.5. *If $M \neq 0$, $x \in R$ is homogeneous of degree $k \geq 1$, and x is a nonzerodivisor of M, then $d(M/xM) = d(M) - 1$.*

Proof. If $d(M) = 0$ then $M_n \neq 0$ for only finitely many n. For the largest such n we have $xM_n = 0$ because $k > 0$, and of course this is impossible. Therefore $d(M) > 0$.

We proceed as in the proof of Theorem F2.2: for each $n \geq 0$ there is an exact sequence

$$0 \to M_n \xrightarrow{x} M_{n+k} \to L_{n+k} \to 0.$$

(That is, $K_n = 0$ because x is a nonzerodivisor.) Continuing with the logic of that argument, we arrive at the equation

$$(1-t^k)P_M(t) = P_{M/xM}(t) + g(t)$$

where $g \in \mathbb{Z}[t]$. Since $1 - t^k = (1-t)(1 + \cdots + t^{k-1})$, the order of the pole of $P_M(t)$ at $t = 1$ is one more than the order of the pole of $P_{M/xM}(t)$. □

F3 The Hilbert Polynomial

Throughout this section R is Noetherian and I is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$. It will be important that for each n, R/I^n is Artinian (Lemmas A14.10 and A14.11). Of course it is also Noetherian.

In this section $\lambda(N)$ denotes the length of an R-module N of finite length. Let M be a finitely generated R-module, and let (M_n) be a stable I-filtration. We will study the lengths of the modules M/M_n.

First of all we must check that each M/M_n has finite length. Since M is finitely generated, so is M/M_n, and consequently (Proposition A4.6) M/M_n is Noetherian. Since $I^n \subset \operatorname{Ann}(M/M_n)$, M/M_n is in effect an R/I^n-module. Since M/M_n is finitely generated and R/I^n is Artinian, Proposition A4.6 implies that M/M_n is an Artinian R/I^n-module. Since M/M_n is both Noetherian and Artinian, Proposition A13.3 implies that it has finite length as an R/I^n-module, hence also as an R-module.

With finite length established, Lemma A13.5 implies that that the length of M/M_n is the sum of the lengths of the M_{i-1}/M_i.

Proposition F3.1. *Suppose R is Noetherian, I is $\mathfrak{m}$-primary for some maximal ideal $\mathfrak{m}$, M is a finitely generated R-module, and (M_n) is a stable I-filtration of M. Fix a system of generators $x_1, \dots, x_s$ for I. Then there is a polynomial $g \in \mathbb{Q}[t]$ of degree at most $s-1$, the so-called* Hilbert polynomial*, such that $\lambda(M_n/M_{n+1}) = g(n)$ for large n.*

Much of our earlier work was in preparation for the following argument.

Proof. Since (M_n) is an I-filtration, the length of each M_{i-1}/M_i as an R-module is the same as its length as an R/I-module (Lemma A13.4). We work with the graded ring $G(R) = \oplus_{n\geq 0} I^n/I^{n+1}$ and the graded $G(R)$-module $G(M) = \oplus_{n\geq 0} M_n/M_{n+1}$. Lemma E4.2 implies that $G(R)$ is Noetherian, and Lemma E5.7 implies that $G(M)$ is finitely generated. Let $\overline{x}_1, \dots, \overline{x}_s$ be the images of $x_1, \dots, x_s$ in I/I^2. Then

$$G(R) = (R/I)[\overline{x}_s, \dots, \overline{x}_s].$$

Since R/I is Artinian, Proposition F2.4 (applied to $G(M)$) implies the claim. □

For large n the length of M/M_n is a polynomial function whose degree is the degree of g plus one.

Proposition F3.2. *Under the hypotheses of the last result, $\lambda(M/M_n)$ is finite for all n, and for sufficiently large n, $\lambda(M/M_n)$ is given by a polynomial $f(n)$ of degree $\leq s$. The degree and leading coefficient of f depend only on M and I, and not on the particular filtration.*

Proof. For $g(t) = t^d$, and thus for any $g \in \mathbb{Z}[t]$ of degree d, $\sum_{m=0}^{n} g(m)$ is a polynomial function of degree $d+1$, so only the final assertion remains to be verified. Let $(\tilde{M}_n)$ be a second stable I-filtration of M, with $\text{length}(M/\tilde{M}_n) = \tilde{f}(n)$ for large n. Lemma E5.2 implies that (M_n) and $(\tilde{M}_n)$ have bounded difference, which is to say that there is an r such that $M_{n+r} \subset \tilde{M}_n$ and $\tilde{M}_{n+r} \subset M_n$ for all n. Therefore $f(n+r) \geq \tilde{f}(n)$ and $\tilde{f}(n+r) \geq f(n)$ for large n. Because f and $\tilde{f}$ are polynomials,

$$\lim_{n\to\infty} f(n)/f(n+r) = \lim_{n\to\infty} \tilde{f}(n)/\tilde{f}(n+r) = 1,$$

so it follows that $\lim_{n\to\infty} f(n)/\tilde{f}(n) = 1$. Therefore f and $\tilde{f}$ have the same degree and leading coefficient. □

Let χ_I^M be the polynomial given by the last result for the particular I-filtration $(I^n M)$. We are mainly interested in the case $M = R$, and we write χ_I in place of χ_I^R. This is the *characteristic polynomial* of the $\mathfrak{m}$-primary ideal I.

Proposition F3.3. *For any $\mathfrak{m}$-primary ideal I, χ_I has the same degree as $\chi_\mathfrak{m}$.*

Proof. Since $\mathfrak{m} \supset I \supset \mathfrak{m}^r$ for some r, $\chi_\mathfrak{m}(n) \leq \chi_I(n) \leq \chi_{\mathfrak{m}^r}(n) = \chi_\mathfrak{m}(rn)$ for large n, which is impossible if χ_I and $\chi_\mathfrak{m}$ have different degrees. □

Let $d(R)$ be the degree of $\chi_\mathfrak{m}$. Combining the results to this point, $d(R)$ is the order of the pole at $t = 1$ of the Hilbert function

$$P_R(t) = \sum_{n=0}^{\infty} \lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1})t^n$$

of $G_\mathfrak{m}(R) = \oplus_n G_{\mathfrak{m},n}(R)$ where $G_{\mathfrak{m},n}(R) = \mathfrak{m}^n/\mathfrak{m}^{n+1}$.

Partly to illustrate these ideas, but also because it is important in itself, we consider the particular case of $R = R_0[X_1, \ldots, X_m]$, where R_0 is an Artinian ring. Here $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is the free R_0-module generated by the monomials of degree n, of which there are $\binom{n+m-1}{n-1}$. (Recall the argument in the proof of Proposition F2.4.) Applying $(*)$ gives

$$P_R(t) = \sum_{n=1}^{\infty} \text{length}(R_0)\binom{n+m-1}{n-1}t^n = \text{length}(R_0)(1-t)^{-m}.$$

We conclude that:

Proposition F3.4. *If R_0 is Artinian, then $d(R_0[X_1,\dots,X_m]) = m$.*

F4 The Dimension of a Local Ring

We now assume that R is local. Our main goal in this section is to show that

$$d(R) = \delta(R) = \dim R.$$

From Proposition F1.2 we know that $\dim R \geq \delta(R)$. In addition, Propositions F3.2 and F3.3 imply that:

Proposition F4.1. *$\delta(R) \geq d(R)$.*

The next result provides the technical basis of the comparison of $d(R)$ and $\dim R$.

Proposition F4.2. *If M is a finitely generated R-module and x is a nonzerodivisor of M, then* $\deg\, \chi_I^{M/xM} \leq \deg\, \chi_I^M - 1$.

Proof. Let $N = xM$ and $M' = M/N$, and for each n let $N_n = N \cap I^n M$. The Artin-Rees lemma (Theorem E5.3) implies that (N_n) is a stable I-filtration of N. Therefore Proposition F3.2 gives a $\chi^N_{(N_n)} \in \mathbb{Q}[t]$ such that the length of N/N_n is $\chi^N_{(N_n)}(n)$ for sufficiently large n.

We claim that there is an exact sequence

$$0 \to N/N_n \to M/I^n M \to M'/I^n M' \to 0.$$

To see this first observe that $N/N_n = N/(N \cap I^n M) = (N + I^n M)/I^n M$ by Lemma A1.2. Now we compute that

$$\frac{M/I^n M}{N/N_n} = \frac{M/I^n M}{(N + I^n M)/I^n M} = \frac{M}{N + I^n M} = \frac{M/N}{(N + I^n M)/N} = \frac{M'}{I^n M'}.$$

(Here the second and third equalities are applications of Lemma A1.1.)

Applied to the exact sequence above, Lemma A13.6 yields

$$\chi^N_{(N_n)}(n) - \chi^M_I(n) + \chi^{M'}_I(n) = 0$$

for large n. As the image of an injective homomorphism with domain M, $N \cong M$. Therefore Proposition F3.2 implies that $\chi^N_{(N_n)}$ and χ^M_I have the same leading term, and the claim follows from this. ∎

Proposition F4.3. *$d(R) \geq \dim R$.*

Proof. We argue by induction on $d = d(R)$. If $d = 0$, then $\chi_{\mathfrak{m}}$ is a constant, which is equal to length$(R/\mathfrak{m}^n)$ for large n. Consequently $\mathfrak{m}^n/\mathfrak{m}^{n+1} = 0$ for large n. That is, $\mathfrak{m} \cdot \mathfrak{m}^n = \mathfrak{m}^n$, so Nakayama's lemma implies that $\mathfrak{m}^n = 0$. Therefore (0) is $\mathfrak{m}$-primary, so R is Artinian (Lemma A14.11) and consequently $\dim R = 0$ by Theorem A14.7. Thus we may suppose that $d > 0$, and that the result has been established for smaller d.

Let $P_0 \subset P_1 \subset \cdots \subset P_r$ be a chain of prime ideals in R. Let $R' = R/P_0$, choose $x \in P_1 \setminus P_0$, and let x' be the image of x in R'. Let $\mathfrak{m}'$ be the maximal ideal of R'. For each n the map $R \to R'$ induces a surjective homomorphism $R/\mathfrak{m}^n \to R'/\mathfrak{m}'^n$, so the length of $R'/\mathfrak{m}'^n$ is not greater than the length $R/\mathfrak{m}^n$. It follows that $d(R') \le d(R)$.

Since R' an integral domain, x is not a zerodivisor, and R' is local, so the last result gives $d(R'/(x')) \le d(R') - 1$. In particular, $d(R'/(x')) < d$, so the induction hypothesis implies that the $\dim R'/(x') \le d - 1$. However, the images of $P_1 \subset \cdots \subset P_r$ in $R'/(x')$ (which are distinct, because their images in R/P_1 are distinct) constitute a chain of length $r - 1$, so $r \le d$. Since this is true for any chain $P_0 \subset P_1 \subset \cdots \subset P_r$ in R, $\dim R \le d$. □

We summarize the main conclusions of the last several sections, which follow from Propositions F1.2, F4.1, and F4.3.

Theorem F4.4. *If R is Noetherian and local, then* $\dim R = d(R) = \delta(R)$, *where these integers are, respectively:*

(a) the maximum length of a chain of prime ideals in R;

(b) the degree of the characteristic polynomial $\chi_{\mathfrak{m}}$;

(c) the minimal number of elements of a system of parameters.

The analysis above has several easily derived consequences that are worth noting. Putting $M = R$ in Proposition F4.2 yields:

Corollary F4.5. *If x is a nonzerodivisor of R, then $d(R/(x)) \le d(R) - 1$.*

In its proof we took some care to avoid assuming the following consequence of Proposition F4.3.

Corollary F4.6. $\dim R < \infty$.

If R is any ring and P is a prime ideal, then R_P is local, of course, and if R is Noetherian, then so is R_P. Therefore:

Corollary F4.7. *If R is Noetherian and P is a prime, then* $\dim R_P < \infty$. *Consequently the primes in a Noetherian ring satisfy the descending chain condition.*

Corollary F4.8. *If R is Noetherian and local, and $\hat{R}$ is the $\mathfrak{m}$-adic completion of R, then* $\dim \hat{R} = \dim R$.

Proof. Proposition E6.3(c) implies that $\hat{\mathfrak{m}}^n/\hat{\mathfrak{m}}^{n+1} \cong \mathfrak{m}^n/\mathfrak{m}^{n+1}$ for each n. The length of $\hat{\mathfrak{m}}^n/\hat{\mathfrak{m}}^{n+1}$ as an $\hat{R}$-module is the same as its length as a $\hat{R}/\hat{\mathfrak{m}}$-module, the length of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ as an R-module is the same as its length as a $R/\mathfrak{m}$-module, and $\hat{R}/\hat{\mathfrak{m}} \cong k \cong R/\mathfrak{m}$ by Proposition E6.3. Therefore $\chi_{\hat{\mathfrak{m}}} = \chi_{\mathfrak{m}}$. □

Corollary F4.9. *If R is Noetherian and local, and I is an $\mathfrak{m}$-primary ideal, then* $\dim G_I(R) = \dim R$.

Proof. Lemma E4.2 implies that $G_I(R)$ is Noetherian, and

$$\mathfrak{m}_I = \mathfrak{m}/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

is its unique maximal ideal (it is an ideal, and all elements outside of it are units) so $G_I(R)$ is local. In view of Theorem F4.4 it suffices to prove that $\delta(G_I(R)) = \delta(R)$.

Fix $x_1, \ldots, x_s \in R$, and for each $i = 1, \ldots, s$ let $\tilde{x}_i$ be the image of x_i in I^{n_i}/I^{n_i+1}, where n_i is the largest integer such that $x_i \in I^{n_i}$. Then the ideal J generated by $x_1, \ldots, x_s$ is $\mathfrak{m}$-primary if and only if $I^n \subset J$ for all large n, which is true if and only if the ideal J_I generated by $\tilde{x}_1, \ldots, \tilde{x}_n$ contains I^n/I^{n+1} for all large n, which is the case if and only if J_I is $\mathfrak{m}_I$-primary. □

Corollary F4.10. *If R is Noetherian and local, then* $\dim R \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

Proof. If the images of $x_1, \ldots, x_s \in \mathfrak{m}$ in $\mathfrak{m}/\mathfrak{m}^2$ are a basis of this vector space over k, then $x_1, \ldots, x_s$ generate $\mathfrak{m}$, by Proposition A2.15, and consequently $s \geq \delta(R)$. □

Corollary F4.11. *If R is Noetherian, $x_1, \ldots, x_r \in R$, and P is a prime that is minimal over $(x_1, \ldots, x_r)$, then* $\operatorname{codim} P \leq r$.

Proof. The image of $(x_1, \ldots, x_r)$ in R_P is P_P-primary, so $r \geq \dim R_P = \operatorname{codim} P$. □

Corollary F4.12. *If R is Noetherian and local, and $x \in \mathfrak{m}$ is not a zerodivisor, then* $\dim R/(x) = \dim R - 1$.

Proof. Let $d = \dim R/(x)$. Combining Corollary F4.5 and the main result above gives $d \leq d(R) - 1 = \dim R - 1$. On the other hand, let $x_1, \ldots, x_d$ be elements of $\mathfrak{m}$ whose images in $R/(x)$ generate an $\mathfrak{m}/(x)$-primary ideal. Then $(x_1, \ldots, x_d, x)$ is an $\mathfrak{m}$-primary ideal, so $d + 1 \geq \dim R$. □

F5 Regular Local Rings

As we mentioned earlier, if R is Noetherian and local, it is *regular* if the minimal number of generators of $\mathfrak{m}$ is the minimal number $\delta(R)$ of elements of a system of parameters. A major event in the history of algebraic geometry was the recognition by Zariski (1947) that regularity of the local ring of an algebraic variety at a point is, in general, the correct measure of whether the point is "smooth" or "simple." Intuitively this means that a neighborhood of the point is the set of simultaneous solutions of a number of polynomial equations equal to the codimension of the variety at the point, with the derivatives of these equations being linearly independent at the point, as per the implicit function theorem. Since Zariski's paper regular local rings have been critically important in algebraic geometry and commutative algebra. They have many nice properties; here we will see that a regular local ring is an integral domain, and eventually we will establish the famous theorem of Nagata and Auslander-Buchsbaum asserting that a regular local ring is a UFD.

The main result below gives conditions that are equivalent to regularity. Its proof depends on a technical result, which in turn depends on the following fact.

Lemma F5.1. *If a polynomial $f \in R[X_1, \dots, X_n]$ is a zerodivisor, then it is annihilated by some $a \in R$.*

Proof. Using exponent vector notation, let $f = \sum_e a_e X^e$, and suppose that $fg = 0$ where $g = \sum_e b_e X^e$. Let the supports of f and g be $S_f = \{ e : a_e \neq 0 \}$ and $S_g = \{ e : b_e \neq 0 \}$. The claim follows if we can show that S_g has a single element, so we may assume that it has more than one, and that there is no g' with $fg' = 0$ that has a smaller support.

Let $S'_f = \{ e \in S_f : a_e g \neq 0 \}$. If $S'_f = \emptyset$, then $a_e b_{e'} = 0$ for all $e \in S_f$ and $e' \in S_g$, so that for any $e' \in S_g$, $b_{e'}$ annihilates f, as desired. Therefore we may assume that S'_f is nonempty. Let $f' = \sum_{e \in S'_f} a_e X^e$.

Let $p \in \mathbb{R}^n$ be a vector such that $\langle p, e \rangle \neq 0$ for all nonzero $e \in \mathbb{Z}^n$. (Unfortunately the obvious proof that such a p exists—observe that $\bigcup_{e \neq 0} \{ p : \langle p, e \rangle = 0 \}$ has Lesbesque measure zero—is out of line with the character of our material.) Let $e_f = \operatorname{argmax}_{e \in S'_f} \langle p, e \rangle$ and $e_g = \operatorname{argmax}_{e \in S_g} \langle p, e \rangle$. Then the coefficient of $X^{e_f + e_g}$ in $fg = f'g$ is $a_{e_f} b_{e_g}$, so this product is zero. Now $f \cdot a_{e_f} g = 0$, $a_{e_f} g \neq 0$, and the support of $a_{e_f} g$ is a proper subset of S_g. This contradiction completes the proof. $\square$

Proposition F5.2. *Suppose that R is Noetherian and local, $x_1, \dots, x_d$ is a system of parameters for R, where $d = \dim R$, and $I = (x_1, \dots, x_d)$. If f is a homogeneous polynomial of degree n in the variables $X_1, \dots, X_d$ with coefficients in R, and $f(x_1, \dots, x_d) \in I^{n+1}$, then the coefficients of f lie in $\mathfrak{m}$.*

Proof. There is an surjective homomorphism

$$\alpha : (R/I)[X_1, \ldots, X_d] \to G_I(R)$$

that takes each monomial $(r+I)X_1^{m_1} \cdots X_d^{m_d}$ to the image of $rx_1^{m_1} \cdots x_d^{m_d}$ in I^m/I^{m+1}, where $m = m_1 + \cdots + m_d$. Let $\overline{f}$ be the element of $(R/I)[X_1, \ldots, X_d]$ obtained by replacing each coefficient of f with its image in R/I. Then the hypothesis on f is that $\overline{f}$ is in the kernel of α.

Aiming at a contradiction, suppose that f has a coefficient that is not in $\mathfrak{m}$ and is consequently a unit. In view of Lemma F5.1, it follows that $\overline{f}$ is not a zerodivisor. Now we have

$$d(G_I(R)) \leq d\big((R/I)[X_1, \ldots, X_d]/(\overline{f})\big) = d\big((R/I)[X_1, \ldots, X_d]\big) - 1 = d - 1.$$

Here the inequality follows from the fact that $\overline{f}$ is in the kernel of the surjection α, the first equality is from Proposition F2.5, and the second equality is from Proposition F3.4, which can be applied because R/I is Artinian (Lemma A14.11). But Corollary F4.9 gives $d(G_I(R)) = d$, so we have the desired contradiction. □

Theorem F5.3. *If R is Noetherian and local, and $d(R) = d$, then the following are equivalent:*

(a) *R is regular;*

(b) *$G_\mathfrak{m}(R) \cong k[X_1, \ldots, X_d]$ as graded rings;*

(c) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$.

Proof. That (b) implies (c) follows from a comparison of the homogeneous components $\mathfrak{m}/\mathfrak{m}^2$ and $k_1[X_1, \ldots, X_d]$ of degree one. Nakayama's lemma (specifically, Proposition A2.15) implies that (a) follows from (c). To show that (a) implies (b) we suppose that $\mathfrak{m}$ is generated by $x_1, \ldots, x_d$ and consider the homomorphism $\alpha : k[X_1, \ldots, X_d] \to G_\mathfrak{m}(R)$ that takes each $\overline{f} \in k_n[X_1, \ldots, X_d]$ to the image of $\overline{f}(x_1, \ldots, x_d)$ in $\mathfrak{m}^n/\mathfrak{m}^{n+1}$. This is obviously surjective, and any nonzero element of its kernel could be lifted to a counterexample to the last result, so it is also injective. □

Corollary F5.4. *If R is a regular local ring, then it is an integral domain.*

Proof. Since it is isomorphic to some $k[X_1, \ldots, X_d]$, $G_\mathfrak{m}(R)$ is an integral domain, so Corollary E5.6 implies the claim. □

If R is a regular local ring, a minimal set of generators $x_1, \ldots, x_d$ for $\mathfrak{m}$ is called a *regular system of parameters.* A sequence of elements $x_1, \ldots, x_d$ in an arbitrary ring R is called an *R-sequence*, or a *regular sequence* of R, if $(x_1, \ldots, x_d)$ is a proper ideal and, for each $i = 1, \ldots, d$,

$$((x_1, \ldots, x_{i-1}) : x_i) = (x_1, \ldots, x_{i-1}).$$

That is, the image of x_i in $R/(x_1, \ldots, x_{i-1})$ is a nonzerodivisor.

Corollary F5.5. *If R is a regular local ring, a regular system of parameters $x_1, \dots, x_d$ is an R-sequence.*

Proof. Since R is an integral domain, $((0) : x_1) = (0)$. Now $R/(x_1)$ is a Noetherian local ring with maximal ideal $\mathfrak{m}/(x_1)$. Corollary F4.12 implies that $\dim R/(x_1) = \dim R - 1$, and since $\mathfrak{m}/(x_1)$ is generated by the images $\tilde{x}_2, \dots, \tilde{x}_d$ of $x_2, \dots, x_d$, $R/(x_1)$ is regular and $\tilde{x}_2, \dots, \tilde{x}_d$ is a regular system of parameters. By induction on d, $\tilde{x}_2, \dots, \tilde{x}_d$ is an $R/(x_1)$-sequence, so for each $i = 2, \dots, d$ we have $((\tilde{x}_2, \dots, \tilde{x}_{i-1}) : \tilde{x}_i) = (\tilde{x}_2, \dots, \tilde{x}_{i-1})$, which means precisely that $((x_1, \dots, x_{i-1}) : x_i) = (x_1, \dots, x_{i-1})$. □

F6 The Principal Ideal Theorem

This section's main result is due to Krull. It is in a sense a partner of Proposition F1.2, which asserts in effect that if R is Noetherian and local, and $c \le \dim R$, then there are $x_1, \dots, x_c$ such that any prime containing $(x_1, \dots, x_c)$ has codimension at least c.

Theorem F6.1 (Principal Ideal Theorem)**.** *If R is Noetherian, $x_1, \dots, x_c \in R$, and P is minimal among the primes that contain these elements, then* $\operatorname{codim} P \le c$.

Proof. The hypotheses are satisfied with $x_1/1, \dots, x_c/1 \in R_P$ and P_P in place of $x_1, \dots, x_c$ and P, and, in view of the bijection between prime ideals contained in P and prime ideals of P_P, the codimension of P is $\le c$ if and only if the codimension of P_P is $\le c$. Thus it suffices to prove the result for $x/1$ and P_P, which means that we may assume that R is a local ring, $\mathfrak{m} = P$ is its unique maximal ideal, and $x_1, \dots, x_c$ is a system of parameters.

The nilradical of $R/(x_1, \dots, x_c)$ is $\mathfrak{m}$ (Corollary A2.9) which is nilpotent because it is finitely generated (Lemma A14.3). Now let P be a prime that is maximal among those other than $\mathfrak{m}$; it will suffice to show that $\operatorname{codim} P \le c-1$. By hypothesis P does not contain some x_i, so we may suppose that $x_1 \notin P$. Let (P, x_1) be the smallest ideal containing P and x_1.

The image of $\mathfrak{m}$ is the only prime in $R/(P, x_1)$, so, as above, $\mathfrak{m}$ is the nilradical of this quotient. In particular, for each $i = 2, \dots, c$ there are $a_i \in R$, $y_i \in P$, and an integer k_i, such that $x_i^{k_i} = y_i + a_i x_1$. The image of $\mathfrak{m}$ is nilpotent in $R/(x_1, y_2, \dots, y_c)$, so Corollary F4.11 implies that the image of $\mathfrak{m}$ in $R/(y_2, \dots, y_c)$ has codimension is at most one. Therefore the image of P in $R/(y_2, \dots, y_c)$ is minimal, which is to say that P is minimal among primes containing $y_2, \dots, y_c$. By induction (with the case $c = 1$ given by Corollary F4.11) the codimension of P is $\le c - 1$, as desired. □

A somewhat stronger assumption delivers a stronger conclusion.

Corollary F6.2 (Krull's Principal Ideal Theorem)**.** *If R is Noetherian and x is an element that is neither a zerodivisor nor a unit, and P is a prime that is minimal over (x), then* $\operatorname{codim} P = 1$.

Proof. Corollary F4.11 gives $\operatorname{codim} P \leq 1$. If $\operatorname{codim} P = 0$, then P is minimal over (0), and consequently it is an associated prime of (0). But Proposition A12.6 would then imply that each element of P is a zerodivisor, contrary to assumption. □

What it means for $x_1, \dots, x_c$ to be an R-sequence is precisely that each x_i is neither a zero divisor nor a unit in $R/(x_1, \dots, x_{i-1})$. When this is the case, for any prime P_c that contains $(x_1, \dots, x_c)$, repeated application of this result gives a sequence of distinct prime ideals $P_0 \subset \cdots \subset P_c$ with $(x_1, \dots, x_{i-1}) \subset P_i$ for all i. Thus:

Proposition F6.3. *If $x_1, \dots, x_c$ is an R-sequence, then the codimension of $(x_1, \dots, x_n)$ is c.*

The principal ideal theorem has a useful converse.

Proposition F6.4. *Any prime P of codimension c is minimal over an ideal generated by c elements.*

Proof. We use induction, supposing that for some $0 \leq r < c$ we already have $x_1, \dots, x_r$ such that the codimension of any prime containing $(x_1, \dots, x_r)$ is at least r. Proposition A4.10 implies that there are finitely many primes Q that are minimal over $(x_1, \dots, x_r)$. For each such Q we have $\operatorname{codim} Q \geq r$, and the principal ideal theorem implies that $\operatorname{codim} Q \leq r$. Therefore P is not one of these minimal primes, so prime avoidance implies that P is not contained in their union. Choose $x_{r+1} \in P$ that is not in any of the minimal primes. If a prime Q is minimal over $(x_1, \dots, x_{r+1})$, then $\operatorname{codim} Q \geq r + 1$ because Q is not minimal over $(x_1, \dots, x_r)$, and the principal ideal theorem implies that $\operatorname{codim} Q \leq r + 1$. □

We now give two applications of Krull's principal ideal theorem.

Proposition F6.5. *If R is a Noetherian integral domain, then R is factorial if and only if every codimension* 1 *prime is principal.*

Proof. First suppose that R is factorial, and let P be a codimension 1 prime ideal. Any nonzero $a \in P$ is a product of primes, and at least one such prime, say p, is in P. Then (p) is a nonzero prime ideal contained in P, and the codimension of P is 1, so $P = (p)$.

Now suppose that every codimension 1 prime is principal. That R is factorial will follow from Proposition A7.5 if we show that any prime that is minimal over a principal ideal is itself principal. Since R is a domain, the unique prime minimal over (0) is (0) itself. Krull's principal ideal theorem implies that $\operatorname{codim} P = 1$ whenever P is minimal over $(a) \neq (0)$. □

The following is applied by Serre in no. 76.

Theorem F6.6. *If R is a normal Noetherian integral domain, then for every prime P associated to a principal ideal (a), P is principal and minimal over (a).*

Proof. In the first part of the proof we will work in R_P, which is of course a local integral domain with maximal ideal P_P. In addition it is Noetherian (Corollary A5.7) and Proposition A7.18 implies that it is normal.

Suppose that $P = ((a) : b)$ for some $b \notin (a)$. Then $P_P = ((\frac{a}{1}) : \frac{b}{1})$. (One containment is clear, and if $\frac{r}{s} \cdot \frac{b}{1} = \frac{t}{u} \cdot \frac{a}{1}$, then $(ru)b \in (a)$ and $u \notin P$, so $r \in P$ and $\frac{r}{s} \in P_P$.) Thus the hypotheses are satisfied with R_P, P_P, and $\frac{a}{1}$ in place of R, P, and a, and in addition R_P is local with maximal ideal P_P.

If the result has been proven with the additional hypotheses that R is local and P is its maximal ideal, then P_P is principal and minimal over $(\frac{a}{1})$. Because elements of R_P of the form $\frac{1}{s}$ for $s \notin P$ are units, $P_P = (\frac{r}{1})$ for some $r \in P$. Of course $a \in (r)$ because otherwise we would have $\frac{a}{1} \notin P_P$. To see that (r) is prime, observe that if $st \in (r)$ and $s, t \notin (r)$, then $\frac{s}{1} \cdot \frac{t}{1} \in (\frac{r}{1})$, but $\frac{s}{1}, \frac{t}{1} \notin (\frac{r}{1})$, contradicting the primality of P_P. If (r) was contained in a prime Q that was a proper subset of P, then $(\frac{a}{1}) \subset Q_P \subset P_P$ gives a violation of the minimality of P_P. Therefore $(r) = P$ and P is minimal over (a).

The upshot of this discussion is that we may assume that R is local and $\mathfrak{m} = P$ is its maximal ideal. Let K be the field of fractions of R, and let $\mathfrak{m}^{-1} = \{ a \in K : a\mathfrak{m} \subset R \}$. Let $\mathfrak{m}^{-1}\mathfrak{m}$ be the set of sums of products of an element of $\mathfrak{m}^{-1}$ with an element of $\mathfrak{m}$. This is a (not necessarily proper) ideal of R, and $\mathfrak{m} \subset \mathfrak{m}^{-1}\mathfrak{m} \subset R$, so, since $\mathfrak{m}$ is maximal, either $\mathfrak{m}^{-1}\mathfrak{m} = \mathfrak{m}$ or $\mathfrak{m}^{-1}\mathfrak{m} = R$.

If $\mathfrak{m}^{-1}\mathfrak{m} = \mathfrak{m}$, then Proposition A7.8 implies that the elements of $\mathfrak{m}^{-1}$ are integral over R, so $\mathfrak{m}^{-1} = R$ because R is normal. We have $\mathfrak{m}b \subset (a)$ and thus $b/a \in \mathfrak{m}^{-1} = R$, which is to say that $b \in (a)$, contradicting our assumption that the image of b in $R/(a)$ is nonzero.

Therefore $\mathfrak{m}^{-1}\mathfrak{m} = R$. For each $x \in \mathfrak{m}^{-1}$, $x\mathfrak{m}$ is a (not necessarily proper) ideal of R, and it cannot be the case that $x\mathfrak{m} \subset \mathfrak{m}$ for all such x, so $x\mathfrak{m} = R$ for some x. Thus $\mathfrak{m} = x^{-1}R$ is principal. Since x^{-1} is neither a zerodivisor (R is a domain) nor a unit ($x^{-1} \in \mathfrak{m}$) Krull's principal ideal theorem (Corollary F6.2) implies that the codimension of $\mathfrak{m}$ is one, so it is minimal over (a). □

Chapter G

The Koszul Complex

The Koszul complex is a rather large scale piece of machinery that can drastically simplify certain otherwise forbidding computations. Historically it was originated around 1950 by Koszul, who used it in the study of the cohomology of Lie groups, but its utility is much more general. It pervades the applications of homology to commutative algebra. This chapter presents the basic background material and some preliminary applications.

By way of introduction we now give a direct definition of the Koszul complex, even though the actual analysis will be based on a different definition that builds up the complex in a step-by-step manner. Let symbols $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be given. For each $k = 0, \ldots, n$ let $\wedge^k R$ be the free module with generators $\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k}$ where $1 \le i_1 < \ldots < i_k \le n$. (Here $\wedge^0 R$ is simply R.) For any $i_1, \ldots, i_k \in \{1, \ldots, n\}$, let $\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k} = 0 \in \wedge^k R$ if these indices are not distinct, and if they are distinct let

$$\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k} = (-1)^{\mathrm{sgn}(\tau)} \mathbf{e}_{i_{\tau(1)}} \wedge \ldots \wedge \mathbf{e}_{i_{\tau(k)}} \in \wedge^k R$$

where $\tau \in S_k$ is the permutation such that $i_{\tau(1)} < \ldots < i_{\tau(k)}$.

Let $x = (x_1, \ldots, x_n) \in R^n$ be given. The *Koszul complex* $K(x)$ is the cochain complex

$$0 \to K^0(x) \xrightarrow{d} K^1(x) \xrightarrow{d} \cdots \xrightarrow{d} K^{n-1}(x) \xrightarrow{d} K^n(x) \to 0$$

where $K^k(x) = \wedge^k R$ and $d : K^k(x) \to K^{k+1}(x)$ is the R-linear function that takes the generator $\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k}$ to

$$d(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k}) = \sum_{i=1}^{n} x_i \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k} \wedge \mathbf{e}_i.$$

(To see that $d^2 = 0$ one may write out the formula for $d^2(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k})$, then observe that the "diagonal" terms are zero while the "off diagonal" terms cancel in pairs.) Roughly, we will be interested in the cohomology of complexes $M \otimes_R K(x)$ where M is an R-module, especially when $x_1, \ldots, x_n$ is a regular sequence on M.

G1 Tensor Products of Cochain Complexes

The Koszul complex can be constructed by repeatedly taking tensor products of simple cochain complexes, and analysis of it often reduces to the elementary properties of this construction. If X and Y are the cochain complexes

$$\cdots \to X^{-1} \to X^0 \to X^1 \to \cdots \quad \text{and} \quad \cdots \to Y^{-1} \to Y^0 \to Y^1 \to \cdots$$

we define $X \otimes Y$ to be the cochain complex

$$\cdots \to \oplus_{i+j=k} X^i \otimes Y^j \to \oplus_{i+j=k+1} X^i \otimes Y^j \to \cdots$$

where the coboundary operator is given by

$$d(x_i \otimes y_j) \mapsto dx_i \otimes y_j + (-1)^i x_i \otimes dy_j.$$

The reader may easily check that the composition of this coboundary operator with itself is zero.

If $\eta : X \to X'$ and $\iota : Y \to Y'$ are cochain maps, there is a cochain map

$$\eta \otimes \iota : X \otimes Y \to X' \otimes Y'$$

given by $(\eta \otimes \iota)_k(x_i \otimes y_j) = \eta_i(x_i) \otimes \iota_j(y_j)$ when $i + j = k$. In this sense the tensor product is a functor from pairs of cochain complexes to cochain complexes.

We establish the basic properties of this tensor product.

Lemma G1.1. *The tensor product of complexes commutes up to natural isomorphism: the map $h : x_i \otimes y_j \mapsto (-1)^{ij} y_j \otimes x_i$ is a natural isomorphism between $X \otimes Y$ and $Y \otimes X$.*

Proof. This map commutes with the coboundary operator:

$$d(h(x_i \otimes y_j)) = (-1)^{ij}(dy_j \otimes x_i + (-1)^j y_j \otimes dx_i)$$

while

$$h(d(x_i \otimes y_j)) = (-1)^{(i+1)j} y_j \otimes dx_i + (-1)^{i(j+1)}(-1)^i dy_j \otimes x_i.$$

Naturality is evident without computation: if $(\eta, \iota) : (X, Y) \to (X', Y')$ is a map of pairs of complexes, then $h \circ (\eta \otimes \iota) = (\iota \otimes \eta) \circ h$. □

Lemma G1.2. *The tensor product of complexes is associative up to natural isomorphism: the map $(x_i \otimes y_j) \otimes z_k \mapsto x_i \otimes (y_j \otimes z_k)$ is a natural isomorphism between $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$.*

Proof. We compute the boundary operator in the two cases:

$$\begin{aligned} d((x_i \otimes y_j) \otimes z_k)) &= d(x_i \otimes y_j) \otimes z_k + (-1)^{i+j}(x_i \otimes y_j) \otimes dz_k \\ &= (dx_i \otimes y_j) \otimes z_k + (-1)^i (x_i \otimes dy_j) \otimes z_k + (-1)^{i+j}(x_i \otimes y_j) \otimes dz_k \end{aligned}$$

while

$$\begin{aligned} d(x_i \otimes (y_j \otimes z_k)) &= dx_i \otimes (y_j \otimes z_k) + (-1)^i x_i \otimes d(y_j \otimes z_k) \\ &= dx_i \otimes (y_j \otimes z_k) + (-1)^i x_i \otimes (dy_j \otimes z_k) + (-1)^{i+j} x_i \otimes (y_j \otimes dz_k). \end{aligned}$$

As in the last proof, naturality is obvious. □

Lemma G1.3. *If X is a cochain complex, then $X \otimes -$ and $- \otimes X$ are right exact functors.*

Proof. This follows from right exactness of the ordinary tensor product because, as an R-module, $X \otimes Y$ is the ordinary tensor product of the R-modules $\oplus_i X_i$ and $\oplus_j Y_j$. □

G2 The Koszul Complex in General

We develop the Koszul complex in two stages. This section studies a general construction, and in the next section we pass to a special case of this.

Fix an R-module N. Let $\wedge^0 N = R$, and for each $k = 1, 2, \ldots$ let $\wedge^k N$ be the set of finite sums of products of the form $y_1 \wedge \cdots \wedge y_k$, where $y_1, \ldots, y_k \in N$, modulo the relations

$$y_1 \wedge y_2 \wedge \cdots \wedge y_k + y_1' \wedge y_2 \wedge \cdots \wedge y_k = (y_1 + y_1') \wedge y_2 \wedge \cdots \wedge y_k$$

and

$$y_{\sigma(1)} \wedge \cdots \wedge y_{\sigma(m)} = \mathrm{sgn}(\sigma) y_1 \wedge \cdots \wedge y_k$$

for permutations σ of $1, \ldots, k$. This is an abelian group, and it becomes an R-module if we define scalar multiplication by setting

$$r(y_1 \wedge y_2 \wedge \cdots \wedge y_k) = (ry_1) \wedge y_2 \wedge \cdots \wedge y_k.$$

Let $\wedge N = \oplus_{k=0}^{\infty} \wedge^k N$. If

$$a = y_1 \wedge \cdots \wedge y_k \in \wedge^k N \quad \text{and} \quad b = z_1 \wedge \cdots \wedge z_\ell \in \wedge^\ell N,$$

let

$$a \wedge b = y_1 \wedge \cdots \wedge y_k \wedge z_1 \wedge \cdots \wedge z_\ell \in \wedge^{k+\ell} N.$$

This product is extended to arbitrary elements of $\wedge N$ by means of the distributive law. It makes $\wedge N$ into a graded R-algebra that is *skew commutative*: if $a \in \wedge^k N$ and $b \in \wedge^\ell N$, then

$$b \wedge a = (-1)^{k\ell} a \wedge b.$$

In particular, $x \wedge x = 0$ for all $x \in \wedge^1 N = N$. We call $\wedge N$ the *exterior algebra* derived from N.

If $\varphi : N \to N'$ is a homomorphism, there is a derived homomorphism $\wedge^k \varphi : \wedge^k N \to \wedge^k N'$ given by

$$\wedge^k \varphi(y_1 \wedge \cdots \wedge y_k) = \varphi(y_1) \wedge \cdots \wedge \varphi(y_k).$$

It is evident that we may regard $\wedge^k$ as a covariant functor from the category of R-modules to itself. Similarly, there is an R-module homomorphism $\wedge \varphi$:

$\wedge N \to \wedge N'$ whose restriction to each $\wedge^k N$ is $\wedge^k \varphi$. If $a \in \wedge^k N$, and $b \in \wedge^\ell N$, then

$$\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b).$$

Thus we may regard $\wedge$ as a covariant functor from the category of R-modules to the category of graded skew-commutative R-algebras.

In the construction of the Koszul complex there is a distinguished $x \in N$. For each $i = 0, 1, 2, \dots$ let $K_i^N(x) = \wedge^i N$. The *Koszul complex* $K^N(x)$ is the cochain complex

$$K^N(x): \quad \cdots \to 0 \to K_0^N(x) \to K_1^N(x) \to K_2^n(x) \to \cdots$$

where the coboundary map $K_0^N(x) \to K_1^N(x)$ is $r \mapsto rx$, and for $i \geq 1$ the coboundary map $\wedge^i N \to \wedge^{i+1} N$ is $a \mapsto x \wedge a$. If $\varphi : N \to N'$ is a homomorphism and $\varphi(x) = x'$, then the maps $\wedge^k \varphi$ evidently constitute a cochain homomorphism from $K^N(x)$ to $K^{N'}(x')$. This construction is functorial in the following sense.

Lemma G2.1. *The Koszul complex is a covariant functor from the category of pairs (N, x) where N is an R-module and $x \in N$ (with morphisms $\varphi : (N, x) \to (N', x')$ that are homomorphisms $\varphi : N \to N'$ with $\varphi(x) = x'$) to the category of cochain complexes.*

There is a nice relationship between Koszul complexes, direct sums, and tensor products.

Proposition G2.2. *If $N = N' \oplus N''$ and $x = (x', x'') \in N$, then*

$$K^N(x) \cong K^{N'}(x') \otimes K^{N''}(x'').$$

Proof. There is a homomorphism from $K^{N'}(x') \otimes K^{N''}(x'')$ to $K^N(x)$ that is defined by specifying that an element $y_1' \wedge \cdots \wedge y_i' \otimes y_1'' \wedge \cdots \wedge y_j''$ is mapped to

$$(y_1', 0) \wedge \cdots \wedge (y_i', 0) \wedge (0, y_1'') \wedge \cdots \wedge (0, y_j'').$$

This is well defined because every element of $K^{N'}(x') \otimes K^{N''}(x'')$ is a sum of elements of this form, and because it "respects" the relations that are required by the definition of an exterior algebra. It is surjective because every element of $K^N(x)$ is a sum of elements of this form. It is injective because the operations that might reduce a sum of elements of the indicated forms (that is, the relations in the definition of an exterior algebra) operate in the same way in the domain and the range.

In $K^N(x)$ the coboundary operator is

$$\begin{aligned} d\big((y_1', 0) &\wedge \cdots \wedge (y_i', 0) \wedge (0, y_1'') \wedge \cdots \wedge (0, y_j'')\big) \\ &= \big((x', 0) + (0, x'')\big) \wedge (y_1', 0) \wedge \cdots \wedge (y_i', 0) \wedge (0, y_1'') \wedge \cdots \wedge (0, y_j'') \\ &= (x', 0) \wedge (y_1', 0) \wedge \cdots \wedge (y_i', 0) \wedge (0, y_1'') \wedge \cdots \wedge (0, y_j'') \\ &\qquad + (-1)^i (y_1', 0) \wedge \cdots \wedge (y_i', 0) \wedge (0, x'') \wedge (0, y_1'') \wedge \cdots \wedge (0, y_j'') \end{aligned}$$

while in $K^{N'}(x') \otimes K^{N''}(x'')$ we have

$$\begin{aligned} d(y'_1 \wedge \cdots \wedge y'_i \otimes y''_1 \wedge \cdots \wedge y''_j) &= d(y'_1 \wedge \cdots \wedge y'_i) \otimes y''_1 \wedge \cdots \wedge y''_j \\ &\quad + (-1)^i y'_1 \wedge \cdots \wedge y'_i \otimes d(y''_1 \wedge \cdots \wedge y''_j) \\ &= x' \wedge y'_1 \wedge \cdots \wedge y'_i \otimes y''_1 \wedge \cdots \wedge y''_j \\ &\quad + (-1)^i y'_1 \wedge \cdots \wedge y'_i \otimes x'' \wedge y''_1 \wedge \cdots \wedge y''_j. \end{aligned}$$

Thus the coboundary operators commute with our isomorphism. □

G3 The Koszul Complex

We will be exclusively interested in the specialization of the complex studied in the last section that is obtained by letting N be the free R-module R^n. This is what is generally understood as *the* Koszul complex, although the term is also applied to the more general construction above.

Let R^n be the free module on the generators $\mathbf{e}_1, \dots, \mathbf{e}_n$. Then for $1 \le k \le n$ the R-module $\wedge^k R^n$ is the free R-module whose generators are those symbols $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ with $i_1 < \cdots < i_k$. Also, $\wedge^k R^n = 0$ if $k > n$. For the most part we will be working with a given

$$x = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \in \wedge^1 R^n,$$

and we usually write $K(x)$ or $K(x_1, \dots, x_n)$ in place of $K^{R^n}(x)$.

The Koszul complex when $n = 1$ is simply

$$K(y): \quad \cdots \to 0 \longrightarrow R \xrightarrow{y} R \longrightarrow 0 \to \cdots .$$

We may understand this as a fundamental building block because Proposition G2.2 and the commutativity and associativity of the tensor product give

$$K(x) \cong K(x_1) \otimes \cdots \otimes K(x_n) \cong K(x_{\sigma(1)}) \otimes \cdots \otimes K(x_{\sigma(n)})$$

for any permutation σ of $1, \dots, n$.

For an arbitrary cochain complex X and an integer ℓ, let $X[\ell]$ be X shifted ℓ steps to the left, so that $X[\ell]^i = X^{i+\ell}$, with the coboundary operator given by the appropriate shift of the coboundary operator of X multiplied by $(-1)^\ell$. We identify R with the complex $\cdots \to 0 \to R \to 0 \to \cdots$, where R is the module in position 0. Note that $X[\ell] = R[\ell] \otimes X$. There is a short exact sequence

$$0 \to R[-1] \to K(y) \to R \to 0$$

of complexes given by the diagram

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "1"] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & R \arrow[r, "y"] \arrow[d, "1"] & R \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & R \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots . \end{tikzcd}$$

We now study the cohomology of $K(y) \otimes X$ for an arbitrary complex X. We can tensor the short exact sequence above on the right by X, obtaining

$$0 \to X[-1] \to K(y) \otimes X \to X \to 0.$$

Here

$$(K(y) \otimes X)^i = (K^0(y) \otimes X^i) \oplus (K^1(y) \otimes X^{i-1}) = X^i \oplus X^{i-1},$$

so $X[-1] \to K(y) \otimes X$ is $x_{i-1} \mapsto (0, x_{i-1})$ and $K(y) \otimes X \to X$ is $(x_i, x_{i-1}) \mapsto x_i$. Evidently this sequence is exact. Applying the definition of the boundary operator of the tensor product gives

$$d(r_0 \otimes x_i) = yr_0 \otimes x_i + r_0 \oplus dx_i \quad \text{and} \quad d(r_1 \otimes x_{i-1}) = -r_1 \otimes dx_{i-1},$$

which boils down to $d(x_i, x_{i-1}) = (dx_i, yx_i - dx_{i-1})$. The associated long exact cohomology sequence is

$$\cdots \xrightarrow{y} H^i(X[-1]) \to H^i(K(y) \otimes X) \to H^i(X) \xrightarrow{y} H^{i+1}(X[-1]) \longrightarrow \cdots.$$

Here a diagram chase shows that the connecting homomorphism is multiplication by y: if $[x_i] \in H^i(X)$, then $(x_i, 0)$ is a preimage of x_i, and yx_i is the preimage of $d(x_i, 0) = (dx_i, yx_i) = (0, yx_i)$.

Proposition G3.1. *For a cochain complex X and $y \in R$, there is a long exact sequence*

$$\cdots \xrightarrow{y} H^{i-1}(X) \longrightarrow H^i(K(y) \otimes X) \longrightarrow H^i(X) \xrightarrow{y} H^i(X) \longrightarrow \cdots$$

where the indicated maps are multiplication by y. In addition y annihilates $H^i(K(y) \otimes X)$.

Proof. Of course $H^i(X[-1]) = H^{i-1}(X)$, so we obtain the sequence shown from the one above. If (x_i, x_{i-1}) is a cocycle, then $y(x_i, x_{i-1})$ is a coboundary because $0 = d(x_i, x_{i-1}) = (dx_i, yx_i - dx_{i-1})$, so that $dx_{i-1} = yx_i$, whence

$$d(x_{i-1}, 0) = (dx_{i-1}, yx_{i-1}) = y(x_i, x_{i-1}).$$

□

In view of the formula $K(x) \cong K(x_1) \otimes \cdots \otimes K(x_n)$ and the associativity and commutativity of the tensor product, this has the following consequence.

Corollary G3.2. *If $x = (x_1, \ldots, x_n)$, then the ideal $(x_1, \ldots, x_n)$ annihilates $H^*(K(x) \otimes X)$.*

G4 Regular Sequences and de Rham's Theorem

The notion of a regular sequence will be a central concept going forward. This section presents definitions, establishes elementary properties, and proves one of the early results applying the Koszul sequence due to de Rham (1954), which is cited by Serre.

Fix an R-module M and a sequence $x_1, \dots, x_n$ of points in R. Let $I_0 = (0)$, and for $i = 1, \dots, n$ let $I_i = (x_1, \dots, x_i)$. We say that $x_1, \dots, x_n$ is a *semiregular sequence* (on M) if, for each $i = 1, \dots, n$, x_i is not a zerodivisor of $M/I_{i-1}M$. (This terminology is not standard, but it is useful here because some of the results require only this hypothesis.) We say that $x_1, \dots, x_n$ is a *regular sequence on* M, or simply an M*-sequence*, if it is semiregular and $I_n M \neq M$. An important example is $R = M = K[X_1, \dots, X_n]$ where K is a field, with $x_1 = X_1, \dots, x_n = X_n$. Clearly $x_1, \dots, x_n$ is semiregular: if $f \in R$ and $X_i f$ is in the ideal generated by $X_1, \dots, X_{i-1}$, then so is f. In addition $M/I_n M \cong K$.

There is useful piece of related notation. If N is a submodule of M and I is an ideal, then

$$N{:}I = \{\, m \in M : Im \subset N \,\}.$$

Of course this is a submodule of M that contains N, and $N{:}\,J \subset N{:}\,I$ if $I \subset J$. Usually we will write $N{:}\,r$ rather than $N{:}\,(r)$. Then $x_1, \dots, x_n$ is an M-sequence if:

(a) $I_{i-1}M : x_i = I_{i-1}M$ for all $i = 1, \dots, n$;

(b) $I_n M \neq M$.

We are interested in the cohomology of the cochain complex $M \otimes K(x)$. In the present context the long exact sequence of Proposition G3.1 is:

Proposition G4.1. *If $x = (x', y)$ where $x' = (x_1, \dots, x_{n-1})$ and $y = x_n$, then there is a long exact sequence*

$$\cdots \xrightarrow{\;y\;} H^{i-1}(M \otimes K(x')) \longrightarrow H^i(M \otimes K(x)) \longrightarrow H^i(M \otimes K(x'))$$
$$\xrightarrow{\;y\;} H^i(M \otimes K(x')) \longrightarrow \cdots$$

where the indicated maps are multiplication by y, and $H^i(M \otimes K(x))$ is annihilated by y.

Lemma G4.2. $H^n(M \otimes K(x)) = M/I_n M$.

Proof. The final nonzero terms of the Koszul complex are in effect $\cdots \to R^n \xrightarrow{x\wedge} R \to 0$, and the image of $M \otimes R^n \xrightarrow{x\wedge} M$ is $I_n M$. $\square$

Most of the analytic substance of our work here is contained in the proof of the following result.

Proposition G4.3. *For $i = 0, \dots, n$, if $x_1, \dots, x_i$ is a semiregular sequence on M, then*

$$H^i(M \otimes K(x)) = (I_iM : I_n)/I_iM.$$

Proof. The initial terms of the Koszul complex are $0 \to R \xrightarrow{x\wedge} R^n \to \cdots$, and the kernel of $M \xrightarrow{x\wedge} M \otimes R^n = M^n$ is precisely the set of m that are annihilated by all x_i, so

$$H^0(M \otimes K(x)) = (0)M : I_n = ((0)M : I_n)/(0)M.$$

Thus the claim holds when $i = 0$. We may assume that $i > 0$, and that the claim holds for all smaller i. For the given i we argue by induction on n. Since $I_nM : I_n = M$, the result above is the claim when $n = i$, so we may assume that $n > i$, and that the claim holds with n replaced by $n - 1$.

As above let $x' = (x_1, \dots, x_{n-1})$. Since $(x_i) \subset I_{n-1}$ and x_i is a nonzerodivisor of $M/I_{i-1}M$, the induction hypothesis for i gives

$$H^{i-1}(M \otimes K(x')) = (I_{i-1}M : I_{n-1})/I_{i-1}M \subset (I_{i-1}M : x_i)/I_{i-1}M = 0.$$

The induction hypothesis on n implies that

$$H^i(M \otimes K(x')) = (I_iM : I_{n-1})/I_iM.$$

Therefore the long exact sequence of the last result becomes

$$0 \to H^i(M \otimes K(x)) \to (I_iM : I_{n-1})/I_iM \xrightarrow{x_n} (I_iM : I_{n-1})/I_iM \to \cdots.$$

To finish up note that $I_nm \subset I_iM$ if and only if $I_{n-1}m \subset I_iM$ and $x_nm \in I_iM$, so $(I_iM : I_n)/I_iM$ is the kernel of this multiplication by x_n. □

To be precise, the specific result established by de Rham consists of the first assertion and the case $j = n$ of the second assertion of the following result.

Theorem G4.4 (de Rham)**.** *$H^n(M \otimes K(x)) = M/I_nM$. For all $i = 0, \dots, n$, if $x_1, \dots, x_i$ is a semiregular sequence on M, then $H^j(M \otimes K(x)) = 0$ for $0 \le j < i$.*

Proof. The first assertion is Lemma G4.2 above. For the second we apply the last result, observing that $I_jM : I_n = I_jM$ because x_{j+1} is not a zerodivisor of M/I_jM. □

If J is an ideal, an M-sequence $y_1, \dots, y_r$ is *maximal in J* if $y_1, \dots, y_r \in J$ and there is no $y \in J$ such that $y_1, \dots, y_r, y$ is an M-sequence. In the next section we will consider situations in which $i < n$ and $(x_1, \dots, x_i)$ is a maximal M-sequence in I_n.

Proposition G4.5. *If R is Noetherian, M is finitely generated, and $x_1, \dots, x_i$ is a maximal M-sequence in I_n, then $H^i(M \otimes K(x)) \neq 0$.*

Proof. In view of Proposition G4.3 we need to show that $(I_iM : I_n)/I_iM \neq 0$, which amounts to $I_iM : I_n$ being a proper superset of I_iM. If $I_i = I_n$, then $I_iM : I_n = M$, which is a proper superset of I_iM because $x_1, \dots, x_i$ is an M-sequence. Therefore suppose that I_i is a proper subset of I_n. Since $x_1, \dots, x_i$ is an M-sequence, $I_iM \neq M$, and since it is maximal in I_n, every element of $I_n \setminus I_i$ is a zerodivisor of M/I_iM. The set of zerodivisors of M/I_iM is the union of the ideals $\mathrm{Ann}(\tilde{m})$ for $0 \neq \tilde{m} \in M/I_iM$. Since R is Noetherian, each such ideal is contained in a maximal such ideal. (This need not be a maximal ideal in the usual sense.) Lemma A10.2 implies that each such maximal ideal is an associated prime of M/I_nM. From Theorem A10.16 there are finitely many associated primes, and since I_n is contained in their union, prime avoidance (Lemma A10.10) implies that I_n is contained in one of them, say Q. As an associated prime, Q is the annihilator of some nonzero $\tilde{m} = m + I_iM \in M/I_iM$, so that $0 \neq m \in (I_iM : I_n) \setminus I_iM$. □

We come now to a key result.

Theorem G4.6. *Suppose R is Noetherian, M is finitely generated, $H^j(M \otimes K(x)) = 0$ for all $j = 0, \dots, r-1$, and $H^r(M \otimes K(x)) \neq 0$. Then all maximal M-sequences drawn from I_n have length r.*

An immediate consequence, of great interest, is that all maximal M-sequences drawn from I_n have the same length. In addition, if $I_nM \neq M$, then de Rham's theorem gives $H^n(M \otimes K(x)) = M/I_nM \neq 0$, so there is necessarily some r satisfying the hypotheses. That is, $H^*(M \otimes K(x))$ always detects the maximal length of M-sequences in I_n.

Two preliminary results prepare the proof of Theorem G4.6.

Proposition G4.7. *If $y_1, \dots, y_r \in I_n$, then*

$$K(x_1, \dots, x_n, y_1, \dots, y_r) \cong K(x) \otimes K(0, \dots, 0).$$

Proof. Suppose that $y_i = \sum_j a_{ij}x_j$; let A be the $r \times n$ matrix with entries a_{ij}. Let $N = R^n \oplus R^r$, and let $\alpha : N \to N$ be the homomorphism with matrix $\begin{pmatrix} I & 0 \\ -A & I \end{pmatrix}$. This is an isomorphism because its inverse is the homomorphism with matrix $\begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$. Since

$$\alpha(x_1, \dots, x_n, y_1, \dots, y_r) = (x_1, \dots, x_n, 0, \dots, 0)$$

the functorial nature of the Koszul complex (Lemma G2.1) implies that

$$K(x_1, \dots, x_n, y_1, \dots, y_r) \cong K(x_1, \dots, x_n, 0, \dots, 0),$$

after which the claim follows from Proposition G2.2. □

The Koszul complex of $(0,\dots,0) \in R^r$ is $0 \to R \to R^r \to \wedge^2 R^r \to \cdots$ with zero coboundary operator. For any cochain complex and integer n there is the computation $H^*(X \otimes_R R^n) = H^*(X^n) = (H^*(X))^n = H^*(X) \otimes_R R^n$, so:

Proposition G4.8. *If $y_1,\dots,y_r \in I_n$, then*

$$H^*(M \otimes K(x_1,\dots,x_n,y_1,\dots,y_r)) = H^*(M \otimes K(x)) \otimes \wedge R^r.$$

In particular,

$$H^i(M \otimes K(x_1,\dots,x_n,y_1,\dots,y_r)) = 0$$

if and only if $H^k(M \otimes K(x_1,\dots,x_n)) = 0$ for all k such that $i-r \le k \le i$.

Proof of Theorem G4.6. Let $y_1,\dots,y_r$ be a maximal M-sequence in I_n, and let s be the smallest integer such that $H^s(M \otimes K(x)) \neq 0$. In view of the last result,

$$H^i(M \otimes K(x)) = 0 \quad (i = 0,\dots,s-1)$$

if and only if

$$H^i(M \otimes K(x_1,\dots,x_n,y_1,\dots,y_r)) = 0 \quad (i = 0,\dots,s-1).$$

Therefore s is the smallest integer such that

$$H^s(M \otimes K(x_1,\dots,x_n,y_1,\dots,y_r)) \neq 0.$$

Proposition G2.2 and the commutativity of the tensor product imply that

$$K(x_1,\dots,x_n,y_1,\dots,y_r) \cong K(y_1,\dots,y_r,x_1,\dots,x_n).$$

Now Theorem G4.4 and Proposition G4.5 imply that $r = s$. □

G5 Regular Sequences in Local Rings

As in the last section we fix an R-module M and a sequence $x_1,\dots,x_n$, setting $I_0 = (0)$ and $I_i = (x_1,\dots,x_i)$ for $1 \le i \le n$. The most important applications of the results above are to local rings, for which stronger results hold.

Proposition G5.1. *Suppose that R is Noetherian and local, $x_1,\dots,x_n \in \mathfrak{m}$, and M is finitely generated. Let $x = (x_1,\dots,x_n)$ and $x' = (x_1,\dots,x_{n-1})$. If $H^k(M \otimes K(x)) = 0$ for some $k \le n$, then*

$$H^j(M \otimes K(x)) = 0 \quad \textit{and} \quad H^j(M \otimes K(x')) = 0 \quad (j = 0,\dots,k-1).$$

Proof. We argue by induction on n; the case $n = 0$ is trivial, so we may suppose that $n > 0$. Since $H^k(M \otimes K(x)) = 0$ the long exact sequence of Proposition G4.1 contains

$$\cdots \to H^{k-1}(M \otimes K(x')) \xrightarrow{x_n} H^{k-1}(M \otimes K(x')) \to 0.$$

Since $x_n \in \mathfrak{m}$ this surjectivity implies that

$$\mathfrak{m}H^{k-1}(M \otimes K(x')) = H^{k-1}(M \otimes K(x')).$$

Since M is finitely generated, $H^{k-1}(M \otimes K(x'))$ is finitely generated, so Nakayama's lemma gives $H^{k-1}(M \otimes K(x')) = 0$. The induction hypothesis implies that $H^j(M \otimes K(x')) = 0$ for all $j \le k-1$. For $j < k$ we now have $H^j(M \otimes K(x)) = 0$ because in the long exact sequence of Proposition G4.1 the terms on either side of it vanish. □

Theorem G5.2. *Suppose that R is Noetherian and local, $x_1, \ldots, x_n \in \mathfrak{m}$, and $M \neq 0$ is finitely generated. If $H^{n-1}(M \otimes K(x_1, \ldots, x_n)) = 0$, then $x_1, \ldots, x_n$ is an M-sequence.*

Proof. First of all, $I_n M$ is a proper submodule because $I_n M = M$ would imply $\mathfrak{m}M = M$, after which Nakayama's lemma would give $M = 0$, contrary to hypothesis.

Let $x = (x_1, \ldots, x_n)$ and $x' = (x_1, \ldots, x_{n-1})$. We argue by induction on n, beginning with $n = 1$. In this case $M \otimes K(x_1)$ is $0 \to M \xrightarrow{x_1} M \to 0$, and $H^0(M \otimes K(x_1)) = 0$ means precisely that x_1 is a not a zerodivisor of M. Suppose that $n > 1$. Above we showed that $H^{n-2}(M \otimes K(x')) = 0$, so the induction hypothesis implies that $x_1, \ldots, x_{n-1}$ is an M-sequence. Proposition G4.3 gives

$$0 = H^{n-1}(M \otimes K(x)) = (I_{n-1}M : I_n)/I_{n-1}M = (I_{n-1}M : x_n)/I_{n-1}M,$$

i.e., x_n is not a zerodivisor of $M/I_{n-1}M$. □

Corollary G5.3. *If R is Noetherian and local, $M \neq 0$ is finitely generated, and I_n is proper and contains an M-sequence of length n, then $x_1, \ldots, x_n$ is itself an M-sequence.*

Proof. In view of the last result it suffices to show that $H^{n-1}(M \otimes K(x)) = 0$. If this was not the case Theorem G4.6 would imply that the length of all maximal M-sequences was less than n, contrary to hypothesis. □

Proposition G5.4. *If R is Noetherian, M is finitely generated, $x_1, \ldots, x_n$ is an M-sequence, and $t_1, \ldots, t_n$ are positive integers, then $x_1^{t_1}, \ldots, x_n^{t_n}$ is an M-sequence.*

Proof. We first establish the result when R is local and $x_1, \dots, x_n \in \mathfrak{m}$. Since a power of a nonzerodivisor is a nonzerodivisor, $x_1, \dots, x_{n-1}, x_n^{t_n}$ is a semiregular sequence. In addition $(x_1, \dots, x_{n-1}, x_n^{t_n})M \subset I_nM \neq M$, so $x_1, \dots, x_{n-1}, x_n^{t_n}$ is an M-sequence. The ideal generated by $x_n^{t_n}, x_1, \dots, x_{n-1}$ contains an M-sequence of length n, so the last result implies that it is an M-sequence. Similarly, $x_{n-1}^{t_{n-1}}, x_n^{t_n}, x_1, \dots, x_{n-2}$ is an M-sequence, and so forth.

We now take up the general case. Of course $(x_1^{t_1}, \dots, x_n^{t_n})M \subset I_nM \neq M$, so it suffices to show that $x_1^{t_1}, \dots, x_n^{t_n}$ is a semiregular sequence. We argue by induction on n. When $n = 1$ the claim follows from the fact that a power of a nonzerodivisor is a nonzerodivisor. Therefore we may suppose that $n > 1$, and that $x_1^{t_1}, \dots, x_{n-1}^{t_{n-1}}$ is a semiregular sequence. Let

$$\varphi : M/(x_1^{t_1}, \dots, x_{n-1}^{t_{n-1}})M \to M/(x_1^{t_1}, \dots, x_{n-1}^{t_{n-1}})M$$

be multiplication by x_n. Since a power of a nonzerodivisor is a nonzerodivisor, it suffices to show that $\operatorname{Ker} \varphi = 0$.

In view of Lemma A5.9, showing that $\operatorname{Ker} \varphi_{\mathfrak{m}} = 0$ for a given maximal ideal $\mathfrak{m}$ is enough. For $i = 1, \dots, n$ let $\tilde{x}_i = x_i/1 \in R_{\mathfrak{m}}$. By definition

$$\varphi_{\mathfrak{m}} : (M/(x_1^{t_1}, \dots, x_{n-1}^{t_{n-1}})M)_m \to (M/(x_1^{t_1}, \dots, x_{n-1}^{t_{n-1}})M)_m$$

is multiplication by $\tilde{x}_n$. Corollary A5.2 implies that

$$(M/(x_1^{t_1}, \dots, x_{n-1}^{t_{n-1}})M)_m \cong M_{\mathfrak{m}}/((x_1^{t_1}, \dots, x_{n-1}^{t_{n-1}})M)_m,$$

and $((x_1^{t_1}, \dots, x_{n-1}^{t_{n-1}})M)_m = (\tilde{x}_1^{t_1}, \dots, \tilde{x}_{n-1}^{t_{n-1}})M_m$, so our goal is to show that $\tilde{x}_n$ is a nonzerodivisor of $M_{\mathfrak{m}}/(\tilde{x}_1^{t_1}, \dots, \tilde{x}_{n-1}^{t_{n-1}})M_m$.

We may assume that $M_{\mathfrak{m}}/(\tilde{x}_1^{t_1}, \dots, \tilde{x}_{n-1}^{t_{n-1}})M_{\mathfrak{m}} \neq 0$ (otherwise $\tilde{x}_n$ is automatically a nonzerodivisor) so $x_1, \dots, x_{n-1} \in \mathfrak{m}$. Also, we may assume that $x_n \in \mathfrak{m}$ because otherwise $\tilde{x}_n$ is a unit, and thus a nonzerodivisor. Therefore $\tilde{x}_1, \dots, \tilde{x}_n \in \mathfrak{m}_{\mathfrak{m}}$. We may assume that $(\tilde{x}_1, \dots, \tilde{x}_n)M_{\mathfrak{m}} \neq M_{\mathfrak{m}}$ because otherwise $\mathfrak{m}_{\mathfrak{m}}M_{\mathfrak{m}} = M_{\mathfrak{m}}$, when Nakayama's lemma would give $M_{\mathfrak{m}} = 0$, in which case $\tilde{x}_n$ is automatically a nonzerodivisor. For each $i = 1, \dots, n$, x_i is a nonzerodivisor of $M/I_{i-1}M$, so (Lemma A5.9) $\tilde{x}_i$ is a nonzerodivisor of $M_{\mathfrak{m}}/(\tilde{x}_1, \dots, \tilde{x}_{i-1})M_{\mathfrak{m}}$. Thus $\tilde{x}_1, \dots, \tilde{x}_n$ is an $M_{\mathfrak{m}}$-sequence, and the hypotheses hold with R, M, and $x_1, \dots, x_n$ replaced by $R_{\mathfrak{m}}$, $M_{\mathfrak{m}}$, and $\tilde{x}_1, \dots, \tilde{x}_n$. The special case established at the beginning now implies that $x_1^{t_1}/1, \dots, x_n^{t_n}/1$ is an $M_{\mathfrak{m}}$-sequence. In particular, $x_n^{t_n}/1$ is a nonzerodivisor of $M_{\mathfrak{m}}/(x_1^{t_1}, \dots, x_{n-1}^{t_{n-1}})M_{\mathfrak{m}}$, so $\tilde{x}_n$ is also a nonzerodivisor, as desired. $\square$

G6 A Variant of the Koszul Complex

This section develops one of the results cited by Serre. In addition to presenting a relatively sophisticated application of the Koszul complex, it has another point of interest, namely that the Koszul complex can appear in a different guise.

As was the case earlier, we are given $x = (x_1, \ldots, x_n) \in R^n$. For $0 \le k \le n$ let $\tilde{K}_k(x) = \wedge^k R^n$, and define $\delta_k : \tilde{K}_k(x) \to \tilde{K}_{k-1}(x)$ by R-linearly extending the formula

$$\delta_k(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}) = \sum_{h=1}^{k} (-1)^{h+1} x_{i_h} \mathbf{e}_{i_1} \wedge \cdots \wedge \hat{\mathbf{e}}_{i_h} \wedge \cdots \wedge \mathbf{e}_{i_k}.$$

It is straightforward to verify that $\delta_{k-1} \circ \delta_k = 0$. In the literature one will sometimes see

$$\tilde{K}(x) : 0 \to \tilde{K}_n(x) \to \cdots \to \tilde{K}_0(x) \to 0$$

described as the Koszul complex. We first show that this is not really a different concept because $K(x)$ and $\tilde{K}(x)$ are effectively isomorphic.

Let $\langle \cdot, \cdot \rangle$ be the inner product on $\wedge^k R^n$ that has the $\mathbf{e}_{1_1} \wedge \ldots \wedge \mathbf{e}_{i_k}$ as an orthonormal basis. The *Hodge star operator* is the function $* : \wedge^k R^n \to \wedge^{n-k} R^n$ defined by requiring that

$$\lambda \wedge \mu = \langle *\lambda, \mu \rangle \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$$

for all $\lambda \in \wedge^k R^n$ and all $\mu \in \wedge^{n-k} R^n$. Let $\sigma \in S_n$ be the permutation $j \mapsto i_j$. If we set $\lambda = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ and $\mu = \mathbf{e}_{i_{k+1}} \wedge \cdots \wedge \mathbf{e}_{i_n}$, then clearly $*\lambda$ is some scalar multiple of μ, and the formula above gives $*\lambda = \operatorname{sgn}(\sigma)\mu$. That is,

$$*\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} = \operatorname{sgn}(\sigma) \mathbf{e}_{i_{k+1}} \wedge \cdots \wedge \mathbf{e}_{i_n}.$$

Combining this with the definition of the differential gives:

$$\begin{aligned}
*\big(d_k(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k})\big) &= *\big((x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \wedge \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}\big) \\
&= *\Big((-1)^k \sum_{j=k+1}^{n} x_{i_j} \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \wedge \mathbf{e}_{i_j}\Big) \\
&= (-1)^k \sum_{j=k+1}^{n} \operatorname{sgn}(\sigma)(-1)^{j-k+1} x_j \mathbf{e}_{i_{k+1}} \wedge \cdots \wedge \hat{\mathbf{e}}_{i_j} \wedge \cdots \wedge \mathbf{e}_{i_n}
\end{aligned}$$

because $\operatorname{sgn}(\sigma)(-1)^{j-k+1}$ is the sign of the permutation

$$1, \ldots, n \mapsto i_1, \ldots, i_k, i_j, i_{k+1}, \ldots, \hat{i}_j, \ldots, i_n.$$

On the other hand

$$\begin{aligned}
\delta_{n-k}\big(*(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k})\big) &= \delta_{n-k}\big(\operatorname{sgn}(\sigma)\mathbf{e}_{i_{k+1}} \wedge \cdots \wedge \mathbf{e}_{i_n}\big) \\
&= \operatorname{sgn}(\sigma) \sum_{j=k+1}^{n} (-1)^{j-k+1} x_j \mathbf{e}_{i_{k+1}} \wedge \cdots \wedge \hat{\mathbf{e}}_{i_j} \wedge \cdots \wedge \mathbf{e}_{i_n}.
\end{aligned}$$

Thus $\delta_{n-k}* = (-1)^k * d_k$.

We can say a bit more. For any permuation σ there is a function $\tau_\sigma^k : \wedge^k R^n \to \wedge^k R^n$ that is the R-linear extension of the formula

$$\tau_\sigma^k(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}) = \mathbf{e}_{\sigma(i_1)} \wedge \cdots \wedge \mathbf{e}_{\sigma(i_k)}.$$

The formula above becomes

$$*\tau_\sigma^k(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k) = \operatorname{sgn}(\sigma)\tau_\sigma^{n-k}(\mathbf{e}_{k+1} \wedge \cdots \wedge \mathbf{e}_n) = \operatorname{sgn}(\sigma)\tau_{\sigma\sigma_k}^{n-k}(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{n-k}),$$

where σ_k is addition of k mod n, i.e., $j \mapsto j + k$ or $j \mapsto j + k - n$ according to whether $j \le n - k$. We have $\operatorname{sgn}(\sigma\sigma_k) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma_k)$ and $\operatorname{sgn}(\sigma_k) = (-1)^{k(n-k)}$, so applying this formula again with k and $n - k$ swapped gives

$$**\tau_\sigma^k(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k) = (-1)^{k(n-k)}\tau_\sigma^k(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k).$$

This holds for every σ, so $** = (-1)^{k(n-k)}$. Composing the formula from the last paragraph with $*$, both on the left and on the right, gives

$$d_k = (-1)^{k(n-k-1)} * \delta_{n-k} * \quad \text{and} \quad \delta_{n-k} = (-1)^{k(n-k-1)} * d_k * .$$

(Of course for most purposes the signs of the boundary operators are unimportant.)

There are the ideals $I_0 = (0)$ and $I_\ell = (x_1, \ldots, x_\ell)$ for $\ell = 1, \ldots, n$. Let $\tilde{\mathcal{K}}(x)$ be the sequence of homomorphisms

$$0 \to \tilde{K}_n(x) \xrightarrow{\delta_n} \tilde{K}_{n-1}(x) \to \cdots \to \tilde{K}_1(x) \xrightarrow{\delta_1} \tilde{K}_0(x) \xrightarrow{\varepsilon} R/I_n \to 0$$

where $\varepsilon : R \to R/I_n$ is the natural map. This is a chain complex because $\tilde{K}(x)$ is a chain complex and the image of δ_1 is contained in I_n.

Now let M be an R-module. For each $\ell = 0, \ldots, n$ and $k = 0, \ldots, \ell$ let

$$X_k^{(\ell)} = M \otimes \wedge^k R^\ell,$$

and if $k \ge 1$ let $\delta_k : X_k^{(\ell)} \to X_{k-1}^{(\ell)}$ be the homomorphism

$$\delta_k(m \otimes \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}) = \sum_{h=1}^{k} (-1)^{h+1}(x_{i_h} m) \otimes \mathbf{e}_{i_1} \wedge \cdots \wedge \hat{\mathbf{e}}_{i_h} \wedge \cdots \wedge \mathbf{e}_{i_k}.$$

Under the natural identification of $M \otimes \tilde{K}_0(x_1, \ldots, x_\ell)$ with M, the image of $X_1^{(\ell)} \to X_0^{(\ell)}$ is contained in $I_\ell M$, so if we let ε be the natural map from $X_0^{(\ell)} = M$ to $M/I_\ell M$, it is easy to see that

$$0 \to X_\ell^{(\ell)} \xrightarrow{\delta_\ell} X_{\ell-1}^{(\ell)} \to \cdots \to X_1^{(\ell)} \xrightarrow{\delta_1} X_0^{(\ell)} \xrightarrow{\varepsilon} M/I_\ell M \to 0,$$

is a chain complex, which we denote by $X^{(\ell)}$. The image of $X_1^{(\ell)} \to X_0^{(\ell)}$ is contained in $I_\ell M$, so there is another chain complex

$$0 \to X_\ell^{(\ell)} \xrightarrow{\delta_\ell} X_{\ell-1}^{(\ell)} \to \cdots \to X_2^{(\ell)} \xrightarrow{\delta_2} X_1^{(\ell)} \xrightarrow{\delta_1} I_\ell M \to 0,$$

that we denote by $Y^{(\ell)}$.

Proposition G6.1. *If $x_1, \ldots, x_n$ is a semiregular sequence, then $X^{(\ell)}$ and $Y^{(\ell)}$ are acyclic for all $\ell = 0, \ldots, n$.*

Proof. Since $X^{(0)}$ is $0 \to M \xrightarrow{\mathbf{1}_M} M \to 0$ and $Y^{(0)} = 0$, by induction we may assume that $X^{(\ell-1)}$ and $Y^{(\ell-1)}$ are acyclic. Clearly $X^{(\ell)}$ is exact at $M/I_\ell M$. It is exact at $X_0^{(\ell)}$ if and only if the image of δ_1 is all of $I_\ell M$, so $X^{(\ell)}$ is acyclic if and only if $Y^{(\ell)}$ is acyclic. Therefore it suffices to prove that $Y^{(\ell)}$ is acyclic. If we can show that $Y^{(\ell)}/Y^{(\ell-1)}$ is acyclic, then two out of three terms in the long exact homology sequence of the short exact sequence of complexes

$$0 \to Y^{(\ell-1)} \to Y^{(\ell)} \to Y^{(\ell)}/Y^{(\ell-1)} \to 0$$

will be zero, so the terms related to $Y^{(\ell)}$ also vanish.

There is a chain map

$$\begin{tikzcd}
0 \arrow[r] & X_{\ell-1}^{(\ell-1)} \arrow[r] \arrow[d] & X_{\ell-2}^{(\ell-1)} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & X_\ell^{(\ell)} \arrow[r] & X_{\ell-1}^{(\ell)}/X_{\ell-1}^{(\ell-1)} \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & X_0^{(\ell-1)} \arrow[r] \arrow[d] & M/I_{\ell-1}M \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & X_1^{(\ell)}/X_1^{(\ell-1)} \arrow[r] & I_\ell M/I_{\ell-1}M \arrow[r] & 0
\end{tikzcd}$$

from $X^{(\ell-1)}$ to $Y^{(\ell)}/Y^{(\ell-1)}$ in which $M/I_{\ell-1}M \to I_\ell M/I_{\ell-1}M$ is induced by $m \mapsto x_\ell m$ and the other vertical maps are induced by $\wedge x_\ell \mathbf{e}_\ell$. Consideration of the defining formula for δ_k shows that the difference between the compositions

$$X_i^{(\ell-1)} \xrightarrow{\mathbf{1}_M \otimes \delta_i} X_{i-1}^{(\ell-1)} \xrightarrow{\wedge x_\ell \mathbf{e}_\ell} X_i^{(\ell)} \quad \text{and} \quad X_i^{(\ell-1)} \xrightarrow{\wedge x_\ell \mathbf{e}_\ell} X_{i+1}^{(\ell)} \xrightarrow{\mathbf{1}_M \otimes \delta_{i+1}} X_i^{(\ell)}$$

lies in $X_i^{(\ell-1)}$, and similarly for the final box, so this diagram commutes. It is easy to see that the maps $\wedge x_\ell \mathbf{e}_\ell$ are all isomorphisms. The map $M/I_{\ell-1}M \to I_\ell M/I_{\ell-1}M$ is surjective because $(x_\ell)+I_{\ell-1} = I_\ell$ and injective because $I_{\ell-1}M : x_\ell = I_{\ell-1}M$. Since the top row is exact, so is the bottom row. □

Since $M \otimes_R R/I_n = M/I_n M$ (Lemma A6.3) we have $X^{(n)} = M \otimes_R \tilde{\mathcal{K}}(x)$, which leads to the most important case of the last result.

Theorem G6.2. *If $x_1, \ldots, x_n$ is a semiregular sequence, then $M \otimes_R \tilde{\mathcal{K}}(x)$ is acyclic.*

An important special case is $M = R$ with I_n a proper subset of R. Let $Q = R/I_n$. We have $R \otimes_R \tilde{\mathcal{K}}(x) = \tilde{\mathcal{K}}(x)$, so the result above asserts that $\tilde{\mathcal{K}}(x)$

is acyclic. In addition $\tilde{K}_i = R \otimes_{\mathbb{Z}} E_i(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is a free R-module, so $\tilde{\mathcal{K}}(x)$ is a free resolution of Q.

Let N be an R-module. Then

$$\mathrm{Hom}_R(\delta_n, N) : \mathrm{Hom}_R(\tilde{K}_{n-1}(x), N) \to \mathrm{Hom}_R(\tilde{K}_n(x), N)$$

is the map $g \mapsto \sum_{h=1}^{n} (-1)^{h+1} x_h g_h$ if we identify $g \in \mathrm{Hom}_R(\tilde{K}_{n-1}(x), N)$ with $g = (g_1, \ldots, g_n) \in N^n$ where each g_h is the image of $\mathbf{e}_1 \wedge \cdots \wedge \hat{\mathbf{e}}_h \wedge \cdots \wedge \mathbf{e}_n$, and we identify an element of $\mathrm{Hom}_R(\tilde{K}_n(x), N)$ with the image of $\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$.

The image of $\mathrm{Hom}_R(\delta_n, N)$ is contained in $\mathrm{Hom}_R(\tilde{K}_n(x), I_n N)$, so when $N = Q$ we find that $\mathrm{Hom}_R(\delta_n, Q) = 0$. Combining all this with the definition of Ext, we have

$$\begin{aligned}\mathrm{Ext}^n_R(Q, Q) &= H^n(\mathrm{Hom}_R(\tilde{\mathcal{K}}(x), Q)) = \mathrm{Ker}(\mathrm{Hom}_R(\delta_n, Q)) \\ &= \mathrm{Hom}_R(\tilde{K}_{n-1}(x), Q) = \mathrm{Hom}_R(R^n, Q) = Q^n \neq 0.\end{aligned}$$

The *projective dimension* $\mathrm{pd}_R M$ of M is, by definition, the smallest n such that there is a projective resolution

$$\cdots \to 0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

with $X_{n+1} = 0$. The *free dimension* is defined in the same way, using free resolutions instead of projective resolutions. Because free modules are projective, the free dimension is never less than the projective dimension.

Theorem G6.3. *If $x_1, \ldots, x_n$ is a regular sequence and $Q = R/(x_1, \ldots, x_n)$, then $\mathrm{pd}_R Q = n$.*

Proof. Theorem G6.2 with $M = R$ gives $\tilde{\mathcal{K}}(x)$, a resolution

$$0 \to \tilde{K}_n(x) \xrightarrow{\delta_n} \tilde{K}_{n-1}(x) \to \cdots \to \tilde{K}_1(x) \xrightarrow{\delta_1} \tilde{K}_0(x) \xrightarrow{\varepsilon} Q \to 0$$

of Q, in which each $\tilde{K}_i(x) = R \otimes_{\mathbb{Z}} E_i(y_1, \ldots, y_n)$ is free and $R \otimes_R \tilde{\mathcal{K}}(x) = \tilde{\mathcal{K}}(x)$ is acyclic. Thus the free dimension of Q is not greater than n. On the other hand we showed above that $\mathrm{Ext}^n_R(Q, Q) \neq 0$. Since $\mathrm{Ext}_R(Q, Q)$ can be computed using a projective resolution of Q in the first variable, it follows that the projective dimension of Q is at least n. Since the free dimension is at least as large as the projective dimension, the result follows. □

Chapter H

Depth and Cohen-Macaulay Rings

We have seen that the local ring at a smooth point of an algebraic variety is regular. While there are some special properties that distinguish some smooth points from others, the corresponding algebraic analysis has not been the central focus of commutative algebra since FAC. Instead, a main concern has been to study points of varieties that are not smooth, but which nonetheless enjoy properties that make them well behaved, at least relative to the wealth of unpleasant things that can happen at singular points. The corresponding algebraic endeavor has been to study local rings that are not regular, but which still enjoy attractive properties. In some cases we study rings which are not themselves local, but whose localizations at maximal ideals have the desirable features. Thus a regular ring is one whose localization at each maximal ideal is a regular local ring.

Suppose that R is Noetherian and local, and let d be its Krull dimension. Proposition H1.7 below implies that there are no R-sequences with more than d elements. If R is regular, then $\mathfrak{m}$ is generated by d elements $x_1, \ldots, x_d$, and in this circumstance (Corollary F5.5) $x_1, \ldots, x_d$ is an R-sequence. The notion of a local Cohen-Macaulay ring generalizes this by requiring that there is an R-sequence with d elements, without requiring that $\mathfrak{m}$ is generated by such a sequence. If R is not local, it is a Cohen-Macaulay ring if, for each maximal ideal $\mathfrak{m}$, $R_{\mathfrak{m}}$ is a local Cohen-Macaulay ring.

Even though they will not be studied here, we should also mention Gorenstein rings, which are more general than regular rings and less general than Cohen-Macaulay rings. (Daniel Gorenstein was fond of saying that he didn't understand the definition of a Gorenstein ring, and we won't bother with it.) These have also been studied extensively, and encompass a large percentage of the rings that arise in nature.

H1 Depth

Throughout this section we work with a given R-module M and a given ideal I, which may be improper. The *depth of I on M*, denoted by $\operatorname{depth}(I, M)$, is the maximal length of a regular sequence on M whose elements lie in I. The *depth* of I is $\operatorname{depth}(I) = \operatorname{depth}(I, R)$, and the *depth* of R is $\operatorname{depth}(R) = \operatorname{depth}(R, R)$. It is common to define the depth of M to be $\operatorname{depth}(R, M)$, but when M is an ideal this terminology becomes ambiguous, so we will not use the term in this sense. Note that if R is local (the most important case) then $\operatorname{depth}(R) = \operatorname{depth}(\mathfrak{m}, R) = \operatorname{depth}(\mathfrak{m})$.

Earlier we defined the *codimension* of a prime P to be the Krull dimension of R_P, which is the maximal length of chains of prime ideals descending from P. The *codimension* of I is the minimum of the codimensions of the primes containing I. In much of the literature this is called the *height* of I. Our primary aim in this section is to demonstrate a relationship between depth and codimension.

Our first result shows that the depth of I on M is a geometric concept in contexts in which there is a bijection between varieties and radical ideals.

Proposition H1.1. *If R is Noetherian and M is finitely generated, then* $\operatorname{depth}(\operatorname{rad}(I), M) = \operatorname{depth}(I, M)$.

Proof. If $x_1, \dots, x_n$ is an M-sequence in $\operatorname{rad}(I)$, then $x_1^t, \dots, x_n^t$ is an M-sequence for any positive integer t (Proposition G5.4) and for some t it is contained in I. □

Depth can only be increased by localization. The following result has a basic character, and does not require a Noetherian hypotheses.

Lemma H1.2. *If M is finitely generated, P is a prime in its support, and $x_1, \dots, x_n$ is an M-sequence in P, then $(x_1, \dots, x_n)$ is an M_P-sequence. Consequently if $I \subset P$, then* $\operatorname{depth}(I, M) \le \operatorname{depth}(I_P, M_P)$.

Proof. Since P is in the support of M, $P_P M_p$ is a proper subset of M_P: otherwise Nakayama's lemma would imply that $M_P = 0$. Consequently $(x_1, \dots, x_n)M_P \subset P_P M_P \neq M_P$.

It remains to show that for a given $i = 1, \dots, n$, x_i is not a zerodivisor of $M_P/(x_1, \dots, x_{i-1})M_P$. Suppose that $m/s \in (x_1, \dots, x_{i-1})M_P : x_i$, so that

$$x_i(m/s) = x_1(m_1/s_1) + \cdots + x_{i-1}(m_{i-1}/s_{i-1}).$$

Multiplying this by st where $t = s_1 \cdots s_{i-1}$ gives a relationship

$$x_i(tm) = x_1(t_1 m_1) + \cdots + x_{i-1}(t_{i-1} m_{i-1})$$

in M. Since $x_1, \dots, x_i$ is regular on M there are $n_1, \dots, n_{i-1} \in M$ such that

$$tm = x_1 n_1 + \cdots + x_{i-1} n_{i-1}.$$

Now dividing by st shows that $m/s \in (x_1, \dots, x_{i-1})M_P$. □

Depth is a local property, in the sense that it agrees with the maximum depth of its various localizations.

Lemma H1.3. *If R is Noetherian and M is finitely generated, there is some maximal ideal $\mathfrak{m}$ in the support of M such that $I \subset \mathfrak{m}$ and* $\operatorname{depth}(I, M) = \operatorname{depth}(I_{\mathfrak{m}}, M_{\mathfrak{m}})$.

Proof. Let $I = (x_1, \dots, x_n)$ and let $r = \operatorname{depth}(I, M)$. Theorem G4.6 implies that $H^r(M \otimes K(x_1, \dots, x_n)) \neq 0$. Lemma A5.8 implies that the support of $H^r(M \otimes K(x_1, \dots, x_n))$ contains maximal ideals. Any such maximal $\mathfrak{m}$ contains I because I annihilates $H^*(M \otimes K(x_1, \dots, x_n))$ (Corollary G3.2) and for such a $\mathfrak{m}$ we have

$$0 \neq H^r(M \otimes K(x_1, \dots, x_n))_{\mathfrak{m}} = H^r((M \otimes K(x_1, \dots, x_n))_{\mathfrak{m}})$$

$$= H^r(M_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} K(x_1, \dots, x_n)_{\mathfrak{m}})$$

because localization commutes with homology (Proposition A5.5) and tensor products (Proposition A6.5). This implies both that $\mathfrak{m}$ is in the support of M and (by Theorem G4.6) that $\operatorname{depth}(I_{\mathfrak{m}}, M_{\mathfrak{m}}) \leq r$. The last result gives the opposite inequality. □

Taking the ideal I to be maximal, we obtain:

Corollary H1.4. *If R is Noetherian, M is finitely generated, and $\mathfrak{m}$ is a maximal ideal, then* $\operatorname{depth}(\mathfrak{m}, M) = \operatorname{depth}(\mathfrak{m}_{\mathfrak{m}}, M_{\mathfrak{m}})$.

For $y \in R$ let (I, y) denote the ideal generated by y and the elements of I, which is of course the smallest ideal containing both I and y. When R is local, adjoining an element $y \in \mathfrak{m}$ in this way cannot increase depth by more than one.

Lemma H1.5. *If R is Noetherian and local, M is finitely generated, and $y \in \mathfrak{m}$, then*

$$\operatorname{depth}((I, y), M) \leq \operatorname{depth}(I, M) + 1.$$

Proof. Suppose $I = (x_1, \dots, x_n)$, and set $r = \operatorname{depth}((I, y), M)$. Theorem G4.6 implies that $H^i(M \otimes K(x_1, \dots, x_n, y)) = 0$ for all $i < r$. The long exact sequence given by Proposition G4.1 becomes

$$0 \to H^i(M \otimes K(x)) \xrightarrow{y} H^i(M \otimes K(x)) \to 0$$

for $i < r-1$. Since $y \in \mathfrak{m}$, Nakayama's lemma implies that $H^i(M \otimes K(x)) = 0$ for $i < r - 1$, so Theorem G4.6 implies that $\operatorname{depth}(I, M) \geq r - 1$. □

Applying the last result, we obtain a relationship between depth and the length of chains of prime ideals.

Proposition H1.6. *Suppose R is Noetherian, M is finitely generated, I contains the annihilator of M, and $P = P_0 \supset P_1 \supset \cdots \supset P_\ell$ is a maximal chain of prime ideals descending from a prime P that is minimal over I to a prime $P_\ell \in \operatorname{Ass}(M)$. Then* $\operatorname{depth}(I, M) \leq \ell$.

Proof. We use induction on ℓ. The case $\ell = 0$ is trivial: since P is associated to M, each of its nonzero elements is a zerodivisor, so there is no M-sequence of positive length contained in I. Suppose that $\ell \geq 1$.

Now $(P_0)_P \supset \cdots \supset (P_\ell)_P$ is a chain of prime ideals of R_P and P_P is a minimal prime over I_P (Proposition A5.6). Proposition A10.6 implies that $(P_\ell)_P$ is an associated prime of M_P. In addition Lemma H1.2 implies that any M-sequence in P goes to a M_P-sequence in P_P. Therefore it suffices to prove the claim when R is local and $I = \mathfrak{m}$, as we now assume. Choose an $x \in \mathfrak{m} \backslash P_1$. Since $\mathfrak{m}$ is the only prime containing (P_1, x), its image in $R/(P_1, x)$ is the nilradical and consequently nilpotent, because R is Noetherian. Proposition H1.1 implies that $\operatorname{depth}(\mathfrak{m}, M) = \operatorname{depth}((P_1, x), M)$, Lemma H1.5 implies that $\operatorname{depth}((P_1, x), M) \leq \operatorname{depth}(P_1, M) + 1$, and the induction hypothesis gives $\operatorname{depth}(P_1, M) \leq \ell - 1$. □

Recall that the codimension of an arbitrary ideal I is the minimum codimension of the primes containing I. The following inequality displays the depth of I as a variant of codimension.

Proposition H1.7. *If R is Noetherian, then* $\operatorname{depth} I \leq \operatorname{codim} I$*.*

Proof. Let P be a prime containing I of minimal codimension; of course P is minimal over I. The annihilator of R is (0), which is contained in I, and R itself is a finitely generated R-module, so the hypotheses of Proposition H1.6 hold with $M = R$. The claim follows from that result because the maximal length of a chain of prime ideals descending from P to an associated prime of R is not greater than the maximal length of any chain of prime ideals descending from P. □

H2 Cohen-Macaulay Rings

It seems quite natural to consider the possibility that the inequality in the last result holds with equality. This thought leads to one of the major definitions of commutative algebra. The ring R is *Cohen-Macaulay* if it is Noetherian and, for every maximal ideal $\mathfrak{m}$, the depth of $\mathfrak{m}$ is equal to the codimension of $\mathfrak{m}$. A regular local ring is Cohen-Macaulay because its unique maximal ideal satisfies this condition, by virtue of the definition of regularity. The importance of Cohen-Macaulay rings results from their many nice properties, which this sections explores a bit, and the fact that they are common in practice.

Proposition H2.1. *If R is Cohen-Macaulay, then* $\operatorname{depth} I = \operatorname{codim} I$ *for every ideal I.*

Proof. Lemma H1.2 gives a maximal ideal $\mathfrak{m}$ containing I such that $\operatorname{depth} I = I_{\mathfrak{m}}$, and of course $\operatorname{codim} I = \operatorname{codim} I_{\mathfrak{m}}$. Thus it suffices to prove the claim under the hypothesis that R is local and $\operatorname{depth}\mathfrak{m} = \operatorname{codim}\mathfrak{m}$. Proposition H1.7 gives $\operatorname{depth} I \leq \operatorname{codim} I$.

When I is $\mathfrak{m}$-primary $\operatorname{codim} I = \operatorname{codim}\mathfrak{m}$, and Proposition H1.1 gives $\operatorname{depth} I = \operatorname{depth}\mathfrak{m}$, so we may assume that this is not the case. Therefore prime avoidance gives an $x \in \mathfrak{m}$ that is not contained in any prime that is minimal over I. Lemma H1.5 gives $\operatorname{depth} I + 1 \geq \operatorname{depth} I + (x)$. Since R is Noetherian, we may assume that I is a maximal element of the set of ideals for which the result fails, so $\operatorname{depth} I + (x) = \operatorname{codim} I + (x)$.

If the image $\tilde{x}$ of x in $R/\operatorname{rad}(I)$ was a zerodivisor, say $\tilde{x}\tilde{y} = 0$ where $\tilde{y} \neq 0$, then $\tilde{y}$ would be the image of some $y \in R \setminus \operatorname{rad}(I)$, and Corollary A2.9 would imply that y was outside some prime P that is minimal over I, so that $xy \in \operatorname{rad}(I) \subset P$, which is impossible. Krull's principal ideal theorem (Corollary F6.2) applied to $R/\operatorname{rad}(I)$ now gives $\operatorname{codim} I + (x) = \operatorname{codim} I + 1$. □

Proposition H2.2. *The ring R is Cohen-Macaulay if and only if $R_\mathfrak{m}$ is Cohen-Macaulay for every maximal ideal $\mathfrak{m}$, in which case R_P is Cohen-Macaulay for every prime ideal P.*

Proof. If R is Cohen-Macualay and P is a prime, then

$$\operatorname{codim} P_P = \operatorname{codim} P = \operatorname{depth} P \leq \operatorname{depth} P_P \leq \operatorname{codim} P_P$$

where the second equality is from the last result, the first inequality is from Proposition H1.2, and the second is from Proposition H1.6. Since the inequality is an equality and P_P is the unique maximal ideal of R_P, R_P is Cohen-Macaulay.

If $R_\mathfrak{m}$ is Cohen-Macaulay for every maximal ideal $\mathfrak{m}$, then for any such $\mathfrak{m}$ we have $\operatorname{depth}(\mathfrak{m}, R) = \operatorname{depth}(\mathfrak{m}_\mathfrak{m}, R_\mathfrak{m})$ by Lemma H1.3. Since $\operatorname{codim}\mathfrak{m} = \operatorname{codim}\mathfrak{m}_\mathfrak{m}$, it follows that R is Cohen-Macaulay. □

Proposition H2.3. *If R is Cohen-Macaulay then so is $R[X]$.*

Proof. From the last result it suffices to prove that $R[X]_{\mathfrak{m}_X}$ is Cohen-Macaulay for a given maximal ideal $\mathfrak{m}_X$ of $R[X]$. The complement of $P = \mathfrak{m}_X \cap R$ in R is contained in the complement of $\mathfrak{m}_X$ in $R[X]$, so $R[X]_{\mathfrak{m}_X} = R_P[X]_{(R\setminus P)^{-1}\mathfrak{m}_X}$. (Very concretely, the set of quotients f/g with $f \in R[X]$ and $g \in R[X] \setminus \mathfrak{m}_X$ is the same as the set of quotients f/g with $f \in (R \setminus P)^{-1} R_P[X]$ and $g \in (R\setminus P)^{-1}(R[X]\setminus\mathfrak{m}_X)$.) In addition $(R\setminus P)^{-1}\mathfrak{m}_X$ is a maximal ideal of $R_P[X]$, and $P_P[X] \subset (R \setminus P)^{-1}\mathfrak{m}_X$ because $P[X] \subset \mathfrak{m}_X$.

Therefore we may assume that R is local with maximal ideal $\mathfrak{m}$, and that $\mathfrak{m}[X] \subset \mathfrak{m}_X$. Let k and k_X be the corresponding residue fields. Our goal is to show that $\operatorname{depth}\mathfrak{m}_X = \operatorname{codim}\mathfrak{m}_X$. Proposition H1.7 gives $\operatorname{depth}\mathfrak{m}_X \leq \operatorname{codim}\mathfrak{m}_X$.

Now $R[X]/\mathfrak{m}[X] = k[X]$ is a principal ideal domain, so $\mathfrak{m}_X/\mathfrak{m}[X]$ is generated by a single polynomial $f \in R[X]$, which is to say that $\mathfrak{m}_X = \mathfrak{m} + (f)$. We may take f such that the image of f in $k[X]$ in monic.

If $x_1, \ldots, x_n$ is an R-sequence in $\mathfrak{m}$, then it is also a $R[X]$-sequence because $R[X]$ is a free R-module. (This is a direct consequence of the definition of a regular sequence.) For any nonzero $g \in R[X]/(x_1, \ldots, x_n)R[X] = (R/(x_1, \ldots, x_n))[X]$, the product of the leading coefficient of f (which is the sum of 1 and an element of $\mathfrak{m}$) and the leading coefficient of g cannot vanish in $R/(x_1, \ldots, x_n)$, so $(x_1, \ldots, x_n, f)$ is an $R[X]$-sequence in $\mathfrak{m}_X$.

Thus $\operatorname{depth} \mathfrak{m}_x + 1 \geq \operatorname{depth} \mathfrak{m} + 1$. Since R is Cohen-Macaulay, $\operatorname{depth} \mathfrak{m} = \operatorname{codim} \mathfrak{m}$. The principal ideal theorem (Theorem F6.1) implies that $\operatorname{codim} \mathfrak{m} + 1 \geq \operatorname{codim} \mathfrak{m}_X$. Combining these gives $\operatorname{depth} \mathfrak{m}_X \geq \operatorname{codim} \mathfrak{m}_X$, as desired. □

A ring R is *catenary* if, for any prime ideals $P \subset Q$, all maximal chains of prime ideals between P and Q have the same length. If I is an ideal of R, the primes of R/I are in inclusion preserving bijection with the primes of R that include I, so two maximal chains between given prime ideals in R/I are, in effect, maximal chains between prime ideals in R. If P is a prime, the primes of R_P are in bijection with the primes of R that are contained in P, so a maximal chain in R_P is, in effect, a maximal chain in R. Thus:

Lemma H2.4. *If R is catenary, then any quotient of R is catenary, and any localization of R is catenary.*

Proposition H2.5. *If R is Cohen-Macaulay and local, then any two maximal chains of prime ideals have the same length, and every associated prime of R is minimal.*

Proof. We claim that all maximal chains of prime ideals descended from the maximal ideal $\mathfrak{m}$ to an associated prime of R have length equal to the Krull dimension of R. By Proposition H1.6 the length of any such chain is at least the depth of $\mathfrak{m}$, and by hypothesis the depth of $\mathfrak{m}$ is the codimension of $\mathfrak{m}$. □

Proposition H2.6. *If R is Cohen-Macaulay, then it is caternary.*

Proof. If P and Q are primes with $Q \subset P$, then any chain of primes descending from P to Q can be extended to a maximal chain of primes descended from P, so it suffices to show that any two such maximal chains have the same length. This is true in R if and only if it is true in R_P, which Proposition H2.2 and the last result show to be the case. □

A full proof would take us too far afield, but it is worth mentioning that the converse of Proposition H2.3 also holds: if $R[X]$ is Cohen-Macaulay, then so is R. Suppose $R[X]$ is Cohen-Macaulay. Then R is Noetherian because $R[X]$ is. Let $\mathfrak{m}$ be a maximal ideal of R. Then $\mathfrak{m} + (X)$ is a maximal ideal of $R[X]$. If $x_1, \ldots, x_n$ is an R-sequence in $\mathfrak{m}$, then it is an $R[X]$-sequence in $\mathfrak{m} + (X)$. Clearly, X is not a zerodivisor in $R[X]/(x_1, \ldots, x_n)R[X]$, so $x_1, \ldots, x_n, X$ is an $R[X]$-sequence in $\mathfrak{m} + (X)$. Thus $\operatorname{depth} \mathfrak{m} + 1 \leq \operatorname{depth} \mathfrak{m} + (X)$, and Lemma H1.5 gives the opposite inequality. The last result implies that any two maximal chains of prime ideals in $R[X]$ have the same length. Therefore $\operatorname{codim} \mathfrak{m} + (X) = \dim R[X]$. If $\mathfrak{m}' \supset P_1 \supset \cdots \supset P_k$ is a maximal chain of prime ideals of R, then $\mathfrak{m}' + (X) \supset \mathfrak{m}'[X] \supset P_1[X] \supset \cdots \supset P_k[X]$ is a maximal chain of prime ideals in $R[X]$. (Since it is bulky and suitably challenging, the verification of this is left as an exercise.) Therefore any two maximal chains of prime ideals in R have the same length, and $\operatorname{codim} \mathfrak{m} = \dim R$. Exercise 7 on p. 126 of Atiyah and McDonald (1969) gives $\dim R[X] + 1 = \dim R + 1$. Since $R[X]$ is Cohen-Macaulay we have

$$\operatorname{codim} \mathfrak{m} + 1 = \operatorname{codim} \mathfrak{m} + (X) = \operatorname{depth} \mathfrak{m} + (X) = \operatorname{depth} \mathfrak{m} + 1.$$

The ring R is *universally catenary* if every finitely generated R-algebra is catenary. Any such algebra is the homomorphic image of $R[X_1, \ldots, X_n]$ for some n, so in view of Lemma H2.4, R is universally catenary if and only if each $R[X_1, \ldots, X_n]$ is catenary. Proposition H2.3 implies that this is the case if R is Cohen-Macaulay, so:

Proposition H2.7. *If R is Cohen-Macaulay, then it is universally caternary.*

Proposition H2.8. *If R is Cohen-Macaulay and the codimension of $I = (x_1, \ldots, x_n)$ is n, then R/I is Cohen-Macaulay.*

Proof. Any maximal ideal of R/I is $\mathfrak{m}/I$ for some maximal ideal $\mathfrak{m}$ of R that contains I. By Proposition H2.2 it suffices to show that for any such $\mathfrak{m}$, $(R/I)_{\mathfrak{m}/I}$ is Cohen-Macaulay. We have $(R/I)_{\mathfrak{m}/I} = R_{\mathfrak{m}}/IR_{\mathfrak{m}}$, and $IR_{\mathfrak{m}}$ is generated by $x_1, \ldots, x_n$. The codimension of $IR_{\mathfrak{m}}$ in $R_{\mathfrak{m}}$ agrees with the codimension of I in R, so it is n. Proposition H2.2 implies that $R_{\mathfrak{m}}$ is Cohen-Macaulay. Thus the hypotheses are satisfied with $R_{\mathfrak{m}}$ and $IR_{\mathfrak{m}}$ in place of R and I, so it suffices to prove the claim with the additional hypothesis that R is local.

Since R is Noetherian, so is R/I. Corollary G5.3 implies that $x_1, \ldots, x_n$ is a regular sequence. Extending this to a maximal regular sequence in $\mathfrak{m}$ shows that the $\operatorname{depth} \mathfrak{m}/I = \operatorname{depth} \mathfrak{m} - n$. Since R is Cohen-Macaulay, $\operatorname{depth} \mathfrak{m} = \operatorname{codim} \mathfrak{m}$. Since R is local, $\operatorname{codim} \mathfrak{m} = \dim R$. We have $\dim R/I \leq \dim R - n$ because for each i, x_i is not contained in any prime that is minimal over $(x_1, \ldots, x_{i-1})$. Since R/I is local, $\operatorname{codim} \mathfrak{m}/I = \dim R/I$. Thus

$$\operatorname{depth} \mathfrak{m}/I = \operatorname{depth} \mathfrak{m} - n = \operatorname{codim} \mathfrak{m} - n = \dim R - n \geq \dim R/I = \operatorname{codim} \mathfrak{m}/I,$$

and Proposition H1.7 gives the reverse inequality. □

Given the intuitive resemblance between the associated primes of an ideal and the minimal primes over that ideal, it is desirable to have results that give precise relationships. Consequently the theorem below is quite prominent. We separate out one part that holds quite generally.

Lemma H2.9. *If $I = (x_1, \dots, x_n)$ and the codimension of I is n, then all minimal primes over I have codimension n.*

Proof. The codimension of I is by definition the minimum of the codimensions of the primes containing I, so the codimension of any such prime is at least n. On the other hand the principal ideal theorem implies that the codimension of any prime that is minimal over I is at most n. □

An ideal I is said to be *unmixed* if all of its associated primes have the same codimension. Of course this implies that there are no embedded associated primes, so, in view of Theorem A10.9, the associated primes are precisely the primes that are minimal over I.

Theorem H2.10 (Unmixedness Theorem)**.** *If R is Noetherian, then it is Cohen-Macaulay if and only if every ideal that is generated by a number of generators equal to its codimension is unmixed.*

Proof. Suppose that R is Cohen-Macaulay and $I = (x_1, \dots, x_r)$ has codimension r. Then R/I is Cohen-Macaulay by Proposition H2.8, and Proposition H2.5 implies that every associated prime of I is minimal over I. The lemma above implies that the the codimension of every such prime is r.

Now suppose that for each r, each ideal of codimension r generated by r elements is unmixed. We will show that for a given prime P, $\operatorname{depth} P = \operatorname{codim} P$. Proposition H1.7 implies that $\operatorname{depth} P \leq \operatorname{codim} P$. Let $r = \operatorname{codim} P$. Proposition F6.4 implies that there are $x_1, \dots, x_r$ such that P is minimal over $(x_1, \dots, x_r)$. For each $i = 1, \dots, r$ the principal ideal theorem (Theorem F6.1) implies that $\operatorname{codim}(x_1, \dots, x_i) \leq i$, and also that the codimension of the ideal in $R/(x_1, \dots, x_i)$ generated by $x_{i+1}, \dots, x_r$ is not greater than $r - i$, but this codimension is $r - \operatorname{codim}(x_1, \dots, x_i)$, so $\operatorname{codim}(x_1, \dots, x_i) = i$. Therefore x_{i+1} is not contained in any prime that is minimal over $(x_1, \dots, x_i)$, so the hypothesis implies that it is not contained in any prime that is associated to $(x_1, \dots, x_i)$, and is consequently (Corollary A10.3) not a zerodivisor of $R/(x_1, \dots, x_i)$. We have shown that $x_1, \dots, x_r$ is an R-sequence in P, so $\operatorname{depth} P \geq r$. □

We can now comment on Serre's application of these ideas in no. 78. Let K be a (not necessarily algebraically complete) field, and let $\mathfrak{m}$ be a maximal ideal of $K[X_1, \dots, X_d]$. The images of $X_1, \dots, X_d$ in $\mathfrak{m}/\mathfrak{m}^2$ generate it, and there is no system of generators with fewer elements, so they are a basis of this vector space. Proposition A2.15 (Nakayama's lemma) implies that $\mathfrak{m}$ is generated by the images of $X_1, \dots, X_d$, so $K[X_1, \dots, X_d]_{\mathfrak{m}}$ is a regular local ring. Since $\mathfrak{m}$ was arbitrary, $K[X_1, \dots, X_d]$ is a regular ring, hence Cohen-Macaulay.

The ring R is a *complete intersection* if $R = K[X_1, \dots, X_d]/(x_1, \dots, x_n)$ for some regular sequence $x_1, \dots, x_n \in K[X_1, \dots, X_d]$. Proposition F6.3 implies that the codimension of $(x_1, \ldots, x_n)$ is n, so Proposition H2.8 implies that R is Cohen-Macaulay. The unmixedness theorem implies that (0) is unmixed, which is to say that all the primes associated to (0) have the same codimension. In general all minimal primes are associated, so the minimal primes are the only associated primes.

Chapter I

Global Dimension

The *global dimension* of R, denoted by $\operatorname{gl\,dim} R$, is defined to be the maximal projective dimension of any R-module. This concept is central in the applications of homological algebra to commutative algebra that were developed in the 1950's, largely as a result of the efforts of Auslander, Buchsbaum, Cartan, Eilenberg, and Serre himself. We will see several famous results. The Hilbert syzygy theorem can be understood as asserting that if R is Noetherian and local, then its global dimension is $\operatorname{pd}_R k$. It will turn out that such an R is regular if and only if these quantities are finite. These findings will be key steps on our path to the book's pinnacle result, which is that a regular local ring is factorial.

I1 Auslander's Theorem

From a technical point of view, the analysis of global dimension revolves around the following result.

Theorem I1.1 (Auslander)**.** *The following are equivalent:*

(a) $\operatorname{gl\,dim} R \leq n$*;*

(b) $\operatorname{pd}_R R/I \leq n$ *for every ideal* $I \subset R$*;*

(c) the injective dimension of every R-module M is $\leq n$*;*

(d) $\operatorname{Ext}^i_R(M, N) = 0$ *for all R-modules M and N and all* $i > n$*.*

The proof utilizes a result of independent interest.

Proposition I1.2. *An R-module X is injective if* $\operatorname{Ext}^1_R(R/I, X) = 0$ *for all ideals* $I \subset R$*.*

Proof. Let I be an ideal of R. The inductive construction in the proof of Lemma C1.1 gives a free resolution

$$\cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R \longrightarrow R/I \to 0.$$

Since $\operatorname{Ext}^1_R(R/I, X) = 0$, the image of $\operatorname{Hom}_R(d_1, X)$ is the entire kernel of $\operatorname{Hom}_R(d_2, X)$, so whenever $\varphi : F_1 \to X$ is a map with $\varphi \circ d_2 = 0$, there is a $\psi : R \to X$ such that $\varphi = \psi \circ d_1$. More concretely, whenever the kernel of φ contains $\operatorname{Im} d_2 = \operatorname{Ker} d_1$, so that φ may be thought of as a map from $F_1/\operatorname{Ker} d_1 = I$ to X, there is an extension to R. This is Baer's criterion. $\square$

Proof of Theorem I1.1. Clearly (a) implies (b), and (c) implies (d) because any injective resolution can be used to compute $\operatorname{Ext}_R(M, N)$. We will show that (b) implies (c) and that (d) implies (a).

Suppose that (b) holds, and let

$$0 \to N \to I_0 \to \cdots \to I_{n-1} \to X \to 0$$

be an exact sequence with $I_0, \ldots, I_{n-1}$ injective. For any ideal I the iterated connecting homomorphism derived from $\operatorname{Ext}_R(R/I, -)$ (in particular the discussion following Proposition D4.2) gives

$$\operatorname{Ext}^1_R(R/I, X) \cong \operatorname{Ext}^{n+1}_R(R/I, N) = 0$$

where the equality follows from the hypothesis. Since this is true for every ideal I, the last result implies that X is injective, so the sequence is an injective resolution of N.

Now suppose that (d) holds. Let M be any R-module, and let

$$0 \to X \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

be an exact sequence in which $P_0, \ldots, P_{n-1}$ are projective. For any R-module N dimension shifting (Proposition D4.2) gives an isomorphism $\operatorname{Ext}^1_R(X, N) \cong \operatorname{Ext}^{n+1}_R(M, N) = 0$, so Proposition D6.1 implies that X is projective. Therefore $\operatorname{pd}_R M \leq n$. $\square$

I2 Minimal Free Resolutions

Insofar as the definition of free dimension asks for smallest i for which there is a free resolution with $F_{i+1} = 0$, it makes sense to study free resolutions that do not have any excess baggage. An R-module homomorphism $\varphi : A \to B$ is *slender* if there is a minimal set of generators of A that is mapped injectively to a minimal set of generators of the image of φ. Fix an R-module M and a free resolution

$$F : \cdots \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0.$$

We say that F is *minimal* if each φ_i is slender.

Lemma I2.1. *If R is Noetherian and M is finitely generated, then M has a minimal free resolution.*

Proof. We begin by choosing a minimal system of generators for M and letting F_0 be the free R-module on these generators. Because R is Noetherian, F_0 is Noetherian (Proposition A4.6) so the kernel of $F_0 \to M$ is finitely generated. We let F_1 be the free R-module on a minimal set of generators for this kernel, and we continue in this fashion. □

Lemma I2.2. *If R is Noetherian, M is finitely generated, and F is minimal, then each F_i is finitely generated.*

Proof. This follows from induction on i, taking $F_{-1} = M$: since φ_i maps a minimal set of generators of F_i injectively to a minimal set of generators of F_{i-1}, if F_{i-1} is finitely generated, then so is F_i. □

Proposition I2.3. *If R is local and M and each F_i are finitely generated, then F is minimal if and only if $\varphi_{i+1}(F_{i+1}) \subset \mathfrak{m}F_i$ for all $i \geq 0$.*

Proof. Fixing a particular i, let γ be the surjection

$$F_i/\mathfrak{m}F_i \to (\operatorname{Coker}\varphi_{i+1})/\mathfrak{m}(\operatorname{Coker}\varphi_{i+1})$$

induced by $F_i \to \operatorname{Coker}\varphi_{i+1}$. The image of γ is $F_i/(\mathfrak{m}F_i + \operatorname{Im}(\varphi_{i+1}))$, so the image of φ_{i+1} is contained in $\mathfrak{m}F_i$ if and only if γ is an isomorphism. Recall that for any R-module N, $N/\mathfrak{m}N$ is a vector space over k, so γ is a linear transformation.

Let $x_1, \ldots, x_k$ be a minimal system of generators for F_i. Let $\tilde{x}_1, \ldots, \tilde{x}_k$ be the images of $x_1, \ldots, x_k$ in $\operatorname{Coker}\varphi_{i+1}$, and let $v_1, \ldots, v_k$ be the images in $F_i/\mathfrak{m}F_i$.

First suppose the resolution is minimal. Since φ_i is slender we can choose $x_1, \ldots, x_k$ that are mapped injectively to a minimal set of generators of the image of φ_i. Since φ_i induces an isomorphism between $\operatorname{Coker}\varphi_{i+1}$ and the image of φ_i, $\tilde{x}_1, \ldots, \tilde{x}_k$ is a minimal set of generators of $\operatorname{Coker}\varphi_{i+1}$. Now $\gamma(\tilde{x}_1), \ldots, \gamma(\tilde{x}_k)$ is a set of generators of the image of γ that is minimal. (Otherwise Nakayama's lemma (Theorem A2.13) would imply that $\tilde{x}_1, \ldots, \tilde{x}_k$ was not minimal for $\operatorname{Coker}\varphi_{i+1}$.) Since γ maps each v_i to $\gamma(\tilde{x}_i)$, it is a linear isomorphism.

Conversely suppose γ is an isomorphism. Nakayama's lemma implies that $v_1, \ldots, v_k$ is minimal for $F_i/\mathfrak{m}F_i$, hence a basis of this space, so $\gamma(v_1), \ldots, \gamma(v_k)$ is a basis. Applying Nakayama's lemma once more, $\tilde{x}_1, \ldots, \tilde{x}_k$ is a minimal collection of generators for $\operatorname{Coker}\varphi_{i+1}$. It follows that $x_1, \ldots, x_k$ map to a minimal collection of generators of the image of φ_i. □

A *trivial complex* is a direct sum of complexes of the form $\cdots \to 0 \to R \xrightarrow{\mathbf{1}_R} R \to 0 \to \cdots$. Evidently a free complex has no homology. When R is local there is a converse.

Lemma I2.4. *If R is local and*

$$H \ : \ \cdots \to H_i \xrightarrow{\rho_i} H_{i-1} \to \cdots \to H_1 \xrightarrow{\rho_1} H_0 \to 0$$

is a free resolution of the zero module with each H_i finitely generated, then H is trivial.

Proof. Let $v_1, \dots, v_k$ be a basis of $H_0/\mathfrak{m}H_0$. The map $H_1/\mathfrak{m}H_1 \to H_0/\mathfrak{m}H_0$ is a linear surjection, so there is a preimage $w_1, \dots, w_k$ of $v_1, \dots, v_k$, and this can be extended to a basis $w_1, \dots, w_n$ of $H_1/\mathfrak{m}H_1$. By Nakayama's lemma a preimage $x_1, \dots, x_n$ of this in H_1 is a minimal system of generators of H_1. Corollary A3.4 implies that these generators generate H_1 freely. Another application of Nakayama's lemma implies that $x_1, \dots, x_k$ map to a system of generators of H_0. Again, these generators generate H_0 freely, so inverting this gives a map that splits ρ_1. Therefore $H_1 = H_1' \oplus H_0'$ where ρ_1 maps H_0' isomorphically and is zero on H_1', and H_1' is freely generated. We can apply the same argument to $\cdots \to H_3 \to H_2 \to H_1' \to 0$, and so forth. □

Proposition I2.5. *If R is local, M is finitely generated, F is a minimal free resolution of M, and*

$$G \ : \ \cdots \to G_i \xrightarrow{\psi_i} G_{i-1} \to \cdots \to G_1 \xrightarrow{\psi_1} G_0 \xrightarrow{\eta} M \to 0$$

is any free resolution of M with each G_i finitely generated, then G is isomorphic to the direct sum of F and a trivial complex. Any two minimal free resolutions are isomorphic.

Proof. Lemma C3.1 gives chain maps $\alpha : F \to G$ and $\beta : G \to F$ that extend the identity on M, and it implies that for any such α and β, $\beta \circ \alpha$ is homotopic to $\mathbf{1}_F$, so there are maps $s_i : F_i \to F_{i+1}$ such that $\mathbf{1}_{F_i} - \beta_i\alpha_i = \varphi_{i+1}s_i + s_{i-1}\varphi_i$. Proposition I2.3 implies that the images of φ_{i+1} and φ_i are contained in $\mathfrak{m}F_i$ and $\mathfrak{m}F_{i-1}$ respectively, and $s_{i-1}(\mathfrak{m}F_{i-1}) \subset \mathfrak{m}F_i$, so the image of $\mathbf{1}_{F_i} - \beta_i\alpha_i$ is contained in $\mathfrak{m}F_i$.

Relative to any system of generators of F_i, $\beta_i\alpha_i$ is represented by a matrix with entries in R. The image in k of the determinant of this matrix is 1, so the determinant is a unit in R and consequently Cramer's rule gives an inverse γ_i. Of course the various γ_i constitute a chain map $\gamma : F \to F$, and $\gamma\beta$ is a splitting chain map for α.

Let H be the cokernel of α. Then $G \cong F \oplus H$ and the homomology of G is the direct sum of the homology of F and the homology of H. Since α induces an isomorphism between the homology of F and the homology of G, H has no homology, and is trivial by Lemma I2.4. □

If G is another minimal free resolution of M, this result gives an isomorphism between F and G. In this sense we may speak of *the* minimal free resolution of M. One may suspect that the requirement that it have no homology makes the minimal free resolution large relative to chain complexes that end at M and do not contain trivial complexes. The following is a result of this sort.

Proposition I2.6. *If R is Noetherian and local, and $x_1, \dots, x_n$ is a minimal set of generators of $\mathfrak{m}$, then the extended Koszul complex*

$$\tilde{K}(x) : 0 \to \tilde{K}_n(x) \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_1} \tilde{K}_0(x) \xrightarrow{\delta_0} k \to 0$$

(where $x = (x_1, \dots, x_n)$) is isomorphic to a subcomplex of the minimal free resolution of k.

Proof. Let $F : \cdots \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} k \to 0$ be the minimal free resolution. Since this complex is exact and each $\tilde{K}_i(x)$ is free, Lemma C3.1 gives a chain map $f : \tilde{K}(x) \to F$ extending the identity function on k. We will show that each $f_i : \tilde{K}_i(x) \to F_i$ injective.

Observe that the image of δ_1 is the kernel $\mathfrak{m}$ of δ_0 and (because $x_1, \dots, x_n$ is minimal) $\tilde{K}_1(x) \to \tilde{K}_0(x) \to k \to 0$ could be extended to a minimal free resolution. The method of proof of Lemma C3.1 is inductive, and could be used to extend f_0 and f_1 to a chain map from this resolution to F, at which point the last result would imply that this chain map was an isomorphism. Consequently f_0 and f_1 are isomorphisms. By induction we may assume that f_{i-1} is injective, and our goal is to show that f_i injective. Since it is true of f_0 and f_1, we may assume that f_{i-1} splits, provided we can show that f_i splits.

It will be enough to show that the quotient map $\tilde{f}_i : \tilde{K}_i(x)/\mathfrak{m}\tilde{K}_i(x) \to F_i/\mathfrak{m}F_i$ is injective. To see why, suppose that this is the case, $\tilde{v}_1, \dots, \tilde{v}_k$ is a basis of the domain, and $\tilde{w}_j = \tilde{f}_i(\tilde{v}_j)$. Choose representatives $v_1, \dots, v_k \in \tilde{K}_i(x)$, let $w_j = f_i(v_j)$, choose $\tilde{w}_{k+1}, \dots, \tilde{w}_n$ such that $\tilde{w}_1, \dots, \tilde{w}_n$ is a basis, and choose representatives $w_{k+1}, \dots, w_n \in F_i$. Nakayama's lemma implies that $w_1, \dots, w_n$ is a system of generators of F_i, and in fact (Corollary A3.4) they generate it freely, so $r_1 w_1 + \cdots r_n w_n \mapsto r_1 v_1 + \cdots r_k v_k$ is a well defined splitting map for f_i.

Since F is minimal, the image of φ_i is contained in $\mathfrak{m}F_{i-1}$, so there is an induced map $\tilde{\varphi}_i : F_i/\mathfrak{m}F_i \to \mathfrak{m}F_{i-1}/\mathfrak{m}^2 F_{i-1}$, and it suffices to show that $\tilde{\varphi}_i \circ \tilde{f}_i$ is injective. From the definition of δ_i we have $\delta_i(\tilde{K}_i(x)) \subset \mathfrak{m}\tilde{K}_{i-1}(x)$, so there is an induced map $\tilde{\delta}_i : \tilde{K}_i(x)/\mathfrak{m}\tilde{K}_i(x) \to \mathfrak{m}\tilde{K}_{i-1}(x)/\mathfrak{m}^2\tilde{K}_{i-1}(x)$. There is also an induced map $\hat{f}_{i-1} : \mathfrak{m}\tilde{K}_{i-1}(x)/\mathfrak{m}^2\tilde{K}_{i-1}(x) \to \mathfrak{m}F_{i-1}/\mathfrak{m}^2 F_{i-1}$. Since f_{i-1} splits, the elements that it maps to $\mathfrak{m}^2 F_{i-1}$ are precisely those in $\mathfrak{m}^2\tilde{K}_{i-1}(x)$, so $\hat{f}_{i-1}$ is injective. Of course $\tilde{\varphi}_i \circ \tilde{f}_i = \hat{f}_{i-1} \circ \tilde{\delta}_i$, so it now suffices to show that $\tilde{\delta}_i$ is injective.

Identifying (Lemma A6.3) $\tilde{K}_i(x)/\mathfrak{m}\tilde{K}_i(x)$ and $\mathfrak{m}\tilde{K}_{i-1}(x)/\mathfrak{m}^2\tilde{K}_{i-1}(x)$ with $k \otimes_R \wedge^i R^n$ and $\mathfrak{m}/\mathfrak{m}^2 \otimes_R \wedge^{i-1} R^n$ respectively, $\tilde{\delta}_i$ is

$$\sum_{j_1 < \cdots < j_i} a_{j_1 \cdots j_k} \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_i} \mapsto \sum_{j_1 < \cdots < j_i} a_{j_1 \cdots j_k} \sum_{s=1}^{i} \tilde{x}_{j_s} \otimes \mathbf{e}_{j_1} \wedge \cdots \wedge \hat{\mathbf{e}}_{j_s} \wedge \cdots \wedge \mathbf{e}_{j_i}$$

where $\tilde{x}_1, \ldots, \tilde{x}_n$ are the images of $x_1, \ldots, x_n$ in $\mathfrak{m}/\mathfrak{m}^2$. Since $x_1, \ldots, x_n$ is minimal, $\tilde{x}_1, \ldots, \tilde{x}_n$ is a basis of this k-vector space, so the right hand side is zero only when every $a_{j_1 \cdots j_i}$ vanishes. □

I3 The Hilbert Syzygy Theorem

In this section we show that $\operatorname{gl\,dim} K[X_1, \ldots, X_n] = n$, which is a variant of Hilbert's syzygy theorem.

Proposition I3.1. *If R is Noetherian and local, M is a finitely generated R-module, and*

$$F\ : 0 \to F_n \to \cdots \to F_0 \xrightarrow{\varepsilon} M \to 0$$

is a minimal free resolution of M, then n is the free dimension of M, which agrees with the projective dimension of M. In addition, $n \leq \operatorname{pd}_R k$.

Proof. Of course the free dimension of M cannot be greater than n. By definition $\operatorname{Tor}^R(k, M)$ is the homology of the complex

$$\cdots \to k \otimes_R F_2 \xrightarrow{\mathbf{1} \otimes_R \varphi_2} k \otimes_R F_1 \xrightarrow{\mathbf{1} \otimes_R \varphi_1} k \otimes_R F_0 \to 0.$$

Proposition I2.3 gives $\varphi_{i+1}(F_{i+1}) \subset \mathfrak{m}F_i$ for all i, which implies that each of the maps in this complex is zero: for example, an image of $\mathbf{1} \otimes_R \varphi_i$ lies in $k \otimes_R \mathfrak{m}F_{i-1} = \mathfrak{m}k \otimes_R F_{i-1}$, and $\mathfrak{m}k = \mathfrak{m}(R/\mathfrak{m}) = 0$. Therefore $\operatorname{Tor}_n^R(k, M) = k \otimes_R F_n \neq 0$. Since any projective resolution of M could be used to compute $\operatorname{Tor}_n^R(k, M)$, it follows that $\operatorname{pd}_R M \geq n$.

On the other hand a projective resolution of k can be used to compute $\operatorname{Tor}^R(k, M)$, so $\operatorname{Tor}_{n+1}^R(k, M) = 0$ implies that $n \leq \operatorname{pd}_R k$. □

Suppose that R is Noetherian and local. For any ideal I, R/I is finitely generated ($1 + I$ is a generator) so Lemma I2.2 implies that R/I has a minimal free resolution, and the last result implies that $\operatorname{pd}_R R/I \leq \operatorname{pd}_R k$. Consequently Auslander's theorem implies that $\operatorname{gl\,dim} R \leq \operatorname{pd}_R k$. The opposite inequality holds by definition, so:

Proposition I3.2. *If R is a Noetherian local ring, then* $\operatorname{gl\,dim}\, R = \operatorname{pd}_R k$.

Corollary I3.3. *If R is regular, then its dimension is equal to its global dimension.*

Proof. If $x_1, \ldots, x_n$ generate $\mathfrak{m}$, then the Koszul complex $K(x_1, \ldots, x_n)$ is a minimal free resolution of k. Thus the dimension n of R agrees with $\operatorname{pd}_R k$, which is the global dimension of R. □

If k is a field, then $k[X_1, \ldots, X_n]$ is Noetherian (Hilbert basis theorem) and local with maximal ideal $(X_1, \ldots, X_n)$ and residue field k. In addition, $X_1, \ldots, X_n$ is a regular sequence on $k[X_1, \ldots, X_n]$ that lies in $(X_1, \ldots, X_n)$. Combining the last result with Theorem G6.3 yields:

Theorem I3.4 (Hilbert Syzygy Theorem)**.** *For any field k and any integer $n \geq 0$,*

$$\operatorname{gl}\dim k[X_1, \ldots, X_n] = \operatorname{pd}_{k[X_1,\ldots,X_n]} k = n.$$

I4 The Auslander-Buchsbaum Formula

Theorem I4.1 (Auslander-Buchsbaum Formula)**.** *If R is Noetherian and local, and $M \neq 0$ is a finitely generated R-module of finite projective dimension, then*

$$\operatorname{pd}_R M = \operatorname{depth}(\mathfrak{m}, R) - \operatorname{depth}(\mathfrak{m}, M).$$

Proof. We argue by induction on $\operatorname{pd}_R M$. This is zero if and only if M is free, in which case the M-sequences in $\mathfrak{m}$ are precisely the R-sequences in $\mathfrak{m}$. Therefore suppose that $\operatorname{pd}_R M > 0$.

Lemma I2.2 gives a minimal free resolution of M, say

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0.$$

Let N be the image of $F_1 \to F_0$. Then

$$0 \to F_n \to \cdots \to F_1 \to N \to 0$$

is a minimal free resolution of N. From Proposition I3.1 we have $\operatorname{pd}_R M = \operatorname{pd}_R N + 1$, so it suffices to show that $\operatorname{depth}(\mathfrak{m}, M) = \operatorname{depth}(\mathfrak{m}, N) - 1$.

Let $d = \operatorname{depth}(\mathfrak{m}, N)$ and $d' = \operatorname{depth}(\mathfrak{m}, R)$. Let $x = (x_1, \ldots, x_r)$ be a system of generators for $\mathfrak{m}$. By Theorem G4.6 it suffices to show that $H^i(M \otimes_R K(x)) = 0$ for all $i < d-1$ and $H^{d-1}(M \otimes_R K(x)) \neq 0$. There is a short exact sequence $0 \to N \xrightarrow{\varphi} F_0 \to M \to 0$ which gives rise to a long exact sequence

$$\cdots \to H^i(F_0 \otimes_R K(x)) \to H^i(M \otimes_R K(x)) \to H^{i+1}(N \otimes_R K(x))$$

$$\to H^{i+1}(F_0 \otimes_R K(x)) \to \cdots .$$

Theorem G4.6 implies that $H^i(N \otimes_R K(x)) = 0$ for all $i < d$ and $H^d(N \otimes_R K(x)) \neq 0$. By the induction hypothesis, $\operatorname{depth}(\mathfrak{m}, R) \geq \operatorname{depth}(\mathfrak{m}, N)$. As we argued at the outset, since F_0 is free we have $\operatorname{depth}(\mathfrak{m}, F_0) = \operatorname{depth}(\mathfrak{m}, R)$. The result in this case has already been established, so $H^i(F_0 \otimes_R K(x)) = 0$ for all $i < d'$ and $H^{d'}(F_0 \otimes_R K(x)) \neq 0$.

For $i < d-1$ the sequence above now gives $H^i(M \otimes_R K(x)) = 0$. In addition, $H^{d-1}(M \otimes_R K(x)) \neq 0$ will follow if we show that $H^d(N \otimes_R K(x)) \to H^d(F_0 \otimes_R K(x))$ is zero. When $d' > d$ this is the case because $H^d(F_0 \otimes_R K(x)) = 0$.

Therefore we may suppose that $d' = d$, so now the induction hypothesis gives $\operatorname{pd}_R N = 0$. In the last section we saw that the projective dimension of N is the free dimension, so N is free. Therefore

$$H^d(N \otimes_R K(x)) = N \otimes_R H^d(K(x)) \text{ and } H^d(F_0 \otimes_R K(x)) = F_0 \otimes_R H^d(K(x)).$$

(The calculation $H^*(R^n \otimes_R X) = H^*(X^n) = (H^*(X))^n = R^n \otimes_R H^*(X)$ is valid for any cochain complex and any n.) Now φ is the inclusion of N in F_0, and since the resolution is minimal, $N \subset \mathfrak{m}F_0$. Corollary G3.2 implies that $H^d(K(x))$ is annihilated by elements of $\mathfrak{m}$, so $N \otimes_R H^d(K(x))$ vanishes inside $F_0 \otimes_R H^d(K(x))$. □

The following famous criterion for regularity of a Noetherian local ring is due to Auslander-Buchsbaum and Serre.

Theorem I4.2. *If R is Noetherian and local, then the following are equivalent:*

(a) R has finite global dimension;

(b) $\operatorname{pd}_R k < \infty$;

(c) R is regular.

Proof. When R is regular its global dimension is equal to its dimension (Corollary I3.3) which is finite. Of course (a) implies (b) by definition. It remains to show that (b) implies (c), so suppose $\operatorname{pd}_R k < \infty$. The Auslander-Buchsbaum formula gives $\operatorname{pd}_R k = \operatorname{depth}(\mathfrak{m}, R)$.

Let $x_1, \ldots, x_n$ be a minimal set of generators of $\mathfrak{m}$; showing that $\dim R = n$ will fulfill the definition of regularity. The principal ideal theorem (specifically Corollary F4.10) gives $\dim R \leq n$. In conjunction with de Rham's theorem (Theorem G4.4) Proposition H1.7 implies that $\operatorname{depth}(\mathfrak{m}, R) \leq \operatorname{codim} \mathfrak{m} = \dim R$, so it suffices to show that $\operatorname{pd}_R k \geq n$, but this follows from Proposition I3.1 ($\operatorname{pd}_R k$ is the free dimension of k) and Proposition I2.6 (the Koszul complex $\tilde{K}(x_1, \ldots, x_n)$ embeds in the minimal free resolution of k). □

Corollary I4.3. *If R is regular and P is a prime, then R_P is regular.*

Proof. In view of the last result it suffices to show that R_P has finite global dimension, and from Proposition I3.1 it suffices to prove that the residue field R_P/P_P has finite projective dimension. Since R is a regular local ring, it has finite global dimension, so R/P has a finite free resolution as an R-module. Since localization at P is an exact functor (Proposition A5.1) and F_P is a free R_P-module whenever F is a free R-module (if $R^n \to F \to 0$ is exact, then so is $R_P^n \to F_P \to 0$) localization of this gives a finite free resolution of $(R/P)_P = R_P/P_P$ as an R_P-module. □

I5 A Characterization of Projectivity

This section develops one of the results supporting the argument in the next section. It is a simple and seemingly quite natural characterization of projective modules, but its proof is quite subtle.

Lemma I5.1. *If S is an R-algebra, there is a natural transformation α from the bifunctor $S \otimes_R \mathrm{Hom}_R(-,-)$ to the bifunctor $\mathrm{Hom}_S(S \otimes_R -, S \otimes_R -)$ given by the S-module homomorphisms*

$$\alpha_{M,N} : S \otimes_R \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)$$

that take $1 \otimes \varphi$ in the domain to itself, regarded as an element of the range.

Proof. Clearly the homomorphisms $\alpha_{M,N}$ are well defined. If $f : M \to M'$ is a R-module homomorphism and $\varphi' \in \mathrm{Hom}_R(M', N)$, then

$$\alpha_{M,N} \circ (S \otimes_R \mathrm{Hom}_R(f,N)) \quad \text{and} \quad \mathrm{Hom}_S(S \otimes_R f, S \otimes_R N) \circ \alpha_{M',N}$$

both take $1 \otimes \varphi'$ to $(1 \otimes \varphi') \circ (1 \otimes f) = 1 \otimes (\varphi' \circ f)$. The proof of naturality with respect to the second argument is similar. □

Proposition I5.2. *Under the hypotheses of the last result, if S is a flat R-module and M is finitely presented, then $\alpha_{M,N}$ is an isomorphism.*

Proof. If $R^q \to R^p \to M \to 0$ is a finite presentation of M, the right exactness of the tensor product gives an exact sequence

$$S \otimes_R R^q \to S \otimes_R R^p \to S \otimes_R M \to 0,$$

and the left exactness of $\mathrm{Hom}_R(-,N)$ and $\mathrm{Hom}_S(-, S \otimes_R N)$ give exact sequences

$$0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(R^p,N) \to \mathrm{Hom}_R(R^q,N)$$

and

$$\begin{aligned} 0 \to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) &\to \mathrm{Hom}_S(S \otimes_R R^p, S \otimes_R N) \\ &\to \mathrm{Hom}_S(S \otimes_R R^q, S \otimes_R N). \end{aligned}$$

Since S is flat, tensoring with the first of these gives an exact sequence

$$0 \to S \otimes_R \mathrm{Hom}_R(M,N) \to S \otimes_R \mathrm{Hom}_R(R^p,N) \to S \otimes_R \mathrm{Hom}_R(R^q,N).$$

We now form the commutative diagram whose top row is the last exact sequence, whose bottom row is the immediately preceeding sequence, and whose vertical maps are given by α.

It is easy to see that $\alpha_{R,N}$ is in effect the identity map on $S \otimes_R N$. Insofar as Hom commutes with direct sums, it follows that for any n, $\alpha_{R^n,N}$ is an isomorphism. Thus the two right hand vertical maps in this diagram are isomorphisms, so the claim follows from the five lemma after we add a pair of zeros on the left. □

If S is a multiplicatively closed subset of R, then $S^{-1}R$ is flat (Proposition A6.8) so the last result gives:

Corollary I5.3. *If S is a multiplicatively closed subset of R and M and N are R-modules with M finitely presented, then*

$$\mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \cong S^{-1}\mathrm{Hom}_R(M, N).$$

The following result was shown by Kaplansky to not require the hypothesis of finite generation, but the proof becomes much harder.

Lemma I5.4. *If R is a local ring and M is a finitely generated projective R-module, then M is free.*

Proof. Let $0 \to K \to F \to M \to 0$ be an exact sequence with F a free module with a minimal number of generators $x_1, \dots, x_n$. Nakayama's lemma (Theorem A2.13) implies that a minimal set of generators of M go to a basis of $M/\mathfrak{m}$. It follows that $F/\mathfrak{m}F \to M/\mathfrak{m}M$ is a linear isomorphism, and in particular, if $\sum_i a_i x_i$ maps to $0 \in M$, then $a_i \in \mathfrak{m}$ for all i. That is, the image of K is contained in $\mathfrak{m}F$. Since M is projective, (b) of Proposition B7.2 implies that the sequence splits, so there is a map $\psi : M \to F$ such that $F = K \oplus \psi(M)$. Thus $K = \mathfrak{m}K$, and there is a surjection $F \to K$, so K is finitely generated. Consequently Nakayama's lemma implies that $K = 0$. $\square$

Theorem I5.5. *If M is a finitely presented R-module, then M is projective if and only if for all maximal ideals $\mathfrak{m}$, $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module.*

Proof. First suppose that M is projective, and let $\mathfrak{m}$ be a maximal ideal. Then M is a direct factor of a free module, so there is a short exact sequence $0 \to K \xrightarrow{i} F \xrightarrow{p} M \to 0$ with F free and a splitting map $q : M \to F$. Since localization at $\mathfrak{m}$ is an exact functor (Proposition A5.1) the sequence $0 \to K_\mathfrak{m} \xrightarrow{i_\mathfrak{m}} F_\mathfrak{m} \xrightarrow{p_\mathfrak{m}} M_\mathfrak{m} \to 0$ is exact, and $p_\mathfrak{m} \circ q_\mathfrak{m} = \mathbf{1}_{M_\mathfrak{m}}$. Of course $F_\mathfrak{m}$ is free, so $M_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module, hence free by the last result.

Now suppose that for any maximal ideal $\mathfrak{m}$, $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module. We need to show that for any surjection $N \to N'$, the cokernel C of $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N')$ is zero. There is an exact sequence

$$\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to C \to 0,$$

and since localization is an exact functor, for each maximal ideal $\mathfrak{m}$ there is an exact sequence

$$\mathrm{Hom}_R(M, N)_\mathfrak{m} \to \mathrm{Hom}_R(M, N')_\mathfrak{m} \to C_\mathfrak{m} \to 0.$$

Corollary I5.3 gives a natural equivalence between the functors $\mathrm{Hom}_R(M, -)_\mathfrak{m}$ and $\mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, -_\mathfrak{m})$, so the first map is surjective if and only if

$$\mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) \to \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, N'_\mathfrak{m})$$

is surjective. But $N_\mathfrak{m} \to N'_\mathfrak{m}$ is surjective because localization at $\mathfrak{m}$ is an exact functor, and $M_\mathfrak{m}$ is free and consequently projective, so this homomorphism is surjective. Thus $C_\mathfrak{m} = 0$. Since this is true for every maximal ideal $\mathfrak{m}$, Lemma A5.8 implies that $C = 0$. □

I6 Factoriality of Regular Local Rings

An R-module is *stably free* if the direct sum with some free R-module is free. The following observation is due to Serre.

Proposition I6.1. *A projective R-module M is stably free if it has a finite free resolution $F\ : 0 \to F_n \to \cdots \to F_0 \to M \to 0$.*

Proof. For $i = 0, 1, 2, \ldots$ let M_i be the image of F_{i+1}. Since M is projective, $F_0 \to M$ splits, so (up to isomorphism) $F_0 = M \oplus M_0$, and M_0 is projective. By induction, for all i, M_i is projective, $F_{i+1} = M_i \oplus M_{i+1}$, and thus M_{i+1} is projective. We now have

$$M \oplus F_1 \oplus F_3 \oplus \cdots = M \oplus (M_0 \oplus M_1) \oplus (M_2 \oplus M_3) \oplus \cdots$$
$$\cong (M \oplus M_0) \oplus (M_1 \oplus M_2) \oplus \cdots = F_0 \oplus F_2 \oplus \cdots .$$

□

In one case a stably free module is free.

Lemma I6.2. *If R is an integral domain and M is an R-module such that, for some n, $M \oplus R^{n-1} \cong R^n$, then $M \cong R$.*

Proof. We have

$$R \cong \wedge^n R^n \cong \wedge^n(M \oplus R^{n-1}) \cong \oplus_{i=0}^n (\wedge^i M \otimes_R \wedge^{n-i} R^{n-1}).$$

Since $M \otimes_R \wedge^{n-1} R^{n-1} = M \otimes_R R = M$, we have $R \cong M \oplus N$ for some R-module N, and it suffices to show that $N = 0$. Let K be the field of fractions of R. Then $K = R \otimes_R K = M \otimes_R K \oplus N \otimes_R K$ is a vector space of rank 1, so either $M = 0$ or $N = 0$. To rule out $M = 0$ we observe that $K^n = R^n \otimes_R K = (M \oplus R^{n-1}) \otimes_R K = M \otimes_R K \oplus K^{n-1} = K^{n-1}$ is impossible by linear algebra. □

Corollary I6.3. *If R is an integral domain and $I \neq 0$ is a stably free ideal, then I is principal.*

Proof. Let K be the field of fractions of R. Since $I \neq 0$, $I \otimes_R K \cong K$. If $I \oplus R^m \cong R^n$, then tensoring with K reveals that $m = n - 1$, and we may apply the last result. □

Lemma I6.4. *If R is a Noetherian integral domain, x is a prime element, P is a codimension 1 prime of R, and $PR[x^{-1}]$ is a (possibly improper) principal ideal of $R[x^{-1}]$, then P is principal.*

Proof. First suppose that $PR[x^{-1}] = R[x^{-1}]$. Then $f(x^{-1}) = 1$ for some polynomial f with coefficients in P, and multiplying by a suitable power of x shows that $x^k \in P$ for some positive k. Since P is prime, $x \in P$, and since x is prime, (x) is a prime ideal. Furthermore, P is minimal over (x) because its codimension is one, whence $P = (x)$.

Therefore we may suppose that $PR[x^{-1}]$ is a proper ideal. By assumption $PR[x^{-1}] = aR[x^{-1}]$ for some a, which may be taken in R because x is a unit in $R[x^{-1}]$. Among such $a \in R$ we can choose one such that aR is maximal.

First suppose that $a = a'x$ for some $a' \in R$. Then $a' \in P$ because P is prime, whence $(a') = (a)$, so $a' = ay$ for some y. This gives $a = ayx$ and $xy = 1$, but by assumption x is not a unit in R.

Therefore $a \notin (x)$. We will show that $aR = P$. Consider $y \in P$. Since $y \in PR[x^{-1}]$, $x^n y = ra$ for some $r \in R$ and $n \geq 0$, which we may assume is minimal. But x is prime and $a \notin (x)$, so if $n > 0$, then r is divisible by x, say $r = qx$. Then $x^{n-1}y = qa$, contrary to minimality. □

Proposition I6.5. *If R is a Noetherian integral domain, x is a prime element, and $R[x^{-1}]$ is factorial, then R is factorial.*

Proof. In view of the 'if' part of Proposition F6.5, it suffices to show that if a given prime P of R has codimension 1, then it is principal. If $x \in P$, then $(x) \subset P$, and (x) is not minimal, so its codimension is positive, and consequently $P = (x)$.

Now suppose that $x \notin P$. Since P is prime, $P \cap \{1, x, x^2, \ldots\} = \emptyset$. Insofar as the primes of $R[x^{-1}]$ are the $QR[x^{-1}]$ for those primes Q of R that do not intersect $\{1, x, x^2, \ldots\}$ (Proposition A5.6) the codimension of $PR[x^{-1}]$ is 1. Since $R[x^{-1}]$ is factorial, the 'only if' part of Proposition F6.5 implies that $PR[x^{-1}]$ is principal, and the last result implies that P is principal. □

Theorem I6.6 (Nagata, Auslander-Buchsbaum)**.** *If R is a regular local ring, then it is factorial.*

Proof. We argue by induction on the dimension of R. If the dimension is zero, then R is a field, and automatically a UFD. By induction we may assume that the claim has already been established for all regular local rings with dimension less than $\dim R$.

Let x be an member of a set of $\dim R$ elements of $\mathfrak{m}$ that generate $\mathfrak{m}$. Then $R/(x)$ is local with maximal ideal $\mathfrak{m}/(x)$. By the principal ideal theorem, the codimension of (x) is at most one, so $\dim R/(x) \geq \dim R - 1$. Since $\mathfrak{m}/(x)$ can be generated by $\dim R - 1$ elements, $\dim R/(x) \leq \dim R - 1$, so $\dim R/(x) = \dim R - 1$, and $R/(x)$ is regular. Therefore $R/(x)$ is factorial.

Since it is an integral domain, (x) is a prime ideal, i.e., x is a prime element. By the last result it suffices to show that $R[x^{-1}]$ is factorial, or equivalently (Proposition F6.5) that each of its codimension 1 primes is principal.

As we mentioned above, the primes of $R[x^{-1}]$ are the $P'R[x^{-1}]$ for those primes P' of R that do not intersect $\{1, x, x^2, \ldots\}$. Suppose that $P = P'R[x^{-1}]$ has codimension one. Below we will show that P is a projective $R[x^{-1}]$-module. To see that this suffices note that P' has a finite free resolution (Proposition I3.1 and Theorem I4.2) and localizing it gives a finite free resolution of P. (This was explained at the end of the proof of Corollary I4.3.) Then Proposition I6.1 implies that P is stably free, after which it is principal by Corollary I6.3.

Of course P is a finitely presented $R[x^{-1}]$-module (Proposition A4.7) so Theorem I5.5 implies that P is projective if and only if, for each maximal ideal $Q = Q'R[x^{-1}]$ of $R[x^{-1}]$, P_Q is a free $R[x^{-1}]_Q$-module. Note that

$$R[x^{-1}]_Q = ((R \setminus Q') \cdot \{1, x, x^2, \ldots\})^{-1} R$$

is a localization of R, and is consequently regular by Corollary I4.3. In view of the characterization of the primes of $R[x^{-1}]$, its dimension is less than the dimension of R, so it is factorial. If $P' \subset Q'$, then P_Q is a codimension 1 prime of $R[x^{-1}]_Q$, so it is principal, and if P' is not contained in Q', then $P_Q = R[x^{-1}] = (1)$. □

This argument is due to Kaplansky. Nagata (1958) had shown that if the result held for rings of dimension three, then it held in all higher dimensions, and Auslander and Buchsbaum (1959) then proved the low dimensional cases. (One may suspect that the standard attribution of the result to Auslander and Buchsbaum unfairly slights Nagata's contribution.) In spite of its complexity and sophistication, the argument presented here is nevertheless a considerable improvement on that convoluted situation. Additional information can be found on pp. 130–131 of Kaplansky (1974).

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What It's All About

Although the definition of a sheaf is fairly straightforward, just why these are useful bundles of mathematical information is not obvious at the beginning. In order to give some concrete sense of where things are headed we develop some simple examples, using them to briefly illustrate the main concepts. Roughly, our objective here is to create an initial acquaintance with the concepts "sheaf," "coherent," and "algebraic," and to indicate the sorts of information one might hope to get from cohomology.

The following discussion is at the beginning of the theory of Riemann surfaces, and we will presume some knowledge of complex analysis, and the definition of a smooth (that is, C^∞) manifold. The presentation is as stripped down as possible; in particular differential k-forms for $k > 2$ could be defined, and all operations and properties generalized beyond the special cases we consider, without any additional concepts, but it would greatly complicate the algebra.

Let X be an n-dimensional C^∞ manifold. For every open $U \subset X$ let C^∞_U be the set of C^∞ functions $f : U \to \mathbb{R}$. This is a commutative ring with unit. If $U' \subset X$ is another open set containing U, there is a unitary ring homomorphism $\varphi^{U'}_U : C^\infty_{U'} \to C^\infty_U$ that takes f to $f|_U$. Intuitively we think of the sheaf of C^∞ real valued functions on X as the collection of all such objects C^∞_U and $\varphi^{U'}_U$.

However, there is a different point of view that Serre takes as primary. Two C^∞ functions $f \in C^\infty_U$ and $f' \in C^\infty_{U'}$ *have the same germ at* $x \in U \cap U'$ if they agree on some neighborhood of x. This is an equivalence relation, and the equivalence classes are the *germs of* C^∞ *functions at* x. Let C^∞_x be this space of germs, and if $f \in C^\infty_U$ and $x \in U$, let f_x be the germ of f at x. With addition and multiplication defined in the obvious ways, C^∞_x is a commutative ring with unit. For Serre the *sheaf of* C^∞ *functions on* X is $\bigcup_{x \in X} C^\infty_x$ endowed with the topology whose open sets are the unions of the sets of the form $\{ f_x : x \in U \}$ where $f \in C^\infty_U$.

To have a richer collection of examples, with some less trivial operations, we develop some spaces of differential forms, using methods that are concrete and quite old fashioned. (Doing things in a more modern manner has up front costs that we wish to avoid.) We regard forms as systems of numerical functions defined on each possible coordinate chart, where the functions on different charts are related by certain transformation laws.

Let $V, V', V'' \subset \mathbb{R}^n$ be open with typical elements $(x_1, \ldots, x_n)$, $(x'_1, \ldots, x'_n)$, and $(x''_1, \ldots x''_n)$. Let $T' : V'' \to V'$ and $T : V' \to V$ be C^∞ diffeomorphisms. Written in terms of the coordinates,

$$T(x'_1, \ldots, x'_n) = (x_1(x'_1, \ldots, x'_n), \ldots, x_n(x'_1, \ldots, x'_n))$$

and

$$T'(x''_1, \ldots, x''_n) = (x'_1(x''_1, \ldots, x''_n), \ldots, x'_n(x''_1, \ldots, x''_n)).$$

Furthermore, there are partial derivatives $\frac{\partial x_i}{\partial x'_j}$ and $\frac{\partial x'_j}{\partial x''_k}$.

A C^∞ *0-form* on V is a C^∞ function $f : V \to \mathbb{R}$. For such an f we let $T^*(f) = f \circ T$ be the induced C^∞ 0-form on V'. Obviously

$$(T \circ T')^*(f) = T'^*(T^*(f)).$$

A C^∞ *1-form on* V is an expression of the form $\omega = \sum_i \omega_i dx_i$ where $\omega_1, \ldots, \omega_n : V \to \mathbb{R}$ are C^∞ functions. We take a formal approach, according to which $dx_1, \ldots, dx_n$ are just symbols in no need of definition. (Intuitively we think of them as C^∞ 1-forms like any others.) For such an ω there is an associated C^∞ 1-form $\omega' = T^*(\omega)$ on V' given by

$$\omega'_j(x') = \sum_{i=1}^n \frac{\partial x_i}{\partial x'_j}(x')\omega_i(x)$$

where $x = T(x')$. We define T'^* similarly, and it is evident that the equation $(T \circ T')^*(\omega) = T'^*(T^*(\omega))$ is a consequence of the chain rule.

A C^∞ *2-form on* V is an expression of the form $\eta = \sum_{i<j} \eta_{ij} dx_i \wedge dx_j$ where the $\eta_{ij} : V \to \mathbb{R}$ are C^∞ functions. Again, formally, the $dx_i \wedge dx_j$ are symbols in no need of definition. Here they serve as the generators of an $\mathbb{R}$-module, and we adopt the rule that $dx_j \wedge dx_i = -dx_i \wedge dx_j$. For a C^∞ 2-form η on V there is an associated C^∞ 2-form $\eta' = T^*(\eta)$ on V' given by

$$\eta_{k\ell}(x') = \sum_{i<j} \begin{vmatrix} \frac{\partial x_i}{\partial x'_k}(x') & \frac{\partial x_i}{\partial x'_\ell}(x') \\ \frac{\partial x_j}{\partial x'_k}(x') & \frac{\partial x_j}{\partial x'_\ell}(x') \end{vmatrix} \eta_{ij}(x)$$

where $x = T(x')$, and again we define T'^* similarly. In this case it is perhaps not so obvious that the chain rule implies that $(T \circ T')^*(\eta) = T'^*(T^*(\eta))$, but it does, and there is little to be learned from the calculation, so we omit it.

A C^∞ *0-form on* X is an assignment of a C^∞ 0-form ω_φ on V to each C^∞ coordinate chart $\varphi : U \to V$ (here $U \subset X$ and $V \subset \mathbb{R}^n$ are open) in such a manner that $f_{\varphi'} = T^*(f_\varphi)$ on $\varphi'(U \cap U')$ whenever $\varphi' : U' \to V'$ is another such chart and

$$T = \varphi \circ \varphi'^{-1} : \varphi'(U \cap U') \to \varphi(U \cap U').$$

Similarly, a C^∞ *1-form (2-form) on X* is an assignment of a C^∞ 1-form ω_φ (2-form η_φ) on V to each C^∞ coordinate chart $\varphi : U \to V$ in such a way that $\omega_{\varphi'} = T^*(\omega_\varphi)$ $(\eta_{\varphi'} = T^*(\eta_\varphi))$ on $\varphi'(U \cap U')$ whenever φ' and T are as above. Note that in order to describe a k-form, it suffices to specify it on any atlas, because the equation $(T \circ T')^*(\omega) = T'^*(T^*(\omega))$ implies that the transformation rule extends it, in a consistent way, to any coordinate chart. For $k = 0, 1, 2$ let $\Omega^k(U)$ be the space of C^∞ k-forms on U.

The rules for transforming k-forms under a change of coordinates are designed to make a k-form something that can be meaningfully integrated over a k-dimensional object in X. (When $k = 0$ "integration" is just function evaluation.) The idea is that such an integral can be defined, locally relative to the chart φ, by integration in V, which is something we already understand, and that this definition does not depend on φ, by virtue of the change of variables formula for integration. We will assume the reader understands (perhaps only intuitively) integrals $\int_\gamma \omega$ of C^∞ 1-forms over oriented curves in X and integrals $\int_S \eta$ of C^∞ 2-forms over oriented surfaces in X.

For each $k = 0, 1, 2$ let $\mathcal{E}^k_X$ be the sheaf of C^∞ k-forms on X. That, is, for each open $U \subset X$, $\mathcal{E}^k_X(U)$ is $\Omega^k(U)$, and $\mathcal{E}^k_X$ is this collection of spaces together with the obvious restriction operators. As we did with C^∞ functions, we can pass to a suitably topologized space of germs. All this is very much like what we did with C^∞ functions, and in fact $\mathcal{E}^0_X$ is just the sheaf of C^∞ functions viewed from a different angle. The novel element is that we may regard each $\mathcal{E}^k_X(U)$ as a C^∞_U-module. Formally, in order for this to make sense, multiplying by a C^∞ function and then transforming under a change of coordinates must give the same result as transforming and then multiplying by the C^∞ function. For the transformation rules above this is obvious.

We now describe the concept of coherence in general. Let A be a general topological space, let R be a sheaf of rings on A, and let M be a sheaf of modules over R. That is, for each open set U there is a commutative ring with unit R_U and an R_U-module M_U, and there are restriction maps $R_{U'} \to R_U$ and $M_{U'} \to M_U$ satisfying a bunch of natural conditions (e.g. $1 \in R_{U'}$ is mapped to $1 \in R_U$ if $U \subset U'$) that Serre will specify in detail soon enough. In order for the sheaf M to be *coherent* (over R) two conditions need to be satisfied:

(a) each point in A has a neighborhood U such that M_U is a finitely generated R_U-module;

(b) if $s_1, \dots, s_m \in M_U$ for some U, then each point in U has a neighborhood $V \subset U$ such that the *module of relations*

$$\{ (r_1, \dots, r_m) \in R^m_V : r_1 s_1|_V + \cdots + r_m s_m|_V = 0 \}$$

is a finitely generated R_V-module.

It turns out that spaces of C^∞ objects are typically *not* coherent. To illustrate this we'll explain why the sheaf of C^∞ functions on $\mathbb{R}$ is not coherent, as a sheaf of modules over itself. Of course it is locally finitely generated, so the problem has to do with (b). Suppose we are given sections $s_1, \ldots, s_m$ over an open U. We can think of this as a C^∞ function $s : U \to \mathbb{R}^m$, and an element of the module of relations over an open $V \subset U$ is then a C^∞ function $r : V \to \mathbb{R}^m$ that maps each t to a point in $s^\perp(t) = \{\, y \in \mathbb{R}^m : \langle y, s(t) \rangle = 0 \,\}$. To be completely concrete, suppose that $0 \in V$, and that $s(0) = 0$. It can easily happen that the subspaces $s^\perp(t)$ spin around as $t \to 0$ in a way that forces any element r of the module of relations to also vanish at 0, in order to be continuous[1]. Moreover, for any k, the k^{th} derivative $r^{(k)}(t)$ cannot converge to a nonzero $v \in \mathbb{R}^m$ as $t \to 0$, because for the smallest such k, $r(t)$ would be well approximated by $t^k v/k!$ when t is small. Given any elements $r^1, \ldots, r^k$ of the module of relations, we can construct[2] an element r that goes to 0 more slowly than any r^i (that is, $\limsup_{t\to 0} \|r(t)\|/\|r^i(t)\| = \infty$ for all i) so r cannot be written as a C_V^∞-linear combination of the r^i.

Apparently coherence can, in general, be quite a subtle issue. In this connection we mention Oka's theorem, which asserts that if $U \subset \mathbb{C}^n$ is open, $\mathcal{H}$ is the ring of holomorphic functions on U, and $\alpha : \mathcal{H}^m \to \mathcal{H}^n$ is an $\mathcal{H}$-module homomorphism, then the kernel of α is a finitely generated $\mathcal{H}$-module. Now it will turn out that, in the applications of interest in FAC, coherence holds automatically, because the rings and modules are Noetherian. In that context it is the consequences of coherence that are subtle.

We now discuss operations deriving new forms from given forms. If f and $\tilde{f}$ are C^∞ 0-forms on V, then $f\tilde{f}$ is also a C^∞ 0-form on V, obviously. It is trivial to verify that $T^*(f\tilde{f}) = T^*(f)T^*(\tilde{f})$. Thus, if we have two 0-forms on X, then their product is also a 0-form on X. If ω is a C^∞ 1-form on V and η is a C^∞ 2-form on V, then $f\omega$ and $f\eta$ are C^∞ 1- and 2-forms. In these cases as well the equations $T^*(f\omega) = T^*(f)T^*(\omega)$ and $T^*(f\eta) = T^*(f)T^*(\eta)$ follow directly from the definitions. Thus we can multiply 1- and 2-forms on X by 0-forms on X.

If $f \in C^\infty(X)$, for each k there is a system of C_U^∞-module homomorphisms $\Omega_k(U) \to \Omega_k(U)$ given by multiplication by $f|_U$. For each U this is a C_U^∞-module homomorphism, and these maps evidently commute with the restriction operators. In general such a thing is called a *sheaf map.* (Again, Serre will soon give the formal details, so we rely on the reader's intuition.) Fortunately there are sheaf maps that are a lot more interesting than multiplication by scalar functions.

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[1]For a fully specified example set $m = 2$, $s_1(t) = \exp(1/|t|)\cos 1/t$ and $s_2(t) = \exp(1/|t|)\sin 1/t$ if $t \neq 0$, and $s_1(0) = s_2(0) = 0$.

[2]For at least one i it works to set $r(t) = \frac{1}{t} r^i(t)$ if $t \neq 0$ and $r(t) = 0$. Since the derivatives of r^i of all orders vanish at 0, for any positive integers k and ℓ the norm of the k^{th} derivative of $r^i(t)$ is bounded by $|t|^\ell$ when t is small. In view of the result of repeated differentiation of r, this implies that the derivatives of r of all orders converge to 0 as $t \to 0$, so r is C^∞.

If f is a C^∞ 0-form on V, there is a C^∞ 1-form df, the *exterior derivative* of f, given by

$$df(x) = \sum_i \frac{\partial f}{\partial x_i}(x)dx_i.$$

To see that $T^*(df) = d(T^*(f))$ we compute that

$$\begin{aligned} T^*(df)_j(x') &= \sum_{i=1}^n \frac{\partial x_i}{\partial x'_j}(x')(df)_i(x) = \sum_{i=1}^n \frac{\partial x_i}{\partial x'_j}(x')\frac{\partial f}{\partial x_i}(x) \\ &= \frac{\partial(f\circ T)}{\partial x'_j}(x') = d(T^*(f))_j(x'). \end{aligned}$$

Thus the formula above defines exterior differentiation for 0-forms on X.

If ω and $\tilde\omega$ are C^∞ 1-forms on V, there is a C^∞ 2-form on V given by the formula

$$\begin{aligned} \omega\wedge\tilde\omega = \Big(\sum_i \omega_i dx_i\Big)\wedge\Big(\sum_j \tilde\omega_j dx_j\Big) &= \sum_i\sum_j \omega_i\tilde\omega_j dx_i\wedge dx_j \\ &= \sum_{i<j}(\omega_i\tilde\omega_j - \omega_j\tilde\omega_i)dx_i\wedge dx_j. \end{aligned}$$

To check that $T^*(\omega\wedge\tilde\omega) = T^*(\omega)\wedge T^*(\tilde\omega)$ we compute that

$$\begin{aligned} (T^*(\omega)\wedge T^*(\tilde\omega))_{k\ell}(x') &= T^*(\omega)_k(x')T^*(\tilde\omega)_\ell(x') - T^*(\omega)_\ell(x')T^*(\tilde\omega)_k(x') \\ &= \sum_{i,j=1}^n \Big(\frac{\partial x_i}{\partial x'_k}(x')\frac{\partial x_j}{\partial x'_\ell}(x') - \frac{\partial x_i}{\partial x'_\ell}(x')\frac{\partial x_j}{\partial x'_k}(x')\Big)\omega_i(x)\tilde\omega_j(x) \\ &= \sum_{i<j}\begin{vmatrix} \frac{\partial x_i}{\partial x'_k}(x') & \frac{\partial x_i}{\partial x'_\ell}(x') \\ \frac{\partial x_j}{\partial x'_k}(x') & \frac{\partial x_j}{\partial x'_\ell}(x') \end{vmatrix}\big(\omega_i(x)\tilde\omega_j(x) - \omega_j(x)\tilde\omega_i(x)\big) \\ &= \sum_{i<j}\begin{vmatrix} \frac{\partial x_i}{\partial x'_k}(x') & \frac{\partial x_i}{\partial x'_\ell}(x') \\ \frac{\partial x_j}{\partial x'_k}(x') & \frac{\partial x_j}{\partial x'_\ell}(x') \end{vmatrix}(\omega\wedge\tilde\omega)_{ij}(x) = T^*(\omega\wedge\tilde\omega)_{k\ell}(x'). \end{aligned}$$

Thus we have defined a wedge product of 1-forms on X.

If ω is a C^∞ 1-form on V, the *exterior derivative* of ω is the C^∞ 2-form $d\omega$ defined by the formula

$$d\big(\sum_j \omega_j dx_j\big) = \sum_j d\omega_j\wedge dx_j.$$

Substituting the definition of $d\omega_j$, then simplifying, gives

$$(d\omega)(x) = \sum_{i<j}\Big(\frac{\partial\omega_j}{\partial x_i}(x) - \frac{\partial\omega_i}{\partial x_j}(x)\Big)dx_i\wedge dx_j.$$

Before checking that $T^*(d\omega) = d(T^*(\omega))$ we derive two extremely important equations. Initially these equations are understood to be valid for forms on V. After they have been used to check that $T^*(d\omega) = d(T^*(\omega))$, so the exterior derivative of 1-forms on X is well defined, their derivations will be valid in any coordinate system, and thus they will true for forms on X. When $\omega = df$ for some C^∞ 0-form f, we obtain $d(df) = 0$:

$$d(df) = \sum_{i<j} \Big(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j}\Big) dx_i \wedge dx_j = 0.$$

On the other hand putting $f\omega$ in place of ω gives $d(f\omega) = df \wedge \omega + f d\omega$:

$$d(f\omega) = \sum_{i<j} \Big(\frac{\partial (f\omega_j)}{\partial x_i} - \frac{\partial (f\omega_i)}{\partial x_j}\Big) dx_i \wedge dx_j$$

$$= \sum_{i<j} \Big(\frac{\partial f}{\partial x_i}\omega_j - \frac{\partial f}{\partial x_j}\omega_i\Big) dx_i \wedge dx_j + f \sum_{i<j} \Big(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j}\Big) dx_i \wedge dx_j$$

$$= \sum_{1 \le i,j \le n} \frac{\partial f}{\partial x_i} \omega_j dx_i \wedge dx_j + f d\omega = \Big(\sum_i \frac{\partial f}{\partial x_i} dx_i\Big) \wedge \Big(\sum_j \omega_j dx_j\Big) + f d\omega.$$

We now check that $T^*(d\omega) = d(T^*(\omega))$. Computing this directly would be extremely laborious, at best. Fortunately the other results above allow a slicker proof. We begin with the observation that d and T^* obviously commute with addition of forms, then apply the equation $T^*(f\omega) = T^*(f)T^*(\omega)$ with ω_i and dx_i in place of f and ω:

$$d(T^*(\omega)) = d(T^*(\sum_i \omega_i dx_i)) = \sum_i d(T^*(\omega_i dx_i)) = \sum_i d(T^*(\omega_i)T^*(dx_i)).$$

Applying the equation $d(f\omega) = df \wedge \omega + f d\omega$ gives

$$d(T^*(\omega_i)T^*(dx_i)) = d(T^*\omega_i) \wedge T^*(dx_i) + (T^*\omega_i) d(T^*(dx_i)).$$

Now $d(T^*(dx_i)) = d(d(T^*(x_i))) = 0$ and

$$d(T^*\omega_i) \wedge T^*(dx_i) = T^*(d\omega_i) \wedge T^*(dx_i) = T^*(d\omega_i \wedge dx_i),$$

so

$$d(T^*(\omega)) = \sum_i T^*(d\omega_i \wedge dx_i) = T^*(\sum_i d\omega_i \wedge dx_i) = T^*(d\omega).$$

Thus we have defined exterior differentiation of 1-forms on X.

Next we describe two famous theorems concerning exterior differentiation, without providing proofs. Suppose that S is a compact 2-dimensional oriented surface in X with one dimensional boundary ∂S. *Stoke's theorem* asserts that if ω is a C^∞ 1 form on X, then

$$\int_S d\omega = \int_{\partial S} \omega,$$

where ∂S is oriented so that proceeding in the positive direction takes you counterclockwise around S (if S is a disk, which is only the simplest case) when one is looking at S from the direction with respect to which it is positively oriented. In particular, $\int_S d\omega = 0$ when $\partial S = \emptyset$.

A C^∞ 1-form ω is *closed* if $d\omega = 0$, and it is *exact* if there is a C^∞ 0-form f such that $\omega = df$. The equation $d(df) = 0$ says precisely that exact forms are closed. The converse is not true in general, but *Poincare's lemma* asserts that it is true "locally." More precisely, if X is an open contractible subset of $\mathbb{R}^n$, and ω is a closed C^∞ 1-form on X, then ω is exact.

In a proper treatment in which C^∞ k-forms, and their exterior derivatives, are defined for all k, Poincare's lemma holds for all k. From this point forward we will assume that $n = 2$, so that X is a surface. If we defined k-forms for $k > 2$, we would find that there are no such nonzero objects, because any $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ has a repeated index and consequently vanishes. So, when $n = 2$, any 2-form is closed, and Poincaré's lemma asserts that it is exact if its domain is an open contractible subset of a Euclidean space.

Like other forms of differentiation, exterior differentiation is a linear operator (that is, an $\mathbb{R}$-module homomorphism) so it induces sheaf maps $d : \mathcal{E}^0_X \to \mathcal{E}^1_X$, $d : \mathcal{E}^1_X \to \mathcal{E}^2_X$, and $d : \mathcal{E}^2_X \to \mathcal{E}^3_X = 0$. In general, a sequence of sheaf maps $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is *exact* if is is exact locally, so that each $x \in X$ has a neighborhood U such that $\mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is an exact sequence of module homomorphisms. Thus Poincare's lemma implies that

$$\mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \mathcal{E}^2_X \to 0$$

is exact. Abusing notation, let $\mathbb{R}$ denote the sheaf of locally constant 0-forms on X; up to formalities, this is the constant sheaf isomorphic to $\mathbb{R}$. It is easily seen to be the kernel of $d : \mathcal{E}^0_X \to \mathcal{E}^1_X$, so when $n = 2$ we have the exact sequence

$$0 \to \mathbb{R} \xrightarrow{\subset} \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \xrightarrow{d} \mathcal{E}^2_X \to 0.$$

It should be stressed that exactness of a sequence of sheaf maps, in the local sense of the definition, does not imply exactness at the level of global sections. Consider a short exact sequence of sheaf maps

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0.$$

Insofar as we may regard $\mathcal{F}$ as contained in $\mathcal{G}$, any global section of $\mathcal{F}$ maps to (actually is) a global section of $\mathcal{G}$, and it maps to zero if and only if it is zero. If a global section of $\mathcal{G}$ comes from (actually is) a global section of $\mathcal{F}$, then it maps to the zero global section of $\mathcal{H}$. Conversely, if a global section of $\mathcal{G}$ maps to zero in $\mathcal{H}$, then it is contained in the image of $\mathcal{F}$ locally, hence also globally, which is to say that it comes from (actually is) a global section of $\mathcal{F}$. Any global section of $\mathcal{G}$ maps to a global section of $\mathcal{H}$, and any global section of $\mathcal{H}$ is in the kernel of $\mathcal{H} \to 0$, of course. However, *there can be global sections of $\mathcal{H}$ that do not come from global sections of $\mathcal{G}$.* Given a global section of $\mathcal{H}$ and a point in X, we can find a preimage in $\mathcal{G}$ of the restriction of the section to some neighborhoods of the point, but there may be no way to paste together these local preimages into a global preimage in $\mathcal{G}$.

As it is applied in FAC, the main point of cohomology is that it provides an "accounting system" for measuring such failures of global exactness, and for systematically relating them to each other, and to other information. The way in which Tor^R and Ext_R provided a detailed analysis of the failures of exactness of $- \otimes_R -$ and Hom_R is a very apt parallel.

We still need to explain the word "algebraic," and we would like to provide a very concrete setting that illustrates the concepts. For the remainder we assume that X is a compact connected Riemann surface, so $n = 2$. Now the typical points of V and V' will be (x, y) and (x', y'), or $x + iy$ and $x' + iy'$. The requirement that $T : V' \to V$ is holomorphic can be expressed by requiring that the Cauchy-Reimann equations $\frac{\partial x}{\partial x'} = \frac{\partial y}{\partial y'}$ and $\frac{\partial y}{\partial x'} = -\frac{\partial x}{\partial y'}$ hold everywhere, for all transition maps between coordinate charts.

Everything in the discussion above can be repeated, with equal validity, for complex valued functions and forms, with "holomorphic" in place of "C^∞." Nevertheless we will continue to be quite interested in objects that are merely C^∞ (in the real rather than the complex sense).

It turns out that certain linear combinations of the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ and the differentials dx and dy make many computations simpler and more expressive. Let

$$dz = dx + idy \quad \text{and} \quad d\overline{z} = dx - idy,$$

and define the partial differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

(These objects should be thought of as inducing new coordinate system on the tangent and cotangent spaces.)

Suppose that $f = u + iv$. Writing out

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)$$

and

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) = 0$$

leads to a couple interesting conclusions. First, by comparing terms we can verify that

$$\overline{\frac{\partial \overline{f}}{\partial \overline{z}}} = \frac{\partial f}{\partial z}.$$

Second, the Cauchy-Riemann equations boil down to $\frac{\partial f}{\partial \overline{z}} = 0$.

Insofar as compositions of holomorphic functions are holomorphic, being holomorphic is a property that is preserved by holomorphic changes of coordinates. It is natural to ask whether the equation $\frac{\partial f}{\partial \overline{z}} = 0$ is also meaningful, in the sense of being invariant, and in fact it is. We say that f is *antiholomorphic* if $\frac{\partial f}{\partial z}$ vanishes. The equation above implies that f is antiholomorphic if and only if $\overline{f}$ is holomorphic, thereby verifying invariance. Geometrically, a holomorphic change of coordinates is conformal (angle preserving) and consequently preserves the orthogonality of the directions of partial differentiation defining what it means to be holomorphic and antiholomorphic.

For $k = 0, 1, 2$, as before, $\mathcal{E}_X^k$ denotes the sheaf of real valued C^∞ k-forms on X. Let $\mathcal{E}_X^{1,0}$ and $\mathcal{E}_X^{0,1}$ be the subsheafs of $\mathcal{E}_X^1$ consisting of the real valued C^∞ 1-forms of the forms $\omega_z dz$ and $\omega_{\overline{z}} d\overline{z}$ respectively. (Intuitively, these subspaces have invariant meaning because if $T : z \to w$ is a holomorphic change of coordinates, then T^* takes dw to dz and $d\overline{w}$ to $d\overline{z}$.)

In the current context a C^∞ 1-*form* is an expression

$$\omega = \omega_z(z)dz + \omega_{\overline{z}}(z)d\overline{z}$$

where $\omega_z, \omega_{\overline{z}} : V \to \mathbb{C}$ are C^∞ functions. A C^∞ 2-*form* is an expression

$$\eta = \underline{\eta}(z)dz \wedge d\overline{z}$$

where $\underline{\eta} : V \to \mathbb{C}$ is C^∞. We can define differential operators ∂ and $\overline{\partial}$ on C^∞ 0- and 1-forms:

$$\partial f = \frac{\partial f}{\partial z} dz, \quad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z}, \quad \partial \omega = (\partial \omega_{\overline{z}}) \wedge d\overline{z}, \quad \overline{\partial} \omega = (\overline{\partial} \omega_z) \wedge dz.$$

We will need the fact that $\partial\overline{\partial} = -\overline{\partial}\partial$, which is the following calculation:

$$\partial\overline{\partial} f = \Big(\partial \frac{\partial f}{\partial \overline{z}}\Big) \wedge d\overline{z} = \frac{\partial^2 f}{\partial z \partial \overline{z}} dz \wedge d\overline{z} = -\Big(\overline{\partial} \frac{\partial f}{\partial z}\Big) \wedge dz = -\overline{\partial}\partial f.$$

Note that the equations $\partial\partial f = 0$ and $\overline{\partial}\overline{\partial} f = 0$ are automatic consequences of the definitions.

For open $U \subset X$, the *Laplace operator* is $\Delta = \partial\overline{\partial} : \mathcal{E}_X^0(U) \to \mathcal{E}_X^2(U)$. For a C^∞ function $h : U \to \mathbb{R}$ we compute that

$$\Delta h = \partial\Big[\frac{1}{2}\Big(\frac{\partial h}{\partial x} + i\frac{\partial h}{\partial y}\Big) d\overline{z}\Big] = \partial\Big[\frac{1}{2}\Big(\frac{\partial h}{\partial x} + i\frac{\partial h}{\partial y}\Big)\Big] \wedge d\overline{z}$$

$$= \frac{1}{4}\Big[\Big(\frac{\partial^2 h}{\partial x^2} - i\frac{\partial^2 h}{\partial x \partial y}\Big) + i\Big(\frac{\partial^2 h}{\partial y \partial x} - i\frac{\partial^2 h}{\partial y^2}\Big)\Big] dz \wedge d\overline{z}$$
$$= \frac{1}{4}\Big(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\Big) dz \wedge d\overline{z}.$$

The function h is *harmonic* if $\Delta h = 0$. Let $\mathcal{H}(U)$ be the set of harmonic functions on U. Win the obvious restriction operators, these spaces constitute a subsheaf $\mathcal{H}$ of $\mathcal{E}^0_X$, By definition the sequence

$$0 \to \mathcal{H} \xrightarrow{\subset} \mathcal{E}^0_X \xrightarrow{\Delta} \mathcal{E}^2_X$$

is exact.

Harmonic and holomorphic functions are closely related. Suppose that u and v are C^∞ functions on U that satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Then

$$\Delta u = \frac{1}{4}\Big[\frac{\partial}{\partial x}\Big(\frac{\partial v}{\partial y}\Big) + \frac{\partial}{\partial y}\Big(-\frac{\partial v}{\partial x}\Big)\Big] = 0$$

and

$$\Delta v = \frac{1}{4}\Big[\frac{\partial}{\partial x}\Big(-\frac{\partial u}{\partial y}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x}\Big)\Big] = 0,$$

so u and v harmonic.

The converse also holds. In general, if $a_x, a_y : U \to \mathbb{R}$ are C^∞ and $\frac{\partial a_x}{\partial y} = \frac{\partial a_x}{\partial y}$, then any point in U has a neighborhood that is the domain of a C^∞ function F with $\frac{\partial F}{\partial x} = a_x$ and $\frac{\partial F}{\partial y} = a_y$. (This is a particular instance of the *Frobenius theorem.*) If u is harmonic, then $\frac{\partial}{\partial x}\big(\frac{\partial u}{\partial x}\big) = \frac{\partial}{\partial y}\Big(-\frac{\partial u}{\partial y}\Big)$, so there is a v that solves the Cauchy-Riemann equations, and consequently u is locally the real part of a holomorphic function.

Suppose that u is harmonic, and $f = u + iv$ is holomorphic as a function of $x + iy$. If (x_0, y_0) is a critical point of u, then the Cauchy-Riemann equations imply that it is a critical point of v, so $z_0 = x_0 + iy_0$ is a critical point of f. The open mapping theorem for holomorphic functions implies that if f is not locally constant near z_0, then it maps any neighborhood of z_0 onto a neighborhood of $f(z_0)$, which implies that u maps any neighborhood of (x_0, y_0) onto a neighborhood of $u(x_0, y_0)$. Thus we obtain the two dimensional version of the *maximum principle* for harmonic functions: an harmonic function is locally constant at any local maximum or local minimum. The corresponding result for homomorphic functions is the *maximum modulus principle*, which asserts that the maximum of the absolute value of a nonconstant holomorphic function on a compact domain is not attained at a point in the domain's interior.

We now wish to show that the sheaf map $\Delta : \mathcal{E}^0_X \to \mathcal{E}^2_X$ is surjective. The key to this is a special case of *Dolbeault's lemma*, which is an analogue and companion of Poincare's lemma. The special case asserts that if $g : \mathbb{C} \to \mathbb{C}$ is C^∞ and has compact support, then there is a C^∞ function $f : \mathbb{C} \to \mathbb{C}$ such that $g = \frac{\partial f}{\partial \overline{z}}$. Let D_ε be the closed disk of radius ε centered at the origin. We construct the desired f by setting

$$f(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{C} \setminus D_\varepsilon} \frac{g(z-\zeta)}{\zeta} \, d\overline{\zeta} \wedge d\zeta.$$

Since g is continuous and compactly supported, the final integral is well defined, and there is a $C > 0$ such that $|g(z)| < C$ for all z. By considering that the area of $D_{2\varepsilon} \setminus D_\varepsilon$ is proportional to ε^2, and that $|g(z-\zeta)/\zeta| < C/\varepsilon$ when ζ is in this annulus, one sees that the limit exists. A variation of this argument, applied to the partial derivatives of f, shows that they can be computed by differentiating under the integral sign, then taking the limit as $\varepsilon \to 0$, so f is C^∞. Moreover, $\frac{\partial f}{\partial \overline{z}}(z)$ is equal to the right hand side of the formula above with $g(z-\zeta)$ replaced by $\partial g(z-\zeta)/\partial \overline{z}$. The operator $\partial/\partial\overline{\zeta}$ satisfies the product rule for differentiation, and $\partial(1/\zeta)/\partial\overline{\zeta} = 0$ because $1/\zeta$ is holomorphic, so the formulas for d, ∂, and $\overline{\partial}$ give

$$d\Big(\frac{g(z-\zeta)}{\zeta} \, d\zeta\Big) = \frac{\partial g(z-\zeta)/\partial\overline{\zeta}}{\zeta} d\overline{\zeta} \wedge d\zeta = -\frac{\partial g(z-\zeta)/\partial\overline{z}}{\zeta} d\overline{\zeta} \wedge d\zeta.$$

Therefore Stoke's theorem gives

$$\frac{\partial f}{\partial \overline{z}}(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{C} \setminus D_\varepsilon} \frac{\partial g(z-\zeta)/\partial\overline{z}}{\zeta} \, d\overline{\zeta} \wedge d\zeta = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\mathbb{C} \setminus D_\varepsilon} d\Big(\frac{g(z-\zeta)}{\zeta} \, d\zeta\Big)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial D_\varepsilon} \frac{g(z-\zeta)}{\zeta} \, d\zeta = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(z-\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \, i\varepsilon e^{i\theta} d\theta = g(z),$$

as desired. (Instead of checking that all the signs are correct, you can be content to observe that if we got one wrong, we could just adjust the definition of f.)

Note that the special case of Dolbeault's lemma above implies also that every C^∞ function $g : \mathbb{C} \to \mathbb{C}$ with compact support is $\frac{\partial f}{\partial z}$ for some C^∞ f, because if $\overline{g} = \frac{\partial \overline{f}}{\partial \overline{z}}$, then $g = \overline{\frac{\partial \overline{f}}{\partial \overline{z}}} = \frac{\partial f}{\partial z}$.

Now consider some $\eta \in \mathcal{E}^2_X(U)$ where $U \subset X$ is open, and fix a point $p \in U$. Poincare's lemma implies that there is some $\omega = \omega_z(z)dz + \omega_{\overline{z}}(z)d\overline{z} \in \mathcal{E}^1_X(U)$ such that $d\omega$ and η agree in some neighborhood of p. Dolbeault's lemma implies that there are $f, \tilde{f} \in \mathcal{E}^0(U)$ such that $\frac{\partial f}{\partial z}$ agrees with ω_z and $\frac{\partial \tilde{f}}{\partial \overline{z}}$ agrees with $\omega_{\overline{z}}$, and consequently $\omega = \partial f + \overline{\partial}\tilde{f}$, near p. Since $\partial\partial f = 0$ and $\overline{\partial}\overline{\partial}\tilde{f} = 0$ we have

$$d\omega = \partial\omega + \overline{\partial}\omega = \overline{\partial}\partial f + \partial\overline{\partial}\tilde{f}.$$

Thus, near p, η is the sum of an image of $\overline{\partial}\partial$ and an image of $\partial\overline{\partial}$, but $\overline{\partial}\partial = -\partial\overline{\partial}$, so these two operators have the same images, and thus η is an image of

$\partial\overline{\partial}$ near p. We have shown that the sequence

$$0 \to \mathcal{H} \xrightarrow{\subset} \mathcal{E}_X^0 \xrightarrow{\Delta} \mathcal{E}_X^2 \to 0$$

is exact. As before, a key issue is the extent to which there can be global sections of $\mathcal{E}_X^2$ that do not come from global sections of $\mathcal{E}_X^0$.

We can now say a few things about how the facts we have developed fit into the larger development of the theory of Riemann surfaces. A fundamental result of that theory is that X admits a nonconstant meromorphic function. In view of our remarks above, this can also be phrased in terms of harmonic functions. The proof of the Riemann existence theorem involves some functional analysis, as applied in the theory of elliptic differential equations. Partly because these methods are not part of the complex analytic toolkit, and partly because the argument is just hard, introductory books on Riemann surfaces often do not provide the proof, in spite of the fundamental character of the result. It turns out that this result can also be understood as a consequence of the finding of Serre's paper "Géometrie Algébrique et Géomt́rie Analytique" (GAGA) which provides results showing that a certain range of holomorphic spaces and sheaves are subsumed by algebraic objects.

Once there is a fairly rich space of meromorphic functions, by using some standard embeddings of products of projective spaces in larger projective spaces, it is not hard to show that X is biholomorphically diffeomorphic to a nonsingular submanifold of a projective space. (Recall that a meromorphic function on X may be thought of as a holomorphic function from X to the Riemann sphere, which is 1-dimensional complex projective space.) Now there is *Chow's theorem*, which asserts that a compact nonsingular holomorphic submanifold of a complex projective space is algebraic, i.e., the solution set of a finite system of homogenous polynomials equations. (For Riemann surfaces there are simpler ways to prove this.) Chow's theorem is perhaps not so frequently cited, but along with GAGA it looms over the theory of complex manifolds, providing considerable additional relevance to more abstract algebraic geometry. Nevertheless, the portion of algebraic geometry that is specific to complex algebraic varieties is very substantial, with the text *Principles of Algebraic Geometry* by Griffiths and Harris (1978) having a status for that field that is comparable to Hartshorne (1977).

The Riemann existence theorem can be understood as a property of one particular sheaf. The role of cohomology, applied to exact sequences such as the ones developed above, is to develop relations between various sheaves, thereby allowing such fundamental facts to be leveraged. Cornalba (1989) is a development of the theory of Riemann surfaces in this spirit.

Coherent Algebraic Sheaves

Introduction

One knows that cohomological methods, and in particular the theory of sheaves, play a crucial role not only in the theory of functions of several complex variables (cf. [5]) but also in classical algebraic geometry. (It suffices to mention the recent work of Kodaira-Spencer on the Riemann-Roch theorem.) The algebraic character of these methods suggests that it might be possible to apply them equally to abstract algebraic geometry; the aim of the present paper is to show that this is indeed the case.

The contents of the various chapters are as follows:

Chapter I is devoted to the general theory of sheaves. It contains the proofs of those results of that theory that are utilized in the other two chapters. The various algebraic operations on sheaves are described in §1; we have followed the exposition of Cartan ([2], [5]). In §2 we study coherent sheaves of modules; these sheaves generalize coherent analytic sheaves (cf. [3], [5]), and their properties are completely analogous. The cohomology groups of a space X with values in a sheaf $\mathcal{F}$ are defined in §3. In the eventual applications, X is an algebraic variety endowed with the Zariski topology, which is not a separated[3] topological space, and the methods used by Leray [10], or Cartan [3] (based on the "partitions of unity," or the "fine" sheaves) are not available; in addition we cannot follow Čech and define the cohomology groups $H^q(X, \mathcal{F})$ by passing to the limit along a sequence of finer and finer open covers. Another difficulty, rooted in the non-separation of X, is encountered in the "exact sequence of cohomology" (cf. nos. 24 and 25): we have only been able to establish that exact sequence in certain particular cases which happen to be sufficient for the applications we have in mind (cf. nos. 24 and 47).

Chapter II begins with a definition of an algebraic variety, analogous to that of Weil ([17], Chap. VII), but encompassing the case of reducible varieties (thereby signalling that, contrary to Weil, we do not reserve the term 'variety' for irreducible varieties); we impose structure on the variety by endowing it with a topology (the Zariski topology) and a presheaf of the sheaf of germs of functions (the sheaf of local rings). A coherent algebraic sheaf on an algebraic

[3] A topological space is separated if it is Hausdorff.

variety is simply a coherent scheme of $\mathcal{O}_V$-modules. $\mathcal{O}_V$ denotes the sheaf of local rings of V; we will give various examples in §2. We devote §3 to affine varieties. The results we obtain parallel analogous results for the varieties of Stein (cf. [3], [5]); if $\mathcal{F}$ is a coherent algebraic sheaf on the affine variety V, one has $H^q(V, \mathcal{F}) = 0$ for all $q > 0$, and $\mathcal{F}_x$ is determined by $H^0(V, \mathcal{F})$ for any $x \in V$. Moreover (§4) $\mathcal{F}$ is completely determined by $H^0(V, \mathcal{F})$ regarded as a module on the coordinate ring of V.

Chapter III, in relation to projective varieties, contains the essential results of this paper. We begin by establishing a correspondence between the coherent algebraic sheaves $\mathcal{F}$ on the projective space $X = \mathbf{P}_r(K)$ and graded S-modules satisfying condition (TF) of no. 56 (S denotes the algebra of polynomials $K[t_0, \ldots, t_r]$); that correspondence is bijective if we identity two S-modules that differ only in their homogeneous components of sufficiently low degree (for precise details see nos. 57, 59, and 65). Consequently any question pertaining to $\mathcal{F}$ can be translated into a question concerning the associated S-module M. In this way we give in §3 a procedure permitting the algebraic derivation of $H^q(X, \mathcal{F})$ from M, and this permits us to study the properties of $H^q(X, \mathcal{F}(n))$ as n tends to $+\infty$ (for the definition of $\mathcal{F}(n)$ see no. 54); the results we obtain are stated in nos. 65 and 66. In §4 we establish a relationship between the groups $H^q(X, \mathcal{F})$ and the functors Ext^q_S introduced by Cartan-Eilenberg [6]; this allows us to study, in §5, the behavior of $H^q(X, \mathcal{F}(n))$ as n tends to $-\infty$, and to give a homological characterization of "k-fois de première espèce" varieties. In §6 we develop several properties of the Euler-Poincaré characteristic of a projective variety with values in a coherent algebraic sheaf.

We also mention how one can apply the general results of this paper to a variety of particular problems, extending the "duality theorem" of [15] to the abstract case, and then generalizing a group of results of Kodaira-Spencer on the Riemann-Roch theorem; in these applications the theorems of nos. 66, 75, and 76 play as essential role. Equally important, when the base field is $\mathbb{C}$, the theory of coherent algebraic sheaves is essentially identical to the theory of coherent analytic sheaves (cf. [4]).

Table of Contents

Chapter I. Sheaves

§1. Operations on sheaves

1. Definition of a sheaf. Let X be a topological space. A *sheaf of abelian groups* (or simply a *sheaf*) consists of:

(a) A function $x \mapsto \mathcal{F}_x$ associating an abelian group $\mathcal{F}_x$ with each point $x \in X$,

(b) A topology on the disjoint union $\mathcal{F}$ of the various sets $\mathcal{F}_x$.

If $f \in \mathcal{F}_x$, let $\pi(f) := x$; we call π the *projection* of $\mathcal{F}$ on X. Let $\mathcal{F} + \mathcal{F}$ denote the subset of $\mathcal{F} \times \mathcal{F}$ consisting of those pairs (f, g) with $\pi(f) = \pi(g)$.

Given these definitions, we can state the two axioms that constrain the objects given by (a) and (b):

(I) For each $f \in \mathcal{F}$, there is a neighborhood V of f and a neighborhood U of $\pi(f)$ such that the restriction of π to V is a homeomorphism between V and U.

(II) The function $f \mapsto -f$ is a continuous function from $\mathcal{F}$ to itself, and the function $(f, g) \mapsto f + g$ is a continuous function from $\mathcal{F} + \mathcal{F}$ to $\mathcal{F}$.

Observe that even when X is separated (which we have not assumed) $\mathcal{F}$ is not necessarily separated, for example because it could be a sheaf of germs of functions (cf. no. 3).

Example of a sheaf. With G being an abelian group, let $\mathcal{F}_x = G$ for all $x \in X$; the set $\mathcal{F}$ is then the product $X \times G$, and, if we endow it with the product topology of the given topology on X and the discrete topology on G, we obtain a sheaf, called the *constant sheaf* isomorphic to G, and often identified with G.

2. Sections of a sheaf. Let $\mathcal{F}$ be a sheaf on the space X, and let U be a subset of X. By a *section of* $\mathcal{F}$ *on* U we mean a continuous function $s : U \to \mathcal{F}$ such that $\pi \circ s$ is the identity function on U. Of course $s(x) \in \mathcal{F}_x$ for all $x \in X$. The set of sections of $\mathcal{F}$ on U is denoted by $\Gamma(U, \mathcal{F})$; Axiom II implies that $\Gamma(U, \mathcal{F})$ is an abelian group. If $U \subset V$ and s is a section on V, then the restriction of s to U is a section on U; in this way we define a homomorphism $\rho_U^V : \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$.

If U is open in X, then $s(U)$ is open in $\mathcal{F}$, and, when U varies over a base of open sets for X and s varies over the elements of $\Gamma(U, \mathcal{F})$, the sets $s(U)$ vary over a base of open sets for $\mathcal{F}$: this is just another way of expressing Axiom I.

Note another consequence of Axiom I: For all $f \in \mathcal{F}_x$, there is a section s on a neighborhood of s such that $s(x) = f$, and two sections with this property agree on some neighborhood of x. In other words, $\mathcal{F}_x$ is the *inductive limit* of the $\Gamma(U, \mathcal{F})$ along the filtration of neighborhoods of x.

In contemporary terminology 'direct limit' is more common than 'inductive limit.' This concept is defined in Section B.3.

3. Construction of sheaves. Suppose we are given, for each open $U \subset X$, an abelian group $\mathcal{F}_U$, and for any nested pair of open sets $U \subset V$, a homomorphism $\varphi_U^V : \mathcal{F}_V \to \mathcal{F}_U$, such that the transitivity condition $\varphi_U^V \circ \varphi_V^W = \varphi_U^W$ holds whenever $U \subset V \subset W$.

In the language that has become standard since Serre's paper this collection is called a *presheaf* if, in addition, $\mathcal{F}_\emptyset$ is trivial (that is, the group with one element, which is denoted by 0) and φ_U^U is always the identity. It looks like Serre just forgot to state the latter condition, but maybe not: everything below is technically correct without it, since it is subsumed by the condition stated in Proposition 1.

The collection $(\mathcal{F}_U, \varphi_U^V)$ allows one to define a sheaf $\mathcal{F}$ as follows:

(a) We set $\mathcal{F}_x := \lim \mathcal{F}_U$ (the inductive limit following the directed set of open neighborhoods of x). If x is in the open set U there is an obvious canonical homomorphism $\varphi_x^U : \mathcal{F}_U \to \mathcal{F}_x$. As above, let $\mathcal{F}$ be the disjoint union of the various $\mathcal{F}_x$.

(b) For $t \in \mathcal{F}_U$ let $[t, U]$ denote the set of $\varphi_x^U(t)$ for $x \in U$, and endow $\mathcal{F}$ with the topology generated by the sets $[t, U]$. Then $f \in \mathcal{F}_x$ has a neighborhood base consisting of those $[t, U]$ with $x \in U$ and $\varphi_x^U(t) = f$.

One verifies immediately that the given (a) and (b) satisfy Axioms I and II, so $\mathcal{F}$ is in fact a sheaf. We call it the the sheaf *defined by the system* $(\mathcal{F}_U, \varphi_U^V)$.

If $t \in \mathcal{F}_U$, the function $x \to \varphi_x^U(t)$ is a section of $\mathcal{F}$ above U; thus there is a canonical homomorphism $\iota : \mathcal{F}_U \to \Gamma(U, \mathcal{F})$.

PROPOSITION 1. *In order for $\iota : \mathcal{F}_U \to \Gamma(U, \mathcal{F})$ to be injective it is necessary and sufficient that the following condition is satisfied:*

If an element $t \in \mathcal{F}_U$ is such that there exists an open cover $\{U_i\}$ of U with $\varphi_{U_i}^U(t) = 0$ for all i, then $t = 0$.

If t satisfies the preceeding condition, then

$$\varphi_x^U(t) = \varphi_x^{U_i} \circ \varphi_{U_i}^U(t) = 0$$

whenever $x \in U_i$, which means that $\iota(t) = 0$. Conversely, suppose that we have $\iota(t) = 0$ for some $t \in \mathcal{F}_U$. Because $\varphi_x^U(t) = 0$, for each $x \in U$ there is a neighborhood $U(x) \subset U$ of x such that $\varphi_{U(x)}^U(t) = 0$, by the definition of the inductive limit. The sets $U(x)$ constitute an open cover of U, so the condition in the statement holds.

PROPOSITION 2. *Let U be an open subset of X, and suppose that $\iota : \mathcal{F}_V \to \Gamma(V, \mathcal{F})$ is injective for every open $V \subset U$. In order for $\iota : \mathcal{F}_U \to \Gamma(U, \mathcal{F})$ to be surjective (hence bijective) it is necessary and sufficient that the following condition hold:*

For each open cover $\{U_i\}$ of U, and each system $\{t_i\}$, $t_i \in \mathcal{F}_{U_i}$, such that $\varphi_{U_i \cap U_j}^{U_i}(t_i) = \varphi_{U_i \cap U_j}^{U_j}(t_j)$ for all pairs (i, j), there is a $t \in \mathcal{F}_U$ with $\varphi_{U_i}^U(t) = t_i$ for all i.

The condition is necessary: each t_i defines a section $s_i := \iota(t_i)$ on U_i, and s_i and s_j agree on $U_i \cap U_j$, so there is a section s on U that agrees with each s_i on U_i. If $\iota : \mathcal{F}_U \to \Gamma(U, \mathcal{F})$ is surjective, then there must be a $t \in \mathcal{F}_U$ such that $\iota(t) = s$. If we let $t_i' := \varphi_{U_i}^U(t)$, the section defined by t_i' on U_i is none other than s_i, whence $\iota(t_i' - t_i) = 0$, which implies that $t_i = t_i'$ because ι is injective by assumption.

The condition is sufficient: suppose s is a section of $\mathcal{F}$ on U and there is an open cover $\{U_i\}$ of U and elements $t_i \in \mathcal{F}_{U_i}$ such that each $\iota(t_i)$ is the restriction of s to U_i. It follows that the elements $\varphi_{U_i \cap U_j}^{U_i}(t_i)$ and $\varphi_{U_i \cap U_j}^{U_j}(t_j)$ define the same section on $U_i \cap U_j$, hence are equal as a consequence of the hypothesis on ι. If $t \in \mathcal{F}_U$ is such that $\varphi^U_{U_i}(t) = t_i$, $\iota(t)$ coincides with s on each U_i, hence on all of U, qed.

PROPOSITION 3. *If $\mathcal{F}$ is a sheaf of abelian groups on X, the sheaf defined by the system* $(\Gamma(U, \mathcal{F}), \rho^V_U)$ *is canonically isomorphic to $\mathcal{F}$.*

This follows immediately from the properties of sections given in no. 2.

Proposition 3 implies that any sheaf can be defined by some system $(\mathcal{F}_U, \varphi^V_U)$. It can happen that different systems define the *same* sheaf $\mathcal{F}$. However, if one insists that $(\mathcal{F}_U, \varphi^V_U)$ satisfies the conditions of Propositions 1 and 2 there is only one (up to isomorphism) system possible, namely $(\Gamma(U, \mathcal{F}), \rho^V_U)$.

EXAMPLE. Let G be an abelian group, take for $\mathcal{F}_U$ the set of functions on U with values in G, and define $\varphi^V_U : \mathcal{F}_V \to \mathcal{F}_U$ to be the restriction operator. One obtains in this way a system $(\mathcal{F}_U, \rho^V_U)$, and from this a sheaf $\mathcal{F}$ called the *sheaf of germs of functions* with values in G. We can see right away that the system $(\mathcal{F}_U, \varphi^V_U)$ satisfies the conditions of Propositions 1 and 2, from which it follows that one can identify the sections of $\mathcal{F}$ on an open U with the elements of $\mathcal{F}_U$.

Subsequent authors such as Mumford (1999) and Hartshorne (1977) take a different approach, defining a sheaf to be a presheaf $(\mathcal{F}_U, \varphi^V_U)$ satisfying the conditions identified in Propositions 1 and 2. Starting with any presheaf $(\mathcal{F}_U, \varphi^V_U)$, there is a *sheafification* $(\tilde{\mathcal{F}}_U, \tilde{\varphi}^V_U)$ in which each $\tilde{\mathcal{F}}_U$ is, in effect, the set of sections on U that one can obtain by patching together systems of elements $\{t_i \in \mathcal{F}_{U_i}\}$ defined on some open cover $\{U_i\}$ of U such that $\varphi^{U_i}_{U_i \cap U_j}(t_i) = \varphi^{U_j}_{U_i \cap U_j}(t_j)$ for all i and j. (More precisely, $\tilde{\mathcal{F}}_U$ is the set of equivalence classes of such systems, where two such systems are equivalent if they agree on all overlaps.) The more modern style has the advantage that one may (perhaps, eventually) care about presheaves that are not sheaves, and it shows that the relationship between a presheaf and its sheafification can be understood without reference to the sets $\mathcal{F}_x$, which are called the *stalks* of the presheaf.

4. Gluing of sheaves. Let $\mathcal{F}$ be a sheaf on X, and let U be a subset of X. The set $\pi^{-1}(U) \subset \mathcal{F}$, endowed with the relative topology inherited from $\mathcal{F}$, is a sheaf on U, called the sheaf *induced* by $\mathcal{F}$, and denoted by $\mathcal{F}(U)$ (or just $\mathcal{F}$, is confusion is unlikely).

We will now see how, inversely, one can define a sheaf on X from sheaves on an open cover of X.

Serre now brings isomorphisms of sheaves into the discussion, but he hasn't yet said what a morphism of sheaves is. (In that era people were less aware that a morphism could be something other than a function.) Presumably he regards it as automatic that a morphism from $\mathcal{F}$ to another sheaf $\mathcal{G}$ on X is a continuous function $\eta : \mathcal{F} \to \mathcal{G}$ such that $\eta(\mathcal{F}_x) \subset \mathcal{G}_x$ for each x, and the restriction of η to $\mathcal{F}_x$ is actually a homomorphism from $\mathcal{F}_x$ to $\mathcal{G}_x$. (This will become explicit in no. 8.) In the modern system of definitions a morphism

$\tilde{\eta} : (\mathcal{F}_U, \varphi_U^V) \to (\mathcal{G}_U, \gamma_U^V)$ of presheaves on X is a collection of homomorphisms $\tilde{\eta}_U : \mathcal{F}_U \to \mathcal{G}_U$ such that

$$\gamma_U^V \circ \tilde{\eta}_U = \tilde{\eta}_V \circ \varphi_U^V \tag{$*$}$$

whenever $V \subset U$. The latter equation is precisely what is needed in order for $\tilde{\eta}$ to induce a homomorphism $\tilde{\eta}_x : \mathcal{F}_x \to \mathcal{G}_x$ from each stalk $\mathcal{F}_x := \lim \mathcal{F}_U$ to $\mathcal{G}_x := \lim \mathcal{G}_U$. These functions combine to give a function $\eta : \mathcal{F} \to \mathcal{G}$. In addition $(*)$ implies that η is continuous when $\mathcal{F}$ and $\mathcal{G}$ have the topologies described in (b) of no. 3, so η is a morphism. Clearly η is an isomorphism, in the sense described above, whenever each $\tilde{\eta}_U$ is an isomorphism. The converse is true as well (cf. p. 63 of Hartshorne (1977)) but involves some constructions and verifications.

Proposition 4. *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X, and, for each $i \in I$, let $\mathcal{F}_i$ be a sheaf on U_i. For each pair (i, j), let θ_{ij} be an isomorphism mapping $\mathcal{F}_i(U_i \cap U_j)$ to $\mathcal{F}_j(U_i \cap U_j)$, and suppose that for any $i, j, k \in I$ we have $\theta_{ij} \circ \theta_{jk} = \theta_{ik}$ at each point of $U_i \cap U_j \cap U_k$.*

Then there is a sheaf $\mathcal{F}$ on X and, for each $i \in I$, an isomorphism η_i from $\mathcal{F}(U_i)$ to $\mathcal{F}_i$, such that $\theta_{ij} = \eta_i \circ \eta_j^{-1}$ at each point of $U_i \cap U_j$. Moreover, $\mathcal{F}$ and the isomorphisms η_i are determined, up to isomorphism, by these conditions.

The uniqueness of $\{\mathcal{F}, \eta_i\}$ is evident.

Suppose that $\{\hat{\mathcal{F}}, \hat{\eta}_i\}$ is a second system with the desired properties. For each i and j we have $\eta_i \circ \eta_j^{-1} = \theta_{ij} = \hat{\eta}_i \circ \hat{\eta}_j^{-1}$ and thus $\hat{\eta}_i^{-1} \circ \eta_i = \hat{\eta}_j^{-1} \circ \eta_j$ on $U_i \cap U_j$, so we can define an isomorphism $\alpha : \mathcal{F} \to \hat{\mathcal{F}}$ by requiring that it agree with $\hat{\eta}_i^{-1} \circ \eta_i$ on each U_i.

To demonstrate existence one can define $\mathcal{F}$ as a quotient of the disjoint union of the spaces of the spaces $\mathcal{F}_i$. It is preferable to do this using the results of no. 3: if U is an open subset of X, let $\mathcal{F}_U$ be the group whose elements are the systems $\{s_k\}_{k \in I}$ such that s_k and $\theta_{kj}(s_j)$ agree on $U \cap U_j \cap U_k$; if $U \subset V$, define φ_U^V in the obvious manner. The sheaf defined by the system $(\mathcal{F}_U, \varphi_U^V)$ is the sheaf $\mathcal{F}$ we want. Moreover, if $U \subset U_i$, the function that takes the system $\{s_k\} \in \mathcal{F}_U$ to $s_i \in \Gamma(U_i, \mathcal{F}_i)$ is an isomorphism of $\mathcal{F}_U$ onto $\Gamma(U, \mathcal{F}_i)$, due to the transitivity condition. In this way we obtain an isomorphism $\eta_i : \mathcal{F}(U_i) \to \mathcal{F}_i$ that evidently satisfies the stated condition.

We say that the sheaf $\mathcal{F}$ is obtained by *gluing* the sheaves $\mathcal{F}_i$ according to the isomorphisms θ_{ij}.

5. Extension and restriction of a sheaf. Let X be a topological space, let Y be a closed subspace, and let $\mathcal{F}$ be a sheaf on X. We say that $\mathcal{F}$ is *concentrated* on Y, or is *null*, or *vanishes*, outside of Y, if $\mathcal{F}_x = 0$ for all $x \in X \setminus Y$.

In familiar contexts, say the sheaf of continuous real valued functions on a reasonably well behaved space, a section that vanishes on $X \setminus Y$ will also vanish on the closure of this set. One could have part of Y outside this closure, but actually the more interesting possibilities have a different source. For example, let $X = \mathbb{R}$, let $Y = \{0\}$, and define a presheaf on X by letting $\mathcal{F}_U$ be $\mathbb{R}$ if $0 \in U$ and 0 otherwise, and letting φ_U^V be the identity function on $\mathbb{R}$ when this is possible and the zero homomorphism otherwise. We now construct $\mathcal{F}$ according to the recipe given by (a) and (b) of no. 3, finding that as a set of points

$$\mathcal{F} = \{ (x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0 \}.$$

But the topology is strange, if you aren't acclimated: according to (b), a point of the form $(x, 0)$ with $x \neq 0$ has the usual sort of neighborhood base consisting of the sets of the form $\{ (t, 0) : x - \varepsilon < t < x + \varepsilon \}$, but a point of the form $(0, y)$ has a neighborhood base consisting of the sets

$$\{(0, y)\} \cup \{ (t, 0) : -\varepsilon < t < \varepsilon \text{ and } t \neq 0 \}.$$

This space has a family resemblance to "the line with two origins," which is a standard example of a topological space that is not Hausdorff.

Proposition 5. *If the sheaf $\mathcal{F}$ is concentrated on Y, the homomorphism*

$$\rho_Y^X : \Gamma(X, \mathcal{F}) \to \Gamma(Y, \mathcal{F}(Y))$$

is bijective.

If a section of $\mathcal{F}$ on X is null on Y, it is null because $\mathcal{F}_x = 0$ if $x \notin Y$, which shows that ρ_Y^X is injective. Inversely, let s be a section of $\mathcal{F}(Y)$ on Y, and extend s to X by setting $s(x) = 0$ if $x \notin Y$. The function $x \mapsto s(x)$ is evidently continuous on $X \setminus Y$. If $x \in Y$, there is a section s' of $\mathcal{F}$ on a neighborhood U of x such that $s'(x) = s(x)$. Because s is continuous on Y by hypothesis, there is a neighborhood V of x, contained in U, and such that $s'(y) = s(y)$ for all $y \in V \cap Y$. Since $\mathcal{F}_y = 0$ if $y \notin Y$, we also have $s'(y) = s(y)$ when $y \in V \setminus Y$. Thus s and s' coincide on V, which proves that s is continuous on a neighborhood of Y, hence continuous everywhere. It follows that ρ_Y^X is surjective, which completes the proof.

We will now show that the sheaf $\mathcal{F}(Y)$ unambiguously determines the sheaf $\mathcal{F}$.

Proposition 6. *Let Y be a closed subspace of X, and let $\mathcal{G}$ be a sheaf on Y. Let $\mathcal{F}_x$ be $\mathcal{G}_x$ if $x \in Y$ and 0 if $x \notin Y$, and let $\mathcal{F}$ be the disjoint union of the various $\mathcal{F}_x$. Then we can endow $\mathcal{F}$ with the structure of a sheaf, in a unique way, such that $\mathcal{F}(Y) = \mathcal{G}$.*

Let U be an open subset of X. If s is a section of $\mathcal{G}$, extend s by 0 in $U \setminus Y$. Doing this for each s in $\Gamma(U \cap Y, \mathcal{G})$ yields a group $\mathcal{F}_U$ of functions from U to $\mathcal{F}$. Proposition 5 shows that, if $\mathcal{F}$ is endowed with the structure of a sheaf such that $\mathcal{F}(Y) = \mathcal{G}$, then $\mathcal{F}_U = \Gamma(U, \mathcal{F})$, which proves the uniqueness of the structure in question. Its existence is implied by the results of no. 3, applied to $\mathcal{F}_U$, and to the restriction homomorphisms φ_U^V.

We will say that the sheaf $\mathcal{F}$ is obtained by *extending the sheaf* $\mathcal{G}$ *by* 0 *outside of* Y, and we will denote it by $\mathcal{G}^Y$, or just $\mathcal{G}$ if no confusion can result.

6. Sheaves of rings and sheaves of modules. The notion of sheaf defined in no. 1 is that of a sheaf of *abelian groups*. It is clear that there exist analogous definitions for all algebraic structures. (Equally, one can define a "sheaf of sets" in which $\mathcal{F}_x$ has no algebraic structure and only Axiom I is imposed.) In the following we will be concerned primarily with sheaves of *rings* and sheaves of *modules*.

A sheaf of rings $\mathcal{A}$ is a sheaf of abelian groups $\mathcal{A}_x$, $x \in X$, where each $\mathcal{A}_x$ is endowed with the structure of a ring in such a way that the function $(f, g) \mapsto f \cdot g$ is a continuous function from $\mathcal{A} + \mathcal{A}$ to $\mathcal{A}$ (the notations are those of no. 1). We will always assume that each $\mathcal{A}_x$ has a unit element, varying continuously with x.

If $\mathcal{A}$ is a sheaf of rings satisfying the preceeding conditions, then each $\Gamma(U, \mathcal{A})$ is a ring with unit, and $\rho_U^V : \Gamma(V, \mathcal{A}) \to \Gamma(U, \mathcal{A})$ is a unitary homomorphism when $U \subset V$. Conversely, if we are given rings with unit $\mathcal{A}_U$ and unitary homomorphisms $\varphi_U^V : \mathcal{A}_V \to \mathcal{A}_U$ such that $\varphi_U^V \circ \varphi_V^W = \varphi_U^W$, the sheaf $\mathcal{A}$ defined by the system $(\mathcal{A}_U, \varphi_U^V)$ is a sheaf of rings. For example, if G is a ring with unit, the sheaf of germs of functions with values in G (defined in no. 3) is a sheaf of rings.

Let $\mathcal{A}$ be a sheaf of rings. A sheaf $\mathcal{F}$ is called a *sheaf of* $\mathcal{A}$*-modules* if each $\mathcal{F}_x$ is endowed with the structure of a unitary $\mathcal{A}_x$-module (say a left $\mathcal{A}$-module, to fix ideas), varying "continuously" with x, in the following sense: if $\mathcal{A} + \mathcal{F}$ is the subset of $\mathcal{A} \times \mathcal{F}$ consisting of those pairs (a, f) such that $\pi(a) = \pi(f)$, the function $(a, f) \mapsto a \cdot f$ is a continuous function from $\mathcal{A} + \mathcal{F}$ to $\mathcal{F}$.

If $\mathcal{F}$ is a sheaf of $\mathcal{A}$-modules, then each $\Gamma(U, \mathcal{F})$ is a unitary $\Gamma(U, \mathcal{A})$-module. Conversely, suppose that $\mathcal{A}$ is defined by the system $(\mathcal{A}_U, \varphi_U^V)$ as above, and let $\mathcal{F}$ be a sheaf defined by the system $(\mathcal{F}_U, \psi_U^V)$, where each $\mathcal{F}_U$ is a unitary $\mathcal{A}_U$-module, with $\psi_U^V(a \cdot f) = \varphi_U^V(a) \cdot \psi_U^V(f)$ if $a \in \mathcal{A}_V$, $f \in \mathcal{F}_V$. Then $\mathcal{F}$ is a sheaf of $\mathcal{A}$-modules.

Any sheaf of abelian groups can be regarded as a sheaf of $\mathbb{Z}$-modules, where $\mathbb{Z}$ denotes the constant sheaf, isomorphic to the ring of integers. This permits us, in the following, to restrict our studies to sheaves of modules.

7. Subsheaves and quotient sheaves. Let $\mathcal{A}$ be a sheaf of rings, $\mathcal{F}$ a sheaf of $\mathcal{A}$-modules. For each $x \in X$ let $\mathcal{G}_x$ be a subset of $\mathcal{F}_x$. We say that $\mathcal{G} := \bigcup \mathcal{G}_x$ is a *subsheaf* of $\mathcal{F}$ if:

(a) $\mathcal{G}_x$ is a sub-$\mathcal{A}_x$-module of $\mathcal{F}_x$ for each $x \in X$,

(b) $\mathcal{G}$ is an open subset of $\mathcal{F}$.

Condition (b) can also be explained as follows:

(b′) If x is a point of X, and if s is a section of $\mathcal{F}$ on a neighborhood of x such that $s(x) \in \mathcal{G}_x$, then $s(y) \in \mathcal{G}_y$ for all y sufficiently close to x.

It is clear that, if these conditions hold, $\mathcal{G}$ is a sheaf of $\mathcal{A}$-modules.

Let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$, and set $\mathcal{H}_x := \mathcal{F}_x/\mathcal{G}_x$ for all $x \in X$. Endowing $\mathcal{H} := \bigcup \mathcal{H}_x$ with the quotient topology of the topology of $\mathcal{F}$, we easily see that we obtain in this way a sheaf of $\mathcal{A}$-modules, called the *quotient sheaf* of $\mathcal{F}$ by $\mathcal{G}$.

Serre kindly spares us the rather tedious details of the verification.

Let $q : \mathcal{F} \to \mathcal{H}$ be the function taking $f \in \mathcal{F}_x$ to $f + \mathcal{G}_x$. Recall that the quotient topology induced by q is the finest topology on $\mathcal{H}$ such that q is continuous; concretely a set $V \subset \mathcal{H}$ is open in the quotient topology precisely when $q^{-1}(V)$ is open. We need to show that $\mathcal{H}$, endowed with this topology, satisfies Axioms I and II, and that scalar multiplication is continuous.

The first task is to show that q is an open map. That is, whenever $U \subset \mathcal{F}$ is open, $q(U)$ is also open, which is to say that $q^{-1}(q(U))$ is open. It suffices to show this for all U in some base of the topology of $\mathcal{F}$, so (by Axiom I) we may assume that $\pi(U)$ is open and the restriction of π to U is a homeomorphism between U and $\pi(U)$. Then $\pi^{-1}(\pi(U))$ is open, because π is continuous, and

$$q^{-1}(q(U)) = \{\, f \in \pi^{-1}(\pi(U)) : f - (\pi|_U)^{-1}(\pi(f)) \in \mathcal{G}\,\}$$

is open because $\mathcal{G}$ is open and the functions π, $(\pi|_U)^{-1}$, addition, and negation are all continuous.

To verify Axiom I observe that, with U as above, $\pi\circ(q|_U)^{-1}$ and $q\circ(\pi|_U)^{-1}$ are both continuous, hence inverse homeomorphisms between $q(U)$ and $\pi(U)$. To verify Axiom II (continuity of addition and negation) and the continuity of scalar multiplication, consider the following commutative diagrams:

$$\begin{tikzcd}
\mathcal{F}+\mathcal{F} \arrow[r, "q+q"] \arrow[d, "+"'] & \mathcal{H}+\mathcal{H} \arrow[d, "+"'] \\
\mathcal{F} \arrow[r, "q"] & \mathcal{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}+\mathcal{F} \arrow[r, "1_{\mathcal{A}}+q"] \arrow[d, "\cdot"'] & \mathcal{A}+\mathcal{H} \arrow[d, "\cdot"'] \\
\mathcal{F} \arrow[r, "q"] & \mathcal{H}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "q"] \arrow[d, "-"'] & \mathcal{H} \arrow[d, "-"'] \\
\mathcal{F} \arrow[r, "q"] & \mathcal{H}
\end{tikzcd}$$

In each case we need to show that the right hand vertical map is continuous. Let $U \subset \mathcal{H}$ be open, let V be the set of points that are mapped to U by the left hand vertical map followed by the lower horizontal map, and let W be the image of V under the upper horizontal map. Since the upper horizontal map is a surjection, W is the entire set of points mapped to U by the right hand vertical map, and since the left hand vertical map and the lower horizontal map are continuous, the claim will follow if we show that the upper horizontal map is open, which has already been shown for q. To see that $q+q$ and $1_{\mathcal{A}}+q$ are open we first observe that a cartesian product of two open maps is open. Since $\mathcal{F}+\mathcal{F}$ and $\mathcal{A}+\mathcal{F}$ are subsets of $\mathcal{F}\times\mathcal{F}$ and $\mathcal{A}\times\mathcal{F}$, we also need a simple topological fact: if $f : X \to Y$ is an open map, $A \subset X$, and $f^{-1}(f(A)) = A$, then $f|_A : A \to f(A)$ is an open map. Specifically, if $U \subset X$ is open, then $f(A) \cap f(U) = f(U \cap A)$, so $f(U \cap A)$ is relatively open in $f(A)$.

We can give an alternative definition, utilizing the results of no. 3: if U is an open subset of X, let $\mathcal{H}_U := \Gamma(U, \mathcal{F})/\Gamma(U, \mathcal{G})$, and let φ_U^V be the homomorphism obtained from $\rho_U^V : \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ by passage to quotients. Then the sheaf defined by the system $(\mathcal{H}_U, \varphi_U^V)$ is none other than $\mathcal{H}$.

More precisely the two sheafs are canonically isomorphic. Let $\mathcal{L}$ be the sheaf defined by the system $(\mathcal{H}_U, \varphi_U^V)$, and for $x \in U \subset X$ with U open let $\varphi_x^U : \Gamma(U, \mathcal{F}) \to \mathcal{F}_x$ and $\tilde{\varphi}_x^U : \mathcal{H}_U \to \mathcal{L}_x$ be the canonical homomorphism. If $t \in \mathcal{H}_U$, then $t = s + \Gamma(U, \mathcal{G})$ for some $s \in \Gamma(U, \mathcal{F})$, and it is clear that

$$\varphi_x^U(s) + \mathcal{G}_x \mapsto \tilde{\varphi}_x^U(s + \Gamma(U, \mathcal{G})) \quad \text{and} \quad \tilde{\varphi}_x^U(s + \Gamma(U, \mathcal{G})) \mapsto \varphi_x^U(s) + \mathcal{G}_x$$

are well defined, in the sense of depending only on x and $s(x)$, and they are inverse bijections between $\mathcal{H}$ and $\mathcal{L}$. A set in $\mathcal{H}$ is open if it is covered by sets of the form $\{ \varphi_x^U(s) + \mathcal{G}_x : x \in U \} = \{ q(s(x)) : x \in U \}$, and (by (b) of no. 3) a set in $\mathcal{L}$ is open if is covered by sets of the form $\{ \tilde{\varphi}_x^U(s + \Gamma(U, \mathcal{G})) : x \in U \}$, so these functions are both open mappings, hence inverse homeomorphisms.

Using either definition of $\mathcal{H}$, we can see that whenever s is a section of $\mathcal{H}$ on a neighborhood of x, there is a section t of $\mathcal{F}$ in a neighborhood of x such that the class of $t(y)$ mod $\mathcal{G}_y$ is equal to $s(y)$ for all y near x. But it is important to recognize that this is not true globally, in general: if U is an open subset of X, we have only the exact sequence:

$$0 \to \Gamma(U, \mathcal{G}) \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{H}),$$

and the homomorphism $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{H})$ is not surjective in general (cf. no. 24).

8. Homomorphisms. Let $\mathcal{A}$ be a sheaf of rings, and let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves of $\mathcal{A}$-modules. An $\mathcal{A}$-*homorphism*, or an $\mathcal{A}$-linear homorphism, or simply a homomorphism, of $\mathcal{F}$ into $\mathcal{G}$, is given, for each $x \in X$, by an $\mathcal{A}_x$-homomorphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$, such that the function $\varphi : \mathcal{F} \to \mathcal{G}$ defined by these φ_x is continuous. We can also express this condition by saying that if s is a section of $\mathcal{F}$ on U, then $x \mapsto \varphi_x(s(x))$ is a section of $\mathcal{G}$ on U, which will be denoted by $\varphi(s)$. For example, if $\mathcal{G}$ is a subsheaf of $\mathcal{F}$, the injection $\mathcal{G} \to \mathcal{F}$, and the projection $\mathcal{F} \to \mathcal{F}/\mathcal{G}$, are homomorphisms.

PROPOSITION 7. *Let φ be a homomorphism from $\mathcal{F}$ to $\mathcal{G}$. For each $x \in X$ let $\mathcal{N}_x$ be the kernel of φ_x, and let $\mathcal{I}_x$ be the image of φ_x. Then $\mathcal{N} := \bigcup \mathcal{N}_x$ is a subsheaf of $\mathcal{F}$, $\mathcal{I} := \bigcup \mathcal{I}_x$ is a subsheaf of $\mathcal{G}$, and φ defines an isomorphism between $\mathcal{F}/\mathcal{N}$ and $\mathcal{I}$.*

When φ_x is an $\mathcal{A}_x$-homomorphism, $\mathcal{N}_x$ and $\mathcal{I}_x$ are submodules of $\mathcal{F}_x$ and $\mathcal{G}_x$ respectively, and φ_x defines an isomorphism from $\mathcal{F}_x/\mathcal{N}_x$ to $\mathcal{I}_x$. If s is a local section of $\mathcal{F}$ such that $s(x) \in \mathcal{N}_x$, then $\varphi \circ s(x) = 0$, whence $\varphi \circ s(y) = 0$ and $y \in \mathcal{N}_y$ for y sufficiently close to x, which shows that $\mathcal{N}$ is a subsheaf of $\mathcal{F}$. If t is a local section of $\mathcal{G}$ such that $t(x) \in \mathcal{I}_x$, there is a local section s of $\mathcal{F}$ such that $\varphi \circ s(x) = t(x)$, whence $\varphi \circ s = t$ in a neighborhood of x, which shows that $\mathcal{I}$ is a subsheaf of $\mathcal{G}$, isomorphic to $\mathcal{F}/\mathcal{N}$.

The sheaf $\mathcal{N}$ is called the *kernel* of φ, and is denoted by $\mathrm{Ker}(\varphi)$. The sheaf $\mathcal{G}/\mathcal{I}$ is called the *cokernel* of φ, and is denoted by $\mathrm{Coker}(\varphi)$. A homomorphism φ is said to by *injective* if each φ_x is injective, which is equivalent to $\mathrm{Ker}(\varphi) = 0$. It is said to be *surjective* if each φ_x is surjective, or equivalently $\mathrm{Coker}(\varphi) = 0$. If is said to be *bijective* if it is both injective and surjective, in which case Proposition 7 implies that it is an isomorphism from $\mathcal{F}$ to $\mathcal{G}$, and φ^{-1} is also an isomorphism. All the definitions relative to homomorphisms of modules can be transported to corresponding concepts for sheaves of modules. For example, a sequence of homomorphism is said to be *exact* if the image of each homomorphism coincides with the kernel of the following homomorphism. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a homomorphism, the sequences

$$0 \to \mathrm{Ker}(\varphi) \to \mathcal{F} \to \mathrm{image}(\varphi) \to 0$$

$$0 \to \mathrm{image}(\varphi) \to \mathcal{G} \to \mathrm{Coker}(\varphi) \to 0$$

are exact.

If φ is a homomorphism from $\mathcal{F}$ to $\mathcal{G}$, the function $s \to \varphi \circ s$ is a $\Gamma(U, \mathcal{A})$-homomorphism from $\Gamma(U, \mathcal{F})$ to $\Gamma(U, \mathcal{G})$. Conversely, suppose that $\mathcal{A}$, $\mathcal{F}$, $\mathcal{G}$ are defined by the systems $(\mathcal{A}_U, \varphi_U^V)$, $(\mathcal{F}_U, \psi_U^V)$, $(\mathcal{G}_U, \chi_U^V)$, and as in no. 6 we are given, for each open $U \subset X$, an $\mathcal{A}$-homomorphism $\varphi_U : \mathcal{F}_U \to \mathcal{G}_U$ such that $\chi_U^V \circ \varphi_V = \varphi_U \circ \psi_U^V$. By passage to the inductive limit the φ_U define a homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$.

For $x \in U \subset X$ with U open let $\psi_x^U : \mathcal{F}_U \to \mathcal{F}_x$ and $\chi_x^U : \mathcal{G}_U \to \mathcal{G}_x$ be the canonical maps. Then φ_x is defined by the requirement that diagrams of the form

$$\begin{tikzcd} \mathcal{F}_U \arrow[r, "\varphi_U"] \arrow[d, "\psi_x^U"'] & \mathcal{G}_U \arrow[d, "\chi_x^U"] \\ \mathcal{F}_x \arrow[r, "\varphi_x"] & \mathcal{G}_x \end{tikzcd}$$

always commute. In order for φ_x to be well defined it must be the case that $\chi_x^U(\varphi_U(s)) = \chi_x^{U'}(\varphi_{U'}(s'))$ whenever $\psi_x^U(s) = \psi_x^{U'}(s')$, but when the latter equation holds s and s' agree in some neighborhood of x, and the desired equation follows from the definition of the inductive limit.

Fixing U, for $s \in \mathcal{F}_U$ and $t \in \mathcal{G}_U$ let ψ_s^U and χ_t^U be the functions $x \mapsto \psi_x^U(s)$ and $x \mapsto \chi_x^U(t)$. These functions are homeomorphisms between U and their images, and on the image of ψ_s^U we have $\varphi = \chi_{\varphi_U(s)}^U \circ {\psi_s^U}^{-1}$. Since $\mathcal{F}$ is covered by the images of functions such as ψ_s^U, which are open, it follows that φ is continuous.

9. Direct sum of two sheaves. Let $\mathcal{A}$ be a sheaf of rings, $\mathcal{F}$ and $\mathcal{G}$ two sheaves of $\mathcal{A}$-modules. For each $x \in X$ form the module $\mathcal{F}_x + \mathcal{G}_x$, the *direct sum* of $\mathcal{F}_x$ and $\mathcal{G}_x$. An element of $\mathcal{F}_x + \mathcal{G}_x$ is a pair (f, g) with $f \in \mathcal{F}_x$ and $g \in \mathcal{G}_x$. Let $\mathcal{H}$ be the disjoint union of the sets $\mathcal{F}_x + \mathcal{G}_x$; one can identify $\mathcal{H}$ with the subset of $\mathcal{F} \times \mathcal{G}$ consisting of those pairs (f, g) with $\pi(f) = \pi(g)$. If we endow $\mathcal{H}$ with the topology induced by the product topology on $\mathcal{F} \times \mathcal{G}$, it follows immediately that $\mathcal{H}$ is a sheaf of $\mathcal{A}$-modules, called the *direct sum* of $\mathcal{F}$ and $\mathcal{G}$ and denoted by $\mathcal{F} + \mathcal{G}$. Each section of $\mathcal{F} + \mathcal{G}$ on an open $U \subset X$ is of the form $x \mapsto (s(x), t(x))$ where s and t are sections of $\mathcal{F}$ and $\mathcal{G}$ on U. That is, $\Gamma(U, \mathcal{F} + \mathcal{G})$ is isomorphic to the direct sum $\Gamma(U, \mathcal{F}) + \Gamma(U, \mathcal{G})$.

The definition of the direct sum extends by recurrence to any finite number of $\mathcal{A}$-modules. In particular, the direct sum of p sheaves isomorphic to a give sheaf $\mathcal{F}$ is denoted by $\mathcal{F}^p$.

10. Tensor product of two sheaves. Let $\mathcal{A}$ be a sheaf of rings, $\mathcal{F}$ a sheaf of right $\mathcal{A}$-modules, and $\mathcal{G}$ a sheaf of left $\mathcal{A}$-modules. For each $x \in X$ set $\mathcal{H}_x := \mathcal{F}_x \otimes \mathcal{G}_x$, the tensor product over the ring $\mathcal{A}_x$ (cf. example [6], Chap. 2, §2). Let $\mathcal{H}$ be the disjoint union of the $\mathcal{H}_x$.

PROPOSITION 8. *There is a unique structure of a sheaf on $\mathcal{H}$ such that if s and t are sections of $\mathcal{F}$ and $\mathcal{G}$ on an open U, the function $x \mapsto s(x) \otimes t(x) \in \mathcal{H}_x$ is a section of $\mathcal{H}$ on U.*

The sheaf $\mathcal{H}$ so defined is called the tensor product *(over $\mathcal{A}$) of $\mathcal{F}$ and $\mathcal{G}$ and denoted by $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$. If each $\mathcal{A}_x$ is commutative, it is a sheaf of $\mathcal{A}$-modules.*

If $\mathcal{H}$ is endowed with a structure satisfying the stated condition, and s_i and t_i are sections of $\mathcal{F}$ and $\mathcal{G}$ on an open $U \subset X$, the function $\sum s_i(x) \times t_i(x)$ is a section of $\mathcal{H}$ on U. Each $h \in \mathcal{H}_x$ can be written in the form $h = \sum f_i \otimes g_i$, $f_i \in \mathcal{F}$, $g_i \in \mathcal{G}$, which is of the form $\sum s_i(x) \otimes t_i(x)$ where s_i and t_i are sections defined in a neighborhood of x. From this we see that each section of $\mathcal{H}$ is locally equal to a section of the preceeding form, which demonstrates the uniqueness of the sheaf structure of $\mathcal{H}$.

Let's prove existence. We may suppose that $\mathcal{A}$, $\mathcal{F}$, $\mathcal{G}$ are defined by the systems $(\mathcal{A}_U, \varphi_U^V)$, $(\mathcal{F}_U, \psi_U^V)$, $(\mathcal{G}_U, \chi_U^V)$ as in no. 6. Set $\mathcal{H}_U := \mathcal{F}_U \otimes \mathcal{G}_U$, with the tensor product taken over $\mathcal{A}_U$. The homomorphisms ψ_U^V and χ_U^V define, by passage to the tensor product, a homomorphism $\eta_U^V : \mathcal{H}_V \to \mathcal{H}_U$. Taking inductive limits gives

$$\lim_{x \in U} \mathcal{H}_U = \lim_{x \in U} \mathcal{F}_U \otimes \lim_{x \in U} \mathcal{G}_U = \mathcal{H}_x,$$

with the final tensor product taken over $\mathcal{A}_x$ (for the tensor product commuting with the inductive limits, cf. for example [6], Chap. VI, Exer. 18). The sheaf defined by the system $(\mathcal{H}_U, \eta_U^V)$ can be identified with $\mathcal{H}$, and in this way $\mathcal{H}$ is endowed with a sheaf structure visibly satisfying the desired condition. Finally, if the $\mathcal{A}_x$ are commutative, we may assume that the $\mathcal{A}_U$ are as well (it suffices to take the rings $\Gamma(U, \mathcal{A})$ as the $\mathcal{A}_U$) in which case $\mathcal{H}_U$ is an $\mathcal{A}_U$-module and $\mathcal{H}$ is a sheaf of $\mathcal{A}$-modules.

Even though it will be a bit tedious, let's work through the meaning of the tensor product commuting with passage to direct limits. Let $(I, \leq)$ be a directed set with typical elements U and V. Let (A_U, α^U_V) be a directed system of commutative rings and homomorphisms, and let $A := \lim A_U$. Let (M_U, μ^U_V) and (N_U, ν^U_V) be directed systems in which M_U and N_U are A_U-modules and the group homomorphisms μ^U_V and ν^U_V are linear in the sense that $\mu^U_V(am) = \alpha^U_V(a)\mu^U_V(m)$ and $\nu^U_V(an) = \alpha^U_V(a)\nu^U_V(n)$ for all $a \in A_U$, $m \in M_U$, and $n \in N_U$. Then $M := \lim M_U$ and $N := \lim N_U$ are A-modules with scalar multiplication given by $[a][m] := [am]$ and $[a][n] := [an]$. Recall that if Z is an A-module, then a group homomorphism $f : M \times N \to Z$ is A-*bilinear* if $f(am, n) = f(m, an) = af(m, n)$ for all $a \in A$, $m \in M$, and $n \in N$. Let $P := \lim M_U \otimes_{A_U} N_U$. The claim here is *not* that $M \otimes_A N = P$ in the literal sense of exact equality, but rather that P satisfies the universal property that constitutes the categorical definition of the tensor product. That is, there is an A-bilinear homomorphism $p : M \times N \to P$ such that whenever Z is an A-module and $f : M \times N \to Z$ is group homomorphism that is A-bilinear, there is a homomorphism of A-modules $g : P \to Z$ such that $f = g \circ p$. Since $M \otimes N$ is a tensor product in the categorical sense, it suffices to show that $M \otimes N$ and P are isomorphic as A-modules, and we claim that there is a unique A-module isomorphism $\iota : M \otimes N \to P$ satisfying $\iota([m] \otimes [n]) = [m \otimes n]$. There are a number of things to verify, but each is easy: (a) $[m \otimes n]$ depends only on $[m]$ and $[n]$ and not on the choices of representatives m and n; (b) the formula defines ι uniquely, because the elements of the form $[m] \otimes [n]$ generate $M \otimes N$; (c) ι is an A-module homomorphism; (d) there is a unique A-module homomorphism $\eta : P \to M \otimes N$ satisfying $\eta([m \otimes n]) = [m] \otimes [n]$ (which again involves verifications like (a), (b), and (c)); (e) ι and η are (obviously, at this point) inverses.

Now let φ and ψ be $\mathcal{A}$-homomorphisms from $\mathcal{F}$ to $\mathcal{F}'$ and from $\mathcal{G}$ to $\mathcal{G}'$. Then $\varphi_x \otimes \psi_x$ is a homomorphism (of abelian groups, in general—of $\mathcal{A}_x$-modules, if $\mathcal{A}$ is commutative), and the definition of $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ shows that the collection of the $\varphi_x \otimes \psi_x$ is a homomorphism from $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ to $\mathcal{F}' \otimes_{\mathcal{A}} \mathcal{G}'$, which we denote by $\varphi \otimes \psi$; if ψ is the identity function we write φ in place of $\varphi \otimes 1$.

All the usual properties of the tensor product of two modules are shared by the tensor product of two sheaves of modules. For example, any exact sequence

$$\mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$$

gives rise to an exact sequence

$$\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{A}} \mathcal{G} \to \mathcal{F}'' \otimes_{\mathcal{A}} \mathcal{G} \to 0.$$

"Right exactness" of the tensor product is established by Proposition B6.2, taking into account Lemma B5.3. This fact, applied to the fibers over each point, implies Serre's claim, because, as a matter of definition, $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$.

There are canonical isomorphisms:

$$\mathcal{F} \otimes_{\mathcal{A}} (\mathcal{G}_1 + \mathcal{G}_2) \approx \mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}_1 + \mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}_2, \quad \mathcal{F} \otimes_{\mathcal{A}} \mathcal{A} \approx \mathcal{F},$$

and (supposing that $\mathcal{A}$ is commutative, to simplify the notations):

$$\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G} \approx \mathcal{G} \otimes_{\mathcal{A}} \mathcal{F}, \quad \mathcal{F} \otimes_{\mathcal{A}} (\mathcal{G} \otimes_{\mathcal{A}} \mathcal{H}) \approx (\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}) \otimes_{\mathcal{A}} \mathcal{H}.$$

These identities are direct consequences of the corresponding identities for tensor products of rings, given in Lemma A6.2.

11. Sheaves of germs of homomorphisms from one sheaf to another. Let $\mathcal{A}$ be a sheaf of rings, $\mathcal{F}$ and $\mathcal{G}$ two sheaves of $\mathcal{A}$-modules. If U is an open subset of X, let $\mathcal{H}_U$ be the group of homomorphisms from $\mathcal{F}(U)$ to $\mathcal{G}(U)$. (We also say "homomorphism from $\mathcal{F}$ to $\mathcal{G}$ on U" in place of "homomorphism from $\mathcal{F}(U)$ to $\mathcal{G}(U)$.") The operation of restriction of a homomorphism defines $\varphi_U^V : \mathcal{H}_V \to \mathcal{H}_U$; the sheaf defined by the system $(\mathcal{H}_U, \varphi_U^V)$ is called the *sheaf of germs of homomorphisms from $\mathcal{F}$ to $\mathcal{G}$*, and is denoted by $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$. An element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x$ is a germ of a homomorphism from $\mathcal{F}$ to $\mathcal{G}$ in a neighborhood of x, and it unambiguously defines an $\mathcal{A}_x$-homomorphism from $\mathcal{F}_x$ to $\mathcal{G}_x$, whence there is a canonical homomorphism

$$\rho : \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x).$$

But, contrary to what was the case for the operations studied up to this point, the homomorphism ρ is not generally a bijection; in no. 14 we will give a sufficient condition for this.

If $\varphi : \mathcal{F}' \to \mathcal{F}$ and $\psi : \mathcal{G} \to \mathcal{G}'$ are homomorphisms, there is evidently an induced homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\varphi, \psi) : \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}', \mathcal{G}').$$

Every exact sequence $0 \to \mathcal{G} \to \mathcal{G}' \to \mathcal{G}''$ gives rise to an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}') \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}'').$$

"Left exactness" of the bifunctor Hom is established by Proposition B6.1, again taking into account Lemma B5.3. One might expect that, as above, Serre's claim also follows from application of this to the fibers over each point, but this does not appear to be the case. The diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x \arrow[r] \arrow[d, "\rho"'] & \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}')_x \arrow[r] \arrow[d, "\rho"'] & \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}'')_x \arrow[d, "\rho"'] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x) \arrow[r] & \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}'_x) \arrow[r] & \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}^q_x, \mathcal{G}''_x)
\end{tikzcd}$$

is the key to understanding what is going on here. Because $0 \to \mathcal{G} \to \mathcal{G}' \to \mathcal{G}''$ is exact, $0 \to \mathcal{G}_x \to \mathcal{G}'_x \to \mathcal{G}''_x$ is exact for each x, and the bottom row of the diagram is exact due to results cited above. If this diagram commuted, and each ρ was an isomorphism, one could conclude that the top row was exact. But as Serre has just mentioned, without further assumptions, the vertical maps need not be isomorphisms.

We can prove commutativity. More generally, we will show that if $\varphi : \mathcal{F}' \to \mathcal{F}$ and $\psi : \mathcal{G} \to \mathcal{G}'$ are homomorphisms, then the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G})_x \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(\varphi,\psi)_x"] \arrow[d, "\rho"'] & \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}',\mathcal{G}')_x \arrow[d, "\rho"'] \\ \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x,\mathcal{G}_x) \arrow[r, "\mathrm{Hom}_{\mathcal{A}_x}(\varphi_x,\psi_x)"] & \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}'_x,\mathcal{G}'_x) \end{tikzcd}$$

commutes. An element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G})_x$ is, by definition, the germ of some $\kappa \in \mathrm{Hom}_{\mathcal{A}(U)}(\mathcal{F}(U),\mathcal{G}(U))$, for some neighborhood U of x, and for such a κ the asserted commutativity boils down to

$$\left(\psi|_{\mathcal{G}(U)} \circ \kappa \circ \varphi|_{\mathcal{F}'(U)}\right)_x = \psi_x \circ \kappa_x \circ \varphi_x,$$

which is of course true.

Proposition 5 of no. 14 gives conditions under which the maps ρ are isomorphisms, and provided they hold, the sequence above is exact. Fortunately, these conditions will generally be in force throughout, so while we must be careful to avoid invoking the commutativity above in their absence, there is little reason to expect that this will be necessary.

All these remarks apply equally to the following claim, which one would expect Serre to assert here, and which he invokes in no. 14: if $\mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$ is exact, then for any $\mathcal{G}$ the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}',\mathcal{G}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}'',\mathcal{G})$$

is exact.

There are also canonical isomorphisms: $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{G}) \approx \mathcal{G}$,

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G}_1+\mathcal{G}_2) \approx \mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G}_1) + \mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G}_2)$$

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}_1+\mathcal{F}_2,\mathcal{G}) \approx \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}_1,\mathcal{G}) + \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}_2,\mathcal{G}).$$

The isomorphism $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{G}) \approx \mathcal{G}$ depends on our maintained assumption that the function taking each x to the unit of $\mathcal{A}_x$ is a global section of $\mathcal{A}$.

§2. Coherent sheaves of modules

In this part X is a topological space and $\mathcal{A}$ is a sheaf of rings on X. We assume that each $\mathcal{A}_x$, $x \in X$, is commutative and possesses a unit element that varies continuously with x. All the sheaves considered up to no. 16 are sheaves of $\mathcal{A}$-modules, and all the homomorphisms are $\mathcal{A}$-homomorphisms.

12. Definitions. Let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules, and let $s_1,\ldots,s_p$ be sections of $\mathcal{F}$ on an open $U \subset X$. If we associate with each family $f_1,\ldots,f_p$ of elements of $\mathcal{A}_x$ the element $\sum_{i=1}^{i=p} f_i \cdot s_i(x)$ of $\mathcal{F}_x$, we obtain a homomorphism

$\varphi : \mathcal{A}^p \to \mathcal{F}$ defined on the open set U. (More formally, φ is a homomorphism from $\mathcal{A}^p(U)$ to $\mathcal{F}(U)$, with the notations of no. 4). The kernel $\mathcal{R}(s_1, \dots, s_p)$ of the homomorphism φ is a subsheaf of $\mathcal{A}^p$ called the *sheaf!of relations* between the s_i. The image of φ is a subsheaf of $\mathcal{F}$ determined by the s_i. Conversely, any homomorphism $\varphi : \mathcal{A}^p \to \mathcal{F}$ is defined by sections $s_1, \dots, s_p$ given by the formulas:

$$s_1(x) := \varphi_x(1, 0, \dots, 0) \ , \ \dots \ , \ s_p(x) := \varphi_x(0, \dots, 0, 1).$$

Definition 1. *A sheaf of $\mathcal{A}$-modules is said to be of finite type if it is generated locally by a finite number of sections.*

In other words, for each point $x \in X$ there exists an open U containing x, and a finite number of sections $s_1, \dots, s_p$ of $\mathcal{F}$ on U, such that each element of $\mathcal{F}_y$, $y \in U$, is a linear combination, with coefficients in $\mathcal{A}_y$, of the $s_i(y)$. In view of the remark above, this is the same as saying that the restriction of $\mathcal{F}$ to U is isomorphic to a quotient sheaf of the sheaf $\mathcal{A}^p$.

Proposition 1. *Let $\mathcal{F}$ be a sheaf of finite type. If $s_1, \dots, s_p$ are sections of $\mathcal{F}$, defined on a neighborhood of a point $x \in X$, that generate $\mathcal{F}_x$, then they generate $\mathcal{F}_y$ for all y sufficiently close to x.*

If $\mathcal{F}$ is of finite type, there are a finite number of sections of $\mathcal{F}$ defined in a neighborhood of x, say $t_1, \dots, t_q$, that generate $\mathcal{F}_y$ for y sufficiently close to x. When the $s_j(x)$ generate $\mathcal{F}_x$, there are sections f_{ij} of $\mathcal{A}$ in a neighborhood of x such that $t_i(x) = \sum_{j=1}^{j=p} f_{ij}(x) \cdot s_j(x)$. It follows that, for each y sufficiently close to x, we have:

$$t_i(y) = \sum\nolimits_{j=1}^{j=p} f_{ij}(y) \cdot s_j(y),$$

from which it follows that the $s_j(y)$ generate $\mathcal{F}_y$, qed.

Definition 2. *A sheaf of $\mathcal{A}$-modules $\mathcal{F}$ is said to be coherent if:*

(a) $\mathcal{F}$ is of finite type,

(b) If $s_1, \dots, s_p$ are sections of $\mathcal{F}$ on an open $U \subset X$, the sheaf of relations between the s_i is a sheaf of finite type (on the open set U).

Note that Definitions 1 and 2 are local in character.

Proposition 2. *Locally, a coherent sheaf is isomorphic to the cokernel of a homomorphism $\varphi : \mathcal{A}^q \to \mathcal{A}^p$.*

This follows immediately from the definitions, and the remarks prior to Definition 1.

Proposition 3. *If a subsheaf of a coherent sheaf is a sheaf of finite type, then it is coherent.*

In effect, if a sheaf $\mathcal{F}$ satisfies condition (b) of Definition 2, every subsheaf also evidently satisfies it.

13. Principal properties of coherent sheaves.

THEOREM 1. *Let* $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ *be an exact sequence. If two of the three sheaves* $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ *are coherent, so is the third.*

Suppose $\mathcal{G}$ and $\mathcal{H}$ are coherent. There is locally a surjective homomorphism $\gamma : \mathcal{A}^p \to \mathcal{G}$; let $\mathcal{I}$ be the kernel of $\beta \circ \gamma$. When $\mathcal{H}$ is coherent, $\mathcal{I}$ is a sheaf of finite type (condition (b)). Therefore $\gamma(\mathcal{I})$ is a sheaf of finite type, which is coherent by Proposition 3. Since α is an isomorphism from $\mathcal{F}$ to $\gamma(\mathcal{I})$, it follows that $\mathcal{F}$ is coherent.

Suppose $\mathcal{F}$ and $\mathcal{G}$ are coherent. Since $\mathcal{G}$ is of finite type, $\mathcal{H}$ is also of finite type, and it remains to show that $\mathcal{H}$ satisfies condition (b) of Definition 2. Let $s_1, \ldots, s_p$ be a finite number of sections of $\mathcal{H}$ in a neighborhood of a point $x \in X$. The question being local, we may suppose that there exist sections $s'_1, \ldots, s'_p$ of $\mathcal{G}$ such that $s_i = \beta(s'_i)$. Let $n_1, \ldots, n_q$ be a finite number of sections of $\mathcal{F}$ in a neighborhood of x, generating $\mathcal{F}_y$ for y close enough to x. In order for a family $f_1, \ldots, f_q$ of elements of $\mathcal{A}_y$ to appear in $\mathcal{R}(s_1, \ldots, s_p)_y$, it is necessary and sufficient that there exist $g_1, \ldots, g_q \in \mathcal{A}_y$ such that

$$\sum\nolimits_{i=1}^{i=p} f_i \cdot s'_i = \sum\nolimits_{j=1}^{j=q} g_j \cdot \alpha(n_j) \quad \text{at } y.$$

Now the sheaf of relations between the s'_i and the $\alpha(n_j)$ is of finite type, when $\mathcal{G}$ is coherent. The sheaf $\mathcal{R}(s_1, \ldots, s_p)$, which is the image of the preceeding under the canonical projection of $\mathcal{A}^{p+q}$ onto $\mathcal{A}^q$, is thus of finite type, which completes the proof that $\mathcal{H}$ is coherent.

Suppose $\mathcal{F}$ and $\mathcal{H}$ are coherent. The question being local, we may suppose that $\mathcal{F}$ (resp. $\mathcal{H}$) is generated by a finite number of sections $n_1, \ldots, n_q$ (resp. $s_1, \ldots, s_p$); in addition we may suppose that there are sections s'_i of $\mathcal{G}$ such that $s_i = \beta(s'_i)$. It is then clear that the section s'_i and $\alpha(n_j)$ generate $\mathcal{G}$, which proves that $\mathcal{G}$ is a sheaf of finite type. Now let $t_1, \ldots, t_r$ be a finite number of section of $\mathcal{G}$ in a neighborhood of a point x; since $\mathcal{H}$ is coherent, there exist sections f^i_j of $\mathcal{A}^r$ $(1 \le i \le r,\ 1 \le j \le s)$, defined in a neighborhood of x, which generate the sheaf of relations between the $\beta(t_i)$. Let $u_j := \sum_{i=1}^r f^i_j \cdot t_i$; then $\sum_{i=1}^r f^i_j \cdot \beta(t_i) = 0$, so the u_j are contained in $\alpha(\mathcal{F})$, and, since $\mathcal{F}$ is coherent, the sheaf of relations between the u_j is generated, in a neighborhood of x, by a finite number of sections g^j_k $(1 \le j \le s,\ 1 \le k \le t)$. I claim that the $\sum_{j=1}^s g^j_k \cdot f^i_j$ generates the sheaf $\mathcal{R}(t_1, \ldots, t_r)$ in a neighborhood of x; in effect, if $\sum_{i=1}^r f_i \cdot t_i = 0$ at y, with $f_i \in \mathcal{A}_y$, then $\sum_{i=1}^r f_i \cdot \beta(t_i) = 0$, and there exist $g_j \in \mathcal{A}_y$ with $f_i = \sum_{j=1}^s g_j \cdot f^i_j$; upon writing that $\sum_{i=1}^r f_i \cdot t_i = 0$, we obtain $\sum_{j=1}^s g_j \cdot u_j = 0$, and from that infer that the g_j are linear combinations of the g^j_k, which proves the claim. It follows that $\mathcal{G}$ satisfies (b), which completes the proof.

COROLLARY. *The direct sum of a finite family of coherent sheaves is a coherent sheaf.*

THEOREM 2. *Let φ be a homomorphism from a coherent sheaf $\mathcal{F}$ to a coherent sheaf $\mathcal{G}$. The kernel, cokernel, and image of φ are coherent sheaves.*

Because $\mathcal{F}$ is coherent, image(φ) is of finite type, hence coherent by Proposition 3. Applying Theorem 1 to the exact sequences

$$0 \to \mathrm{Ker}(\varphi) \to \mathcal{F} \to \mathrm{image}(\varphi) \to 0$$

$$0 \to \mathrm{image}(\varphi) \to \mathcal{G} \to \mathrm{Coker}(\varphi) \to 0,$$

one sees that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are coherent.

COROLLARY. *Let $\mathcal{F}$ and $\mathcal{G}$ be coherent subsheaves of a coherent sheaf $\mathcal{H}$. Then the sheaves $\mathcal{F}+\mathcal{G}$ and $\mathcal{F}\cap\mathcal{G}$ are coherent.*

For $\mathcal{F}+\mathcal{G}$, this follows from Proposition 3. As for $\mathcal{F}\cap\mathcal{G}$, it's the kernel of $\mathcal{F} \to \mathcal{H}/\mathcal{G}$.

14. Operations on coherent sheaves. We have seen that the direct sum of finitely many coherent sheaves is a coherent sheaf. Now we will establish analogous results for the functors $\otimes$ and Hom.

PROPOSITION 4. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves, $\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G}$ is a coherent sheaf.*

According to Proposition 2, $\mathcal{F}$ is locally isomorphic to the cokernel of a homomorphism $\varphi : \mathcal{A}^q \to \mathcal{A}^p$; consequently $\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G}$ is locally isomorphic to the cokernel of $\varphi : \mathcal{A}^q\otimes_{\mathcal{A}}\mathcal{G} \to \mathcal{A}^p\otimes_{\mathcal{A}}\mathcal{G}$. But $\mathcal{A}^q\otimes_{\mathcal{A}}\mathcal{G}$ and $\mathcal{A}^p\otimes_{\mathcal{A}}\mathcal{G}$ are respectively isomorphic to $\mathcal{G}^q$ and $\mathcal{G}^p$, which are coherent (Corollary to Theorem 1). Therefore $\mathcal{F}\otimes_{\mathcal{A}}\mathcal{G}$ is coherent (Theorem 2).

PROPOSITION 5. *Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves, with $\mathcal{F}$ coherent. For each $x \in X$, the module $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G})_x$ is isomorphic to $\mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x,\mathcal{G}_x)$.*

More precisely, we will show that the homomorphism

$$\rho : \mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G})_x \to \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x,\mathcal{G}_x),$$

defined in no. 11, is bijective. First of all let $\psi : \mathcal{F} \to \mathcal{G}$ be a homomorphism defined in a neighborhood of x, and null on $\mathcal{F}_x$; since $\mathcal{F}$ is of finite type we immediately conclude that ψ is null on some neighborhood of x, so ρ is injective. To establish surjectivity we need to show that if φ is an $\mathcal{A}_x$-homomorphism from $\mathcal{F}_x$ to $\mathcal{G}_x$, then there is a homomorphism $\psi : \mathcal{F} \to \mathcal{G}$, defined in a neighborhood of x, such that $\psi_x = \varphi$. Let $m_1, \dots, m_p$ be a finite number of sections of $\mathcal{F}$ in a neighborhood of x that generate $\mathcal{F}_y$ for all y sufficiently close to x, and let f^i_j ($1 \le i \le p$, $1 \le j \le q$) be sections of $\mathcal{A}^p$ that generate $\mathcal{R}(m_1, \dots, m_p)$ in a neighborhood of x. There exist sections of $\mathcal{G}$ defined in a neighborhood of x, say $n_1, \dots, n_p$, such that $n_i(x) = \varphi(m_i(x))$. Set $p_j := \sum_{i=1}^p f^i_j \cdot n_i$, $1 \le j \le q$; the p_j are local sections of $\mathcal{G}$ that vanish at x, and therefore also in some neighborhood U of x. It follows that, for $y \in U$, the formula $\sum f_i \cdot m_i(y) = 0$ with $f_i \in \mathcal{A}_y$, entails $\sum f_i \cdot n_i(y) = 0$.

In a bit more detail, the formula $\sum f_i \cdot m_i(y) = 0$ implies the existence of $a_1, \dots, a_q \in \mathcal{A}_y$ such that $f_i(y) = \sum_j a_j f_j^i(y) = 0$, and

$$\sum_i f_i \cdot n_i(y) = \sum_j a_j \big(\sum_i f_j^i \cdot n_i(y) \big) = 0.$$

For each element $m = \sum f_i \cdot m_i(y) \in \mathcal{F}_y$ we may set

$$\psi_y(m) := \sum_{i=1}^{p} f_i \cdot n_i(y) \in \mathcal{G}_y,$$

and this formula defines $\psi_y(m)$ unambiguously. The collection of ψ_y, $y \in Y$, constitutes a homomorphism $\psi : \mathcal{F} \to \mathcal{G}$, defined on U, and such that $\psi_x = \varphi$, which completes the proof.

Proposition 6. *If $\mathcal{F}$ and $\mathcal{G}$ are two coherent sheaves, then* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ *is a coherent sheaf.*

The question being local, we may suppose, in view of Proposition 2, that we have an exact sequence $\mathcal{A}^q \to \mathcal{A}^p \to \mathcal{F} \to 0$. It follows from the preceeding Proposition that

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^p, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^q, \mathcal{G}) \qquad (*)$$

is exact.

Recall the remarks concerning left exactness in no. 11. The phrase "It follows from the preceeding Proposition" seems to be in line with the logic described there.

But the sheaf $\mathrm{Hom}(\mathcal{A}^p, \mathcal{G})$ is isomorphic to $\mathcal{G}^p$, and is consequently coherent, and similarly for $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^q, \mathcal{G})$. Now Theorem 2 implies that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ is coherent.

15. Coherent sheaves of rings. The sheaf of rings $\mathcal{A}$ may be regarded as a sheaf of $\mathcal{A}$-modules; if this sheaf of $\mathcal{A}$-modules is coherent, we say that $\mathcal{A}$ is a *coherent sheaf of rings*. Since $\mathcal{A}$ is evidently of finite type, this signifies that $\mathcal{A}$ satisfies condition (b) of Definition 2. In other words:

Definition 3. *The sheaf $\mathcal{A}$ is a coherent sheaf of rings if the sheaf of relations between any finite number of sections of $\mathcal{A}$ on any open set U is a sheaf of finite type on U.*

Examples (1) If X is a complex analytic variety, the sheaf of germs of homomorphic functions on X is a coherent sheaf of rings, by a theorem of K. Oka (cf. [3], exposé XV, or [5], §5).

(2) If X is an algebraic variety, the sheaf of local rings of X is a coherent sheaf of rings (cf. no. 37, Proposition 1).

When $\mathcal{A}$ is a coherent sheaf of rings we have the following results:

Proposition 7. *In order for a sheaf of $\mathcal{A}$-modules to be coherent it is necessary and sufficient that, locally, it is isomorphic to the cokernel of a homomorphism $\varphi : \mathcal{A}^q \to \mathcal{A}^p$.*

The necessity is none other than Proposition 2; the sufficiency results from the fact that $\mathcal{A}^p$ and $\mathcal{A}^q$ are coherent, and from Theorem 2.

Proposition 8. *In order for a subsheaf of $\mathcal{A}^p$ to be coherent it is necessary and sufficient that it be of finite type.*

This is a special case of Proposition 3.

Corollary. *The sheaf of relations between a finite number of sections of a coherent sheaf is a coherent sheaf.*

The definition of a coherent sheaf requires that the sheaf of relations be of finite type.

Proposition 9. *Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{A}$-modules. For each $x \in X$, let $\mathcal{I}_x$ be the ideal of $\mathcal{A}_x$ consisting of the $a \in \mathcal{A}_x$ such that $af = 0$ for all $f \in \mathcal{F}_x$. The $\mathcal{I}_x$ form a coherent sheaf of ideals (called the* annihilator *of $\mathcal{F}$).*

In effect, $\mathcal{I}_x$ is the kernel of the homomorphism: $\mathcal{A}_x \to \mathrm{Hom}_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{F}_x)$; therefore we can apply Propositions 5 and 6, and Theorem 2.

More generally, the *transporter* $\mathcal{F} : \mathcal{G}$ of a coherent sheaf $\mathcal{G}$ into a coherent subsheaf $\mathcal{F}$ is a coherent sheaf of ideals (it is the annihilator of $\mathcal{G}/\mathcal{F}$).

16. Change of rings. The notions of a sheaf of finite type, and of a coherent sheaf, are relative to a given sheaf of rings. When one considers several sheaves of rings, one says "of finite type over $\mathcal{A}$," or "$\mathcal{A}$-coherent," to indicate the ring in question.

Theorem 3. *Let $\mathcal{A}$ be a coherent sheaf of rings, let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{A}$, and let $\mathcal{F}$ be a sheaf of $\mathcal{A}/\mathcal{I}$-modules. In order for $\mathcal{F}$ to be $\mathcal{A}/\mathcal{I}$-coherent it is necessary and sufficient that it be $\mathcal{A}$-coherent. In particular, $\mathcal{A}/\mathcal{I}$ is a coherent sheaf of rings.*

It is clear that "of finite type over $\mathcal{A}$" is equivalent to "of finite type over $\mathcal{A}/\mathcal{I}$." For the other part, if $\mathcal{F}$ is $\mathcal{A}$-coherent, and if $s_1, \ldots, s_p$ are sections of $\mathcal{F}$ on an open set U, the sheaf of relations between the s_i, with coefficients in $\mathcal{A}$, is of finite type on $\mathcal{A}$; it follows immediately that the sheaf of relations between the s_i, with coefficients in $\mathcal{A}/\mathcal{I}$, if of finite type on $\mathcal{A}/\mathcal{I}$, because it is the image of the preceeding sheaf under the canonical homomorphism $\mathcal{A}^p \to (\mathcal{A}/\mathcal{I})^p$. Therefore $\mathcal{F}$ is $\mathcal{A}/\mathcal{I}$-coherent. In particular, since $\mathcal{A}/\mathcal{I}$ is $\mathcal{A}$-coherent, it is also $\mathcal{A}/\mathcal{I}$-coherent, i.e., $\mathcal{A}/\mathcal{I}$ is a coherent sheaf of rings. In the other direction, if $\mathcal{F}$ is $\mathcal{A}/\mathcal{I}$-coherent, it is locally isomorphic to the cokernel of a homomorphism $\varphi : (\mathcal{A}/\mathcal{I})^q \to (\mathcal{A}/\mathcal{I})^p$, and since $\mathcal{A}/\mathcal{I}$ is $\mathcal{A}$-coherent, $\mathcal{F}$ is $\mathcal{A}$-coherent, by Theorem 2.

17. Extension and restriction of a coherent sheaf. Let Y be a closed subspace of X. If $\mathcal{G}$ is a sheaf on Y, we denote by $\mathcal{G}^X$ the sheaf obtained by extending $\mathcal{G}$ by 0 outside of Y; it is a sheaf on X (cf. no. 5). If $\mathcal{A}$ is a sheaf of rings on Y, $\mathcal{A}^X$ is a sheaf of rings on X, and, if $\mathcal{F}$ is a sheaf of $\mathcal{A}$-modules, $\mathcal{F}^X$ is a sheaf of $\mathcal{A}^X$-modules.

PROPOSITION 10. *For $\mathcal{F}$ to be of finite type on $\mathcal{A}$, it is necessary and sufficient that $\mathcal{F}^X$ is of finite type on $\mathcal{A}^X$.*

Let U be an open subset of X, and let $V := U \cap Y$. Each homomorphism $\varphi : \mathcal{A}^p \to \mathcal{F}$ above V defines a homomorphism $\varphi^X : (\mathcal{A}^X)^p \to \mathcal{F}^X$ above U, and vice versa; in order for φ to be surjective, it is necessary and sufficient that φ^X is. The proposition follows immediately from this.

In the same way one shows that :

PROPOSITION 11. *For $\mathcal{F}$ to be $\mathcal{A}$-coherent, it is necessary and sufficient that $\mathcal{F}^X$ is $\mathcal{A}^X$-coherent.*

From which, on letting $\mathcal{F} = \mathcal{A}$:

COROLLARY. *For $\mathcal{A}$ to be a coherent sheaf of rings, it is necessary and sufficient that $\mathcal{A}^X$ is a coherent sheaf of rings.*

§3. Cohomology of a space with values in a sheaf

In this paragraph, X is a topological space, which may or may not be Hausdorff. By a *cover* of X, we always mean an open cover.

18. Cochains of a cover. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a cover of X. If $s = (i_0, \ldots, i_p)$ is a finite sequence of elements of I, we set:

$$U_s = U_{i_0 \cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}.$$

Let $\mathcal{F}$ be a sheaf of abelian groups on the space X. If p is an integer ≥ 0, a *p-cochain of $\mathfrak{U}$ with values in $\mathcal{F}$* is a function f that takes each sequence $s = (i_0, \ldots, i_p)$ of $p+1$ elements of I to some section $f_s = f_{i_0 \cdots i_p}$ of $\mathcal{F}$ above $U_{i_0 \cdots i_p}$. The p-cochains form an abelian group, denoted by $C^p(\mathfrak{U}, \mathcal{F})$; this is the product group $\prod \Gamma(U_s, \mathcal{F})$, where the product is over all sequences s of $p+1$ elements of I. The family of the $C^p(\mathfrak{U}, \mathcal{F})$, $p = 0, 1, \ldots$, is denoted by $C(\mathfrak{U}, \mathcal{F})$. A p-cochain is also called a *cochain of degree p*.

A p-cochain f is said to be *alternating* if:

(a) $f_{i_0 \cdots i_p} = 0$ whenever two of the indices $i_0, \ldots, i_p$ are equal,

(b) $f_{i_{\sigma 0} \cdots i_{\sigma p}} = \varepsilon_\sigma f_{i_0 \cdots i_p}$ if σ is a permutation of the set $\{0, \ldots, p\}$ (ε_σ denotes the sign of σ).

The alternating p-cochains form a subgroup $C'^p(\mathfrak{U}, \mathcal{F})$ of the group $C^p(\mathfrak{U}, \mathcal{F})$; the family of the $C'^p(\mathfrak{U}, \mathcal{F})$ is denoted by $C'(\mathfrak{U}, \mathcal{F})$.

19. Simplicial operations. Let $S(I)$ be the simplicial complex containing all simplices with vertices in the set I; an (ordered) simplex is a sequence $s = (i_0, \ldots, i_p)$ of elements of I; p is called the *dimension* of s. Let

$K(I) = \sum_{p=0}^{\infty} K_p(I)$ be the complex defined by $S(I)$: by definition, $K_p(I)$ is the free abeliam group generated by the set of simplices of dimension p in $S(I)$.

If s is a simplex in $S(I)$, we denote the set of vertices of s by $|s|$.

A function $h : K_p(I) \to K_q(I)$ is called a *simplicial endomorphism* if

(i) h is a homomorphism,

(ii) For each simplex s of dimension p of $S(I)$, we have

$$h(s) = \sum_{s'} c_s^{s'} \cdot s' \qquad \text{with } c_s^{s'} \in \mathbb{Z},$$

where the sum is over the simplices of dimension q such that $|s'| \subset |s|$.

Let h be a simplicial endomorphism, and let $f \in C^q(\mathfrak{U}, \mathcal{F})$ be a cochain of degree q. For each simplex s of dimension p, set

$$({}^t hf)_s := \sum_{s'} c_s^{s'} \cdot \rho_s^{s'}(f_{s'}),$$

where $\rho_s^{s'}$ denotes the restriction homomorphism $\Gamma(U_{s'}, \mathcal{F}) \to \Gamma(U_s, \mathcal{F})$, which is well defined because $|s'| \subset |s|$. The function $f \to {}^t hf$ is a homomorphism

$${}^t h : C^q(\mathfrak{U}, \mathcal{F}) \to C^p(\mathfrak{U}, \mathcal{F}),$$

and one immediately verifies the formulas:

$${}^t(h_1 + h_2) = {}^t h_1 + {}^t h_2, \quad {}^t(h_1 \circ h_2) = {}^t h_1 \circ {}^t h_2, \quad {}^t\mathbf{1} = \mathbf{1}.$$

Note. In practice one frequently neglects to write the restriction homomorphism $\rho_s^{s'}$.

20. The complexes of cochains. Let's apply the preceeding to the simplicial endomorphism

$$\partial : K_{p+1}(I) \to K_p(I),$$

defined by the usual formula:

$$\partial(i_0, \ldots, i_{p+1}) := \sum_{j=0}^{p+1} (-1)^j (i_0, \ldots, \hat{i_j}, \ldots, i_{p+1}),$$

with the sign ˆ signifying, as usual, that the symbol above which it is found is being omitted.

We obtain in this way a homomorphism ${}^t\partial : C^p(\mathfrak{U}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}, \mathcal{F})$, which we denote by d; by definition, we have

$$(df)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \rho_j (f_{i_0 \cdots \hat{i_j} \cdots i_{p+1}}),$$

where ρ_j is the restriction homomorphism

$$\rho_j : \Gamma(U_{i_0\cdots\hat{i_j}\cdots i_{p+1}}, \mathcal{F}) \to \Gamma(U_{i_0\cdots i_{p+1}}, \mathcal{F}).$$

Because $\partial\circ\partial = 0$, we have $d\circ d = 0$. Thus $C(\mathfrak{U}, \mathcal{F})$ is found to be equipped with a coboundary operator that makes it a complex. The q^{th} cohomology group of the complex $C(\mathfrak{U}, \mathcal{F})$ will be denoted by $H^q(\mathfrak{U}, \mathcal{F})$. We have:

Proposition 1. $H^0(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

A 0-cochain is a system $(f_i)_{i\in I}$, with each f_i being a section of $\mathcal{F}$ above U_i; in order for this cochain to be a cocycle, it is necessary and sufficient that $f_i - f_j = 0$ above $U_i \cap U_j$, i.e., there is a section f of $\mathcal{F}$ on all of X which coincides with f_i on U_i for each $i \in I$. Thus we obtain the Proposition.

(Consequently $H^0(\mathfrak{U}, \mathcal{F})$ is independent of $\mathfrak{U}$, but please be forewarned that it is not the same for $H^q(\mathfrak{U}, \mathcal{F})$ in general.)

It is immediately evident that *df* is alternating if *f* is alternating; in other words, *d* restricts to a coboundary map of $C'(\mathfrak{U}, \mathcal{F})$ which forms a subcomplex of $C(\mathfrak{U}, \mathcal{F})$. The cohomology groups of $C'(\mathfrak{U}, \mathcal{F})$ will be denoted by $H'^q(\mathfrak{U}, \mathcal{F})$.

Proposition 2. *The injection of $C'(\mathfrak{U}, \mathcal{F})$ into $C(\mathfrak{U}, \mathcal{F})$ induces an isomorphism of $H'^q(\mathfrak{U}, \mathcal{F})$ and $H^q(\mathfrak{U}, \mathcal{F})$ for all $q \geq 0$.*

Endow the set I with a total ordering, and let h be the simplicial endomorphism of $K(I)$ defined in the following manner:

$h((i_0, \dots, i_q)) = 0$ if two of the indices $i_0, \dots, i_q$ are equal,

$h((i_0, \dots, i_q)) = \varepsilon_\sigma(i_{\sigma 0}, \dots, i_{\sigma q})$ if all of the indices $i_0, \dots, i_q$ are distinct, where σ is the permutation of $\{0, \dots, q\}$ such that $i_{\sigma 0} < i_{\sigma 1} < \cdots < i_{\sigma q}$.

One verifies right away that h commutes with ∂, and that $h(s) = s$ if $\dim(s) = 0$; consequently (cf. [7], Chap. VI, §5) there is a simplicial endomorphism k, raising the dimension by 1, such that $\mathbf{1} - h = \partial \circ k + k \circ \partial$. From this, on passing to $C(\mathfrak{U}, \mathcal{F})$,

$$\mathbf{1} - {}^t h = {}^t k \circ d + d \circ {}^t k.$$

But one verifies right away that ${}^t h$ is a *projection* of $C(\mathfrak{U}, \mathcal{F})$ on $C'(\mathfrak{U}, \mathcal{F})$; as the preceeding formula shows that it is a homotopy operator, the Proposition is established. (Compare with [7], Chap. VI, th. 6.10).

Everything happens very quickly here, with citations in place of explanation. Here are additional details for several steps.

The verification that h and ∂ commute is mechanical, but less immediate (except insofar as it is standard, and occurs frequently) than Serre indicates. If two of the indices $i_0, \dots, i_q$ are the same, then

$$\partial h(i_0, \dots, i_q) = 0 = h\partial(i_0, \dots, i_q)$$

because every term of $\partial(i_0, \ldots, i_q)$ is either killed by the first clause of the definition of h or appears twice with opposite sign. Suppose the indices are all distinct, let σ be the permutation such that $i_{\sigma 0} < \cdots < i_{\sigma q}$, and for $k = 0, \ldots, q$ let σ_k be the permutation of $\{0, \ldots, \hat{k}, \ldots, q\}$ such that $i_{\sigma_k 0} < \cdots < i_{\sigma_k q}$. Then

$$\partial h(i_0, \ldots, i_q) = \varepsilon_\sigma \sum_{j=1}^{q} (-1)^j (i_{\sigma 0}, \ldots, \widehat{i_{\sigma j}}, \ldots, i_{\sigma q})$$

and

$$h\partial(i_0, \ldots, i_q) = \sum_{k=0}^{q} (-1)^k \varepsilon_{\sigma_k} (i_{\sigma_k 0}, \ldots, \widehat{i_k}, \ldots, i_{\sigma_k q}),$$

so the concrete meaning of $\partial h = h\partial$ is that if $\sigma j = k$, then $(-1)^j \varepsilon_\sigma = (-1)^k \varepsilon_{\sigma_k}$. This is true, and easy enough for the interested reader to pursue in detail.

Let $f = \mathbf{1} - h$. We now want to construct a system of homomorphisms $k_p : K_p \to K_{p+1}$ such that $f = \partial k + k\partial$, which is to say that $\partial k_p = f_p - k_{p-1}\partial$. If k_{p-1} is given, and $(f_p - k_{p-1}\partial)(s)$ is a boundary (image of ∂) for each $s \in K_p$, then a satisfactory k_p can be constructed by choosing a suitable image of each of the generators of K_p. Of course $f_0 - k_{-1}\partial = 0$ because f_0 and k_{-1} are both zero. Proceeding inductively, we may assume that $\partial k_{p-1} = f_{p-1} - k_{p-2}\partial$, in which case

$$\partial(f_p - k_{p-1}\partial) = (f_{p-1} - \partial k_{p-1})\partial = k_{p-2}\partial\partial = 0.$$

Therefore $(f_p - k_{p-1}\partial)(s)$ is always a cycle (element of the kernel of ∂). The homology of K is $\mathbb{Z}$ in dimension 0 and 0 in all other dimensions (you are expected to already know that the homomology of a simplex has this description, which is the case of finite I, and the extension to infinite I is not hard) so a cycle in dimension $p > 0$ is a boundary, and a suitable k_p exists.

Now let ι denote the inclusion map from $C'(\mathfrak{U}, \mathcal{F})$ to $C(\mathfrak{U}, \mathcal{F})$. When Serre says that ${}^t h$ is a "projection," presumably he means that its image is contained in $C'(\mathfrak{U}, \mathcal{F})$ and ${}^t h \circ \iota$ is the identity on $C'(\mathfrak{U}, \mathcal{F})$. It is indeed easy to verify these claims: clearly $({}^t hf)_{i_0 \cdots i_q} = 0$ if two of the indices $i_0, \ldots, i_q$ are the same, and if all of the indices are distinct, then

$$({}^t hf)_{i_0 \cdots i_q} = \varepsilon_\sigma f_{i_{\sigma 0} \cdots i_{\sigma q}}$$

where σ is the permutation such that $i_{\sigma 0} < \cdots < i_{\sigma q}$. If f is alternating, then in either case $({}^t hf)_{i_0 \cdots i_q} = f_{i_0 \cdots i_q}$.

In order to be clear, let η denote ${}^t h$ regarded as a chain map from $C(\mathfrak{U}, \mathcal{F})$ to $C'(\mathfrak{U}, \mathcal{F})$, so that when we think of the range of ${}^t h$ as $C(\mathfrak{U}, \mathcal{F})$ we have ${}^t h = \iota \circ \eta$. The argument showing that $\mathbf{1} - {}^t h$ induces the zero map in cohomology is, of course, quite simple: if f is a cycle, then

$$(\mathbf{1} - {}^t h)f = {}^t kdf + d^t kf = d^t kf$$

is a boundary. Thus $\iota \circ \eta$ induces the identity on each $H^q(\mathfrak{U}, \mathcal{F})$. Since $\eta \circ \iota$ is the identity, it induces the identity on each $H'^q(\mathfrak{U}, \mathcal{F})$. Of course the maps in cohomology induced by $\iota \circ \eta$ and $\eta \circ \iota$ are the respective compositions of the maps induced by η and ι, so we conclude that η and ι induce inverse isomorphisms between $H^q(\mathfrak{U}, \mathcal{F})$ and $H'^q(\mathfrak{U}, \mathcal{F})$ for each q.

COROLLARY. *$H^q(\mathfrak{U}, \mathcal{F}) = 0$ for $q > \dim(\mathfrak{U})$.*

By the definition of $\dim(\mathfrak{U})$, we have $U_{i_0 \cdots i_q} = \emptyset$ for $q > \dim(\mathfrak{U})$, if the indices $i_0, \ldots, i_q$ are distinct, in which case $C'^q(\mathfrak{U}, \mathcal{F}) = 0$, and this implies

$$H^q(\mathfrak{U}, \mathcal{F}) = H'^q(\mathfrak{U}, \mathcal{F}) = 0.$$

21. Passage from a cover to a finer cover. A cover $\mathfrak{U} = \{U_i\}_{i \in I}$ is said to be *finer* than a cover $\mathfrak{V} = \{V_j\}_{j \in J}$ if there is a function $\tau : I \to J$ such that $U_i \subset V_{\tau i}$ for all $i \in I$. If $f \in C^q(\mathfrak{V}, \mathcal{F})$, let:

$$(\tau f)_{i_0 \cdots i_q} = \rho_U^V(f_{\tau i_0 \cdots \tau i_q}),$$

where ρ_U^V denotes the restriction homomorphism defined by the inclusion of $U_{i_0 \cdots i_q}$ in $V_{\tau i_0 \cdots \tau i_q}$. The function $f \mapsto \tau f$ is a homomorphism from $C^q(\mathfrak{V}, \mathcal{F})$ to $C^q(\mathfrak{U}, \mathcal{F})$, defined for all $q \geq 0$ and commuting with d, so there are well defined homomorphisms

$$\tau^* : H^q(\mathfrak{V}, \mathcal{F}) \to H^q(\mathfrak{U}, \mathcal{F}).$$

PROPOSITION 3. *The homomorphisms $\tau^* : H^q(\mathfrak{V}, \mathcal{F}) \to H^q(\mathfrak{U}, \mathcal{F})$ depend only on $\mathfrak{U}$ and $\mathfrak{V}$, and not on the function τ.*

Let τ and τ' be two functions from I to J such that $U_i \subset V_{\tau i}$ and $U_i \subset V_{\tau' i}$; we must show that $\tau^* = \tau'^*$.

Fixing $f \in C^q(\mathfrak{V}, \mathcal{F})$, let

$$(kf)_{i_0 \cdots i_{q-1}} := \sum_{h=0}^{q-1} \rho_h(f_{\tau i_0 \cdots \tau i_h \tau' i_h \cdots \tau' i_{q-1}}),$$

where ρ_h denotes the restriction homomorphism induced by the inclusion of $U_{i_0 \cdots i_{q-1}}$ in $V_{\tau i_0 \cdots \tau i_h \tau' i_h \cdots \tau' i_{q-1}}$.

A direct calculation (cf. [7], Chap. VI, §3) shows that:

$$dkf + k\,df = \tau' f - \tau f,$$

which proves the Proposition.

This equation implies that $\tau' f - \tau f$ is a boundary whenever f is a cycle. As for its proof, here is the dirt. If we omit the restriction maps, the definitions give

$$(dkf)_{i_0 \cdots i_q} = \sum_{j=0}^{q} (-1)^j (kf)_{i_0 \cdots \widehat{i_j} \cdots i_q}$$

$$= \sum_{j=0}^{q} (-1)^j \Big(\sum_{h=0}^{j-1} (-1)^h f_{\tau i_0 \cdots \tau i_h \tau' i_h \cdots \widehat{\tau' i_j} \cdots \tau' i_q} + \sum_{h=j+1}^{q} (-1)^{h-1} f_{\tau i_0 \cdots \widehat{\tau i_j} \cdots \tau i_h \tau' i_h \cdots \tau' i_q} \Big)$$

and

$$(kdf)_{i_0 \cdots i_q} = \sum_{h=0}^{q} (-1)^h (df)_{\tau i_0 \cdots \tau i_h \tau' i_h \cdots \tau' i_q}$$

$$= \sum_{h=0}^{q} (-1)^h \Big(\sum_{j=0}^{h} (-1)^j f_{\tau i_0 \cdots \widehat{\tau i_j} \cdots \tau i_h \tau' i_h \cdots \tau' i_q} + \sum_{j=h}^{q} (-1)^{j+1} f_{\tau i_0 \cdots \tau i_h \tau' i_h \cdots \widehat{\tau' i_j} \cdots \tau' i_q} \Big).$$

The terms with $h \neq j$ appear with opposite sign in the two sums, and cancel when we add them together, so

$$(dkf + kdf)_{i_0\cdots i_q} = \sum_{h=0}^{q} \left(f_{\tau i_0 \cdots \tau i_{h-1} \tau' i_h \cdots \tau' i_q} - f_{\tau i_0 \cdots \tau i_h \tau' i_{h+1} \cdots \tau' i_q} \right)$$
$$= f_{\tau' i_0 \cdots \tau' i_q} - f_{\tau i_0 \cdots \tau i_q}.$$

Therefore, if $\mathfrak{U}$ is finer than $\mathfrak{V}$, for all $q \geq 0$ there is a canonical homomorphism from $H^q(\mathfrak{V}, \mathcal{F})$ to $H^q(\mathfrak{U}, \mathcal{F})$. Throughout the remainder this homomorphism with be denoted by $\sigma(\mathfrak{U}, \mathfrak{V})$.

22. Cohomology groups with values in the sheaf $\mathcal{F}$. The relation "$\mathfrak{U}$ is finer than $\mathfrak{V}$" (which we denote from now on by $\mathfrak{U} \prec \mathfrak{V}$) is a *preorder* of the covers of X. In addition, this relation is a *filtration*, because if $\mathfrak{U} = \{U_i\}_{i\in I}$ and $\mathfrak{V} = \{V_j\}_{j\in J}$ are two covers, $\mathfrak{W} := \{U_i \cap V_j\}_{(i,j)\in I\times J}$ is a cover that is at least as fine as $\mathfrak{U}$ and at least as fine as $\mathfrak{V}$.

We say that two covers $\mathfrak{U}$ and $\mathfrak{V}$ are equivalent if $\mathfrak{U} \prec \mathfrak{V}$ and $\mathfrak{V} \prec \mathfrak{U}$. Each cover $\mathfrak{U}$ is equivalent to a cover $\mathfrak{U}'$ in which the set of indices is a subset of the set of subsets of X; in effect, we can identify $\mathfrak{U}$ with the *set* of open subsets of X contained in the *family* $\mathfrak{U}$. In this way we can speak of the set of equivalence classes of covers, for this equivalence relation; it is a set whose ordering is a filtration[4].

If $\mathfrak{U} \prec \mathfrak{V}$, we have specified at the end of the preceeding no. a well defined homomorphism $\sigma(\mathfrak{U}, \mathfrak{V}) : H^q(\mathfrak{V}, \mathcal{F}) \to H^q(\mathfrak{U}, \mathcal{F})$, for each integer $q \geq 0$ and each sheaf $\mathcal{F}$ on X. Clearly $\sigma(\mathfrak{U}, \mathfrak{U})$ is the identity, and $\sigma(\mathfrak{U}, \mathfrak{V}) \circ \sigma(\mathfrak{V}, \mathfrak{W}) = \sigma(\mathfrak{U}, \mathfrak{W})$ if $\mathfrak{U} \prec \mathfrak{V} \prec \mathfrak{W}$. It follows that, if $\mathfrak{U}$ is equivalent to $\mathfrak{V}$, $\sigma(\mathfrak{U}, \mathfrak{V})$ and $\sigma(\mathfrak{V}, \mathfrak{U})$ are inverse isomorphisms, so $H^q(\mathfrak{U}, \mathcal{F})$ depends only on the equivalence class of the cover $\mathfrak{U}$.

Definition . *The q^{th} cohomology of X with values in the sheaf $\mathcal{F}$, denoted by $H^q(X, \mathcal{F})$, is the inductive limit of the groups $H^q(\mathfrak{U}, \mathcal{F})$, defined by following the filtration of equivalence classes of covers and the homomorphisms $\sigma(\mathfrak{U}, \mathfrak{V})$.*

Inductive limits (also known as direct limits) are defined in Section B3.

In other words, an element of $H^q(X, \mathcal{F})$ may be thought of concretely as a pair $(\mathfrak{U}, x)$ with $x \in H^q(\mathfrak{U}, \mathcal{F})$, where we identify two pairs $(\mathfrak{U}, x)$ and $(\mathfrak{V}, y)$ if there is a cover $\mathfrak{W}$, with $\mathfrak{W} \prec \mathfrak{U}$ and $\mathfrak{W} \prec \mathfrak{V}$, such that $\sigma(\mathfrak{W}, \mathfrak{U})(x) = \sigma(\mathfrak{W}, \mathfrak{V})(y)$ in $H^q(\mathfrak{W}, \mathcal{F})$. For each cover $\mathfrak{U}$ of X there is an associated canonical homomorphism $\sigma(\mathfrak{U}) : H^q(\mathfrak{U}, \mathcal{F}) \to H^q(X, \mathcal{F})$.

Observe that $H^q(X, \mathcal{F})$ can equally well be defined as the inductive limit of the $H^q(\mathfrak{U}, \mathcal{F})$ following any cofinal family of covers $\mathfrak{U}$. Therefore, if X is

[4]In contrast, one cannot speak of the "set" of all covers, because the set of indices of a cover is arbitrary.

quasi-compact (resp. quasi-paracompact) one can restrict consideration to the finite (resp. locally finite) covers.

When $q = 0$, we have, on applying Proposition 1:

PROPOSITION 4. $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

23. Homomorphism of sheaves. Let φ be a homomorphism from a sheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$. If $\mathfrak{U}$ is a cover of X, for each $f \in C^q(\mathfrak{U}, \mathcal{F})$ let $\varphi f \in C^q(\mathfrak{U}, \mathcal{G})$ be the corresponding element defined by the formula $(\varphi f)_s = \varphi(f_s)$. The function $f \mapsto \varphi f$ is a homomorphism from $C(\mathfrak{U}, \mathcal{F})$ to $C(\mathfrak{U}, \mathcal{G})$ that commutes with the coboundary operator, so it induces homomorphisms $\varphi^* : H^q(\mathfrak{U}, \mathcal{F}) \to H^q(\mathfrak{U}, \mathcal{G})$. Since $\varphi^* \circ \sigma(\mathfrak{U}, \mathfrak{V}) = \sigma(\mathfrak{U}, \mathfrak{V}) \circ \varphi^*$, we may pass to the limit, arriving at homomorphisms

$$\varphi^* : H^q(X, \mathcal{F}) \to H^q(X, \mathcal{G}).$$

When $q = 0$, φ^* coincides with the homomorphism from $\Gamma(X, \mathcal{F})$ to $\Gamma(X, \mathcal{G})$ defined in the natural fashion by φ.

In the general case, the homomorphisms φ^* enjoy the usual formal properties:

$$(\varphi + \psi)^* = \varphi^* + \psi^*, \qquad (\varphi \circ \psi)^* = \varphi^* \circ \psi^*, \qquad \mathbf{1}^* = \mathbf{1}.$$

In other words, for all $q \geq 0$, $H^q(X, \mathcal{F})$ is an additive covariant functor of $\mathcal{F}$. A notable consequence of this is that if $\mathcal{F}$ is a direct sum of two sheaves $\mathcal{G}_1$ and $\mathcal{G}_2$, then $H^q(X, \mathcal{F})$ is the direct sum of $H^q(X, \mathcal{G}_1)$ and $H^q(X, \mathcal{G}_2)$.

Suppose that $\mathcal{F}$ is a sheaf of $\mathcal{A}$-modules. Each section of the sheaf $\mathcal{A}$ on all of X defines an endomorphism of $\mathcal{F}$, which in turn induces endomorphisms of the $H^q(X, \mathcal{F})$. It follows that the $H^q(X, \mathcal{F})$ are $\Gamma(X, \mathcal{A})$-modules.

24. Exact sequences of sheaves: the general case. Let

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$$

be an exact sequence of sheaves. If $\mathfrak{U}$ is a cover of X, the sequence

$$0 \to C(\mathfrak{U}, \mathcal{A}) \xrightarrow{\alpha} C(\mathfrak{U}, \mathcal{B}) \xrightarrow{\beta} C(\mathfrak{U}, \mathcal{C})$$

is evidently exact, but the homomorphism β is not surjective in general. Denote by $C_0(\mathfrak{U}, \mathcal{C})$ the image of that homomorphism; this is a subcomplex of $C(\mathfrak{U}, \mathcal{C})$ of which the cohomology groups are denoted by $H_0^q(\mathfrak{U}, \mathcal{C})$. The sequence

$$0 \to C(\mathfrak{U}, \mathcal{A}) \to C(\mathfrak{U}, \mathcal{B}) \to C_0(\mathfrak{U}, \mathcal{C}) \to 0$$

gives rise to a long exact sequence of cohomology groups:

$$\cdots \to H^q(\mathfrak{U}, \mathcal{B}) \to H_0^q(\mathfrak{U}, \mathcal{C}) \xrightarrow{d} H^{q+1}(\mathfrak{U}, \mathcal{A}) \to H^{q+1}(\mathfrak{U}, \mathcal{B}) \to \cdots,$$

where the coboundary operator d is defined in the usual manner.

Homology and cohomology are going to be increasingly important as we go along, requiring understanding of more and more of the homological algebra explained in chapters B, C, and D.

Now let $\mathfrak{U} = \{U_i\}_{i \in I}$ and $\mathfrak{V} = \{V_j\}_{j \in J}$ be two covers, and let $\tau : I \to J$ be a function such that $U_i \subset V_{\tau i}$; thus $\mathfrak{U} \prec \mathfrak{V}$. The commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C(\mathfrak{V}, \mathcal{A}) \arrow[r] \arrow[d, "\tau"'] & C(\mathfrak{V}, \mathcal{B}) \arrow[r] \arrow[d, "\tau"'] & C(\mathfrak{V}, \mathcal{C}) \arrow[d, "\tau"'] \\
0 \arrow[r] & C(\mathfrak{U}, \mathcal{A}) \arrow[r] & C(\mathfrak{U}, \mathcal{B}) \arrow[r] & C(\mathfrak{U}, \mathcal{C})
\end{tikzcd}$$

shows that τ maps $C_0(\mathfrak{V}, \mathcal{C})$ into $C_0(\mathfrak{U}, \mathcal{C})$, so it defines homomorphisms $\tau^* : H_0^q(\mathfrak{V}, \mathcal{C}) \to H_0^q(\mathfrak{U}, \mathcal{C})$. Moreover, the homomorphisms τ^* are independent of the choice of the function τ^*; this follows from the fact that, if $f \in C_0^q(\mathfrak{V}, \mathcal{C})$, then $kf \in C_0^{q-1}(\mathfrak{U}, \mathcal{C})$, with the notations of the proof of Proposition 3.

Presumably there is a typo here: actually τ^* does not depend on the choice of τ. Expanding on this a bit, the point is that since $kf \in C_0^{q-1}(\mathfrak{U}, \mathcal{C})$, the proof of Proposition 3 applies equally to $\tau^* : H_0^q(\mathfrak{V}, \mathcal{C}) \to H_0^q(\mathfrak{U}, \mathcal{C})$.

In this way we obtain canonical homomorphisms $\sigma(\mathfrak{U}, \mathfrak{V}) : H_0^q(\mathfrak{V}, \mathcal{C}) \to H_0^q(\mathfrak{U}, \mathcal{C})$; consequently we can define $H_0^q(X, \mathcal{C})$ to be the inductive limit along the filtration of covers $\mathfrak{U}$ of the groups $H_0^q(\mathfrak{U}, \mathcal{C})$.

Let $(I, <)$ be a directed set, and let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be the inductive limit of a system $\{\mathcal{A}_i \to \mathcal{B}_i \to \mathcal{C}_i\}_{i \in I}$ of exact sequences. We verify the claim Serre establishes by citation below, that $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is exact. Let $[a_i]$ denote the element of $\mathcal{A}$ corresonding to $a_i \in \mathcal{A}_i$. Each $a_i \in \mathcal{A}_i$ maps to $0 \in \mathcal{C}_i$, and each element of $\mathcal{A}$ is $[a_i]$ for some i and a_i, so each element of $\mathcal{A}$ maps to $0 \in \mathcal{C}$.

Suppose that $b \in \mathcal{B}$ maps to $0 \in \mathcal{C}$, and choose i and $b_i \in \mathcal{B}_i$ such that $[b_i] = b$. For every $j > i$ the image of b in $\mathcal{C}$ is $[c_j]$ where b_j is the image of b_i in $\mathcal{B}_j$ and c_j is the image of b_j in $\mathcal{C}_j$. Since b maps to 0, $c_j = 0$ for some j. Choosing $a_j \in \mathcal{A}_j$ that maps to b_j, we find that $b = [b_j]$ is the image of $[a_j]$.

The inductive limit of exact sequences is an exact sequence (cf. [7], Chap. VIII, th. 5.4), so we have

Proposition 5. *The sequence*

$$\cdots \to H^q(X, \mathcal{B}) \xrightarrow{\beta^*} H_0^q(X, \mathcal{C}) \xrightarrow{d} H^{q+1}(X, \mathcal{A}) \xrightarrow{\alpha^*} H^{q+1}(X, \mathcal{B}) \to \cdots$$

is exact.

(d denotes the homomorphism obtained by passage to the limit from the homomorphisms $d : H_0^q(\mathfrak{U}, \mathcal{C}) \to H^{q+1}(\mathfrak{U}, \mathcal{A})$).

In order to be able to apply the last Proposition, it is useful to compare the groups $H_0^q(X, \mathcal{C})$ and $H^q(X, \mathcal{C})$. The injection of $C_0(\mathfrak{U}, \mathcal{C})$ into $C(\mathfrak{U}, \mathcal{C})$

induces the homomorphisms $H_0^q(\mathfrak{U}, \mathcal{C}) \to H^q(\mathfrak{U}, \mathcal{C})$, from which, by passage to the limit over $\mathfrak{U}$, we obtain the homomorphisms

$$H_0^q(X, \mathcal{C}) \to H^q(X, \mathcal{C}).$$

Proposition 6. *The canonical homomorphism $H_0^q(X, \mathcal{C}) \to H^q(X, \mathcal{C})$ is bijective for $q = 0$ and injective for $q = 1$.*

We first prove a lemma:

Lemma 1. *Let $\mathfrak{V} = \{V_j\}_{j \in J}$ be a cover, and let $f = (f_j)$ be an element of $C^0(\mathfrak{V}, \mathcal{C})$. There is a cover $\mathfrak{U} = \{U_i\}_{i \in I}$ and a function $\tau : I \to J$ such that $U_i \subset V_{\tau i}$ and $\tau f \in C_0^0(\mathfrak{U}, \mathcal{C})$.*

For each $x \in X$, choose $\tau x \in J$ such that $x \in V_{\tau x}$. Since $f_{\tau x}$ is a section of $\mathcal{C}$ above $V_{\tau x}$, there is an open neighborhood U_x of x contained in $V_{\tau x}$, and a section b_x of $\mathcal{B}$ above U_x such that $\beta(b_x) = f_{\tau x}$ on U_x. The $\{U_x\}_{x \in X}$ form a cover $\mathfrak{U}$ of X, and the b_x form a 0-cochain b of $\mathfrak{U}$ with values in $\mathcal{B}$; since $\tau f = \beta(b)$, we have $\tau f \in C_0^0(\mathfrak{U}, \mathcal{C})$.

Let's now show that $H_0^1(X, \mathcal{C}) \to H^1(X, \mathcal{C})$ is injective. An element of the kernel of this function can be represented by a 1-cocycle $z = (z_{j_0 j_1}) \in C_0'(\mathfrak{V}, \mathcal{C})$ such that there exists $f = (f_i) \in C^0(\mathfrak{V}, \mathcal{C})$ with $df = z$, applying Lemma 1 to f, we find a cover $\mathfrak{U}$ such that $\tau f \in C_0^0(\mathfrak{U}, \mathcal{C})$, which implies that τz is a boundary in $C_0(\mathfrak{U}, \mathcal{C})$ and is consequently mapped to 0 in $H_0^1(X, \mathcal{C})$. In the same way one shows that $H_0^0(X, \mathcal{C}) \to H^0(X, \mathcal{C})$ is bijective.

Corollary 1. *There is an exact sequence:*

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to H^1(X, \mathcal{A}) \to H^1(X, \mathcal{B}) \to H^1(X, \mathcal{C}).$$

This is an immediate consequence of Propositions 5 and 6.

Corollary 2. *If $H^1(X, \mathcal{A}) = 0$, then $\Gamma(X, \mathcal{B}) \to \Gamma(X, \mathcal{C})$ is surjective.*

25. Exact sequences of sheaves: the case where X is paracompact. Recall that the space X is said to be paracompact if it is separated and if every open cover has a refinement that is also an open cover, and is locally finite. For such a space one can extend Proposition 6 to all values of q (I do not consider whether such an extension is possible for spaces that are not separated):

Proposition 7. *If X is paracompact, the canonical homomorphism*

$$H_0^q(X, \mathcal{C}) \to H^q(X, \mathcal{C})$$

is bijective for all $q \geq 0$.

The Proposition is an immediate consequence of the following lemma, analogous to Lemma 1.

LEMMA 1. *Let $\mathfrak{V} = \{V_j\}_{j\in J}$ be a cover, and let $f = (f_j)$ be an element of $C^q(\mathfrak{V}, \mathcal{C})$. There is a cover $\mathfrak{U} = \{U_i\}_{i\in I}$ and a function $\tau : I \to J$ such that $U_i \subset V_{\tau i}$ and $\tau f \in C_0^q(\mathfrak{U}, \mathcal{C})$.*

Because X is paracompact, we can suppose that $\mathfrak{V}$ is locally finite. Consequently there is an open cover $\{W_j\}_{j\in J}$ such that $\overline{W}_j \subset V_j$.

There is some work to do here. First we show that X is regular: every open neighborhood of a point contains a closed neighborhood. Suppose that $x \in U$ with U open. For each $y \in X \setminus U$ we may choose a neighborhood V_y of y whose closure does not contain x, because X is separated. Let $\{W_i\}_{i\in I}$ be an open locally finite refinement of $\{U\} \cup \{V_y\}_{y\notin U}$. If W_i is not contained in some V_y, it is contained in U; replacing each such W_i with U gives an open cover that is locally finite, with each element not containing x in its closure unless it is U itself. Let $C = X \setminus \bigcup_{i:W_i\neq U} W_i$. Of course C is closed and $x \in C \subset U$. Since x has a neighborhood that intersects only finitely many elements of $\{W_i\}$, C is a neighborhood of x.

We may now choose a cover of X by open sets whose closures are each contained in some element of $\mathfrak{V}$. Let $\{W_k\}_{k\in K}$ be a locally finite refinement, and observe that $\{\overline{W}_k\}$ is also locally finite. For each k choose $\tau' k \in J$ such that $\overline{W}_k \subset V_{\tau' k}$. Now perhaps the easiest approach is to replace $\mathfrak{V}$ with $\mathfrak{V}' = \{V_{\tau' k}\}_{k\in K}$, after which we can continue with the argument. (To literally obtain $\{W_j\}_{j\in J}$ with the indicated properties seems much harder, if it is even possible.)

For each $x \in X$, choose an open neighborhood U_x of x such that:

(a) If $x \in V_j$ (resp. $x \in W_j$), we have $U_x \subset V_j$ (resp. $U_x \subset W_j$).

(b) If $U_x \cap W_j \neq \emptyset$, we have $U_x \subset V_j$.

(c) If $x \in V_{j_0,\dots,j_q}$, there is a section b of $\mathcal{B}$ above U_x such that $\beta(b) = f_{j_0,\dots,j_q}$ above U_x.

Condition (c) can be realized: look at the definition of a quotient sheaf, and the fact that x is contained in only finitely many sets $V_{j_0,\dots,j_q}$. Once (c) is verified, it suffices to consider only U_x satisfying (a) and (b).

Concretely, start with an open neighborhood of x satisfying (c), intersect it with the complement of the union of the $\overline{W}_j$ that do not contain x, then intersect with all the U_j and W_j that contain x.

The family $\{U_x\}_{x\in X}$ is a cover $\mathfrak{U}$; for each each $x \in X$ we choose $\tau x \in J$ such that $x \in W_{\tau x}$. Now we verify that τf belongs to $C_0^q(U, \mathcal{C})$; in other words $f_{\tau x_0,\dots,\tau x_q}$ is the image, under β, of a section of $\mathfrak{V}$ above $U_{x_0} \cap \cdots \cap U_{x_q}$. If $U_{x_0} \cap \cdots \cap U_{x_q}$ is empty this is evident; otherwise, we have $U_{x_0} \cap U_{x_k} \neq \emptyset$ for $0 \le k \le q$, and since $U_{x_k} \subset W_{\tau x_k}$, we have $U_{x_0} \cap W_{\tau x_k} \neq \emptyset$ which, in view of (b), implies that $U_{x_0} \subset V_{\tau x_k}$, and then $x_0 \in V_{\tau x_0,\dots,\tau x_q}$. Applying (c), we see that there is a section b of $\mathfrak{V}$ above U_{x_0} such that $\beta(b)_x = f_{\tau x_1,\ldots,\tau x_q}$ above U_{x_0}, and therefore also above $U_{x_0} \cap \cdots \cap U_{x_q}$, which completes the proof.

The preceeding exact sequence in called the *exact sequence of cohomology* defined by the given exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$. It is derived, more generally, in various circumstances in which one can show that $H^q_0(X, \mathcal{C}) \to H^q(X, \mathcal{C})$ is bijective (we will see in no. 47 that this is the case when X is an algebraic variety and $\mathcal{A}$ is a coherent algebraic sheaf).

26. Cohomology of a closed subspace. Let $\mathcal{F}$ be a sheaf on the space X, and let Y be a subspace. Let $\mathcal{F}(Y)$ be the sheaf induced by $\mathcal{F}$ on Y, in the sense of no. 4. If $\mathfrak{U} = \{U_i\}_{i\in I}$ is a cover of X, the $U'_i = Y \cap U_i$ form a cover $\mathfrak{U}'$ of Y; if $f_{i_0,\ldots,i_q}$ is a section of $\mathcal{F}$ above $U_{i_0,\ldots,i_q}$, the restriction of $f_{i_0,\ldots,i_q}$ to $U'_{i_0,\ldots,i_q} = Y \cap U_{i_0,\ldots,i_q}$ is a section of $\mathcal{F}(Y)$. The operation of restriction is a homomorphism $\rho : C(\mathfrak{U}, \mathcal{F}) \to C(\mathfrak{U}', \mathcal{F}(Y))$, commuting with d, from which we can define $\rho^* : H^q(\mathfrak{U}, \mathcal{F}) \to H^q(\mathfrak{U}', \mathcal{F}(Y))$. If $\mathfrak{U} \prec \mathfrak{V}$, one has $\mathfrak{U}' \prec \mathfrak{V}'$, and $\rho^* \circ \sigma(\mathfrak{U}, \mathfrak{V}) = \sigma(\mathfrak{U}', \mathfrak{V}') \circ \rho^*$; thus the homomorphisms ρ^* define, by passage to the limit over $\mathfrak{U}$, the homomorphisms $\rho^* : H^q(X, \mathcal{F}) \to H^q(Y, \mathcal{F}(Y))$.

Proposition 8. *Suppose that Y is closed in X, and that $\mathcal{F}$ vanishes outside of Y. Then $\rho^* : H^q(X, \mathcal{F}) \to H^q(Y, \mathcal{F}(Y))$ is bijective for all $q \geq 0$.*

The Proposition is obtained from the following two facts:

(a) Each cover $\mathfrak{W} = \{W_i\}_{i\in I}$ of Y is of the form $\mathfrak{U}'$ where $\mathfrak{U}$ is a cover of X.

In effect, it suffices to set $U_i = W_i \cup (X \setminus Y)$, since Y is closed in X.

(b) For each cover $\mathfrak{U}$ of X, $\rho : C(\mathfrak{U}, \mathcal{F}) \to C(\mathfrak{U}', \mathcal{F}(Y))$ is bijective.

In effect, this follows from Proposition 5 of no. 5, applied to $U_{i_0,\ldots,i_q}$ and the sheaf $\mathcal{F}$.

One can also explain Proposition 8 in the following manner: If $\mathcal{G}$ is a sheaf on Y, and if $\mathcal{G}^X$ is the sheaf obtained by extending $\mathcal{G}$ by 0 outside of Y, then $H^q(Y, \mathcal{G}) = H^q(X, \mathcal{G}^X)$ for all $q \geq 0$; in other words, the identification of $\mathcal{G}$ with $\mathcal{G}^X$ is compatible with the passage to cohomology groups.

§4. Comparison of cohomology groups of different covers

In this section, X denotes a topological space, and $\mathcal{F}$ is a sheaf on X. We will give conditions on the structure of a cover $\mathfrak{U}$ of X which imply that $H^n(\mathfrak{U}, \mathcal{F}) = H^n(X, \mathcal{F})$ for all $n \geq 0$.

27. Double complexes. A double complex (cf. [6], Chap. IV, §4) is a bigraded abelian group

$$K = \sum_{p,q} K^{p,q}, \qquad p \geq 0, q \geq 0,$$

endowed with two endomorphisms d' and d'' with the following properties:

$$\begin{cases} d' \text{ maps } K^{p,q} \text{ to } K^{p+1,q} \text{ and } d'' \text{ maps } K^{p,q} \text{ to } K^{p,q+1}, \\ d' \circ d' = 0,\ d' \circ d'' + d'' \circ d' = 0,\ d'' \circ d'' = 0. \end{cases}$$

An element of $K^{p,q}$ is said to be bihomogeneous, of bidegree (p, q), and of total degree $p + q$. The endomorphism $d = d' + d''$ satisfies $d \circ d = 0$, and the cohomology groups of K, endowed with that coboundary operator, are denoted by $H^n(K)$, with n the total degree.

Equally, we can endow K with the coboundary operator d'; as d' is compatible with the bigradation of K, one obtains cohomology groups, denoted by $H_I^{p,q}(K)$; with d'' one has the groups $H_{II}^{p,q}(K)$.

We denote by K_{II}^q the subgroup of $K^{0,q}$ consisting of the elements x such that $d'(x) = 0$, and by K_{II} the direct sum of the K_{II}^q $(q = 0, 1, \ldots)$. Define $K_I = \sum_{p=0}^{\infty} K_I^p$ analogously. Note that

$$K_{II}^q = H_I^{0,q}(K) \quad \text{and} \quad K_I^p = H_{II}^{p,0}(K).$$

K_{II} is a subcomplex of K, and the operator d coincides on K_{II} with the operator d''.

PROPOSITION 1. *If $H_I^{p,q}(K) = 0$ for $p > 0$ and $q \geq 0$, the injection $K_{II} \to K$ induces a bijection between $H^n(K_{II})$ and $H^n(K)$ for all $n \geq 0$.*

(Cf. [4], section XVII-6, from which we have taken the proof below.)

After replacing K with K/K_{II}, the proof reduces to showing that if $H_I^{p,q}(K) = 0$ for $p \geq 0$ and $q \geq 0$, then $H^n(K) = 0$ for all $n \geq 0$. Set

$$K_h = \sum_{q \geq h} K^{p,q}.$$

The K_h $(h = 0, 1, \ldots)$ are subcomplexes embedded in K, and K_h/K_{h+1} is isomorphic to $\sum_{p=0}^{\infty} K^{p,h}$, endowed with the coboundary operator d'. Thus we have $H^n(K_h/K_{h+1}) = H_I^{h,n-h}(K) = 0$ for such n and h, and therefore $H^n(K_h) = H^n(K_{h+1})$. Since $H^n(K_h) = 0$ if $h > n$, one deduces, by descent on h, that $H^n(K_h) = 0$ for such n and h, and since K_0 is equal to K, the Proposition is proved.

It took me a bit of time to work through all this, but there is only one point that seems to reward further explanation, namely that our remarks in no. 24 concerning the exact sequence $0 \to A \to B \to B/A \to 0$ are being applied here, first to infer that each $H^n(K_{II}) \to H^n(K)$ is an isomorphism if $H^n(K/K_{II}) = 0$ for all n, then to infer that each $H^n(K_{h+1}) \to H^n(K_h)$ is an isomorphism if $H^n(K_h/K_{h+1}) = 0$ for all n.

Serre was certainly intimately familiar with the homological algebra of double complexes, since they are central in the theory of spectral sequences, for which he was awarded the Fields Medal in 1954. He is the youngest Fields Medalist ever, and has no rival for the importance of the work done after receiving the prize.

28. Double complexes defined by two covers. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ and $\mathfrak{V} = \{V_j\}_{j\in J}$ be two covers of X. If s is a p-simplex of $S(I)$, and s' is a q-simplex of $S(J)$, we denote by U_s the intersection of the U_i for $i \in s$, by $V_{s'}$ the intersection of the V_j for $j \in s'$, by $\mathfrak{V}_s$ the cover of U_s formed by the $\{U_s \cap V_j\}_{j\in J}$, and by $\mathfrak{U}_{s'}$ the cover of $V_{s'}$ formed by the $\{V_{s'} \cap U_i\}_{i\in I}$.

Define a double complex $C(\mathfrak{U}, \mathfrak{V}; \mathcal{F}) = \sum_{p,q} C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$ in the following manner:

$C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{F}) = \prod \Gamma(U_s \cap V_{s'}, \mathcal{F})$, where the product is over all pairs (s, s') in which s is a simplex of dimension p in $S(I)$ and s' is a simplex of dimension q in $S(J)$.

An element $f \in C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$ is then a system $(f_{s,s'})$ of sections of $\mathcal{F}$ on the $U_s \cap V_{s'}$, where again, in the notations of no. 18, this is a system

$$f_{i_0\cdots i_p, j_0\cdots j_q} \in \Gamma(U_{i_0\cdots i_p} \cap V_{j_0\cdots j_q}, \mathcal{F}).$$

We can also identify $C^{p,q}(\mathfrak{U}, \mathfrak{V}, \mathcal{F})$ with $\prod_{s'} C^p(\mathfrak{U}_{s'}, \mathcal{F})$; since, for each s', there is a coboundary operator $d : C^p(\mathfrak{U}_{s'}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}_{s'}, \mathcal{F})$, we have a derived homomorphism

$$d_{\mathfrak{U}} : C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{F}) \to C^{p+1,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{F}).$$

Expanding the definition of $d_{\mathfrak{U}}$, we obtain:

$$(d_{\mathfrak{U}} f)_{i_0\cdots i_{p+1}, j_0\cdots j_q} = \sum_{k=0}^{p+1} (-1)^k \rho_k (f_{i_0\cdots \hat{i}_k \cdots i_{p+1}, j_0\cdots j_q}),$$

where ρ_k is the restriction homomorphism defined by the inclusion of

$$U_{i_0\cdots i_{p+1}} \cap V_{j_0\cdots j_q} \quad \text{in} \quad U_{i_0\cdots \hat{i}_k \cdots i_{p+1}} \cap V_{j_0\cdots j_q}.$$

In the same way we define $d_{\mathfrak{V}} : C^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{F}) \to C^{p,q+1}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$, and we have

$$(d_{\mathfrak{V}} f)_{i_0\cdots i_p, j_0\cdots j_{q+1}} = \sum_{h=0}^{q+1} (-1)^h \rho_h (f_{i_0\cdots i_p, j_0\cdots \hat{j}_h \cdots j_{q+1}}).$$

It is clear that $d_{\mathfrak{U}} \circ d_{\mathfrak{U}} = 0$, $d_{\mathfrak{U}} \circ d_{\mathfrak{V}} = d_{\mathfrak{V}} \circ d_{\mathfrak{U}}$, $d_{\mathfrak{V}} \circ d_{\mathfrak{V}} = 0$. Setting $d' = d_{\mathfrak{U}}$, $d'' = (-1)^p d_{\mathfrak{V}}$, we have endowed $C(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$ with the structure of a double complex. Consequently we can apply to $K = C(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$ the definitions of the preceeding no.; the groups of complexes designated in the general case by $H^n(K)$, $H_I^{p,q}(K)$, $H_{II}^{p,q}(K)$, K_I, K_{II}, are denotes here by $H^n(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$, $H_I^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$, $H_{II}^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$, $C_I(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$ and $C_{II}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$ respectively.

In view of the definitions of d' and d'', we immediately have:

PROPOSITION 2. *$H_I^{p,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$ is isomorphic to $\prod_{s'} H^p(\mathfrak{U}_{s'}, \mathcal{F})$, where the product is over all simplexes of dimension q in $S(J)$. In particular,*

$$C_{II}^q(\mathfrak{U}, \mathfrak{V}; \mathcal{F}) = H_I^{0,q}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$$

is isomorphic to $\prod_{s'} H^0(\mathfrak{U}_{s'}, \mathcal{F}) = C^q(\mathfrak{V}, \mathcal{F})$.

The last equality follows from $H^0(\mathfrak{U}_{s'}, \mathcal{F}) = \Gamma(V_{s'}, \mathcal{F})$, which is Proposition 1 of no. 20 with $V_{s'}$ in place of X.

We denote by ι'' the canonical isomorphism: $C(\mathfrak{V}, \mathcal{F}) \to C_{II}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$. If $(f_{j_0 \cdots j_q})$ is an element of $C^q(\mathfrak{V}, \mathcal{F})$, we now have

$$(\iota'' f)_{i_0, j_0 \cdots j_q} = \rho_{i_0}(f_{j_0 \cdots j_q}),$$

where ρ_{i_0} denotes the restriction homomorphism defined by the inclusion

$$U_{i_0} \cap V_{j_0 \cdots j_q} \quad \text{in} \quad V_{j_0 \cdots j_q}.$$

Please note that a result analogous to Proposition 2 holds for $H^{p,q}_{II}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$, and there is an isomorphism $\iota' : C(\mathfrak{U}, \mathcal{F}) \to C_I(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$.

29. Applications. With the notations as in the preceeding no., we have:

PROPOSITION 3. *Suppose that $H^p(\mathfrak{U}_{s'}, \mathcal{F}) = 0$ for all s' and all $p > 0$. Then the homomorphism $H^n(\mathfrak{V}, \mathcal{F}) \to H^n(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$, defined by ι'', is bijective for all $n \geq 0$.*

This is an immediate consequence of Propositions 1 and 2.

Before stating Proposition 4, we give a lemma:

LEMMA 1. *Let $\mathfrak{V} = \{W_i\}_{i \in I}$ be a cover of a space Y, and let $\mathcal{F}$ be a sheaf on Y. If there is an $i \in I$ such that $W_i = Y$, then $H^p(\mathfrak{V}, \mathcal{F}) = 0$ for all $p > 0$.*

Let $\mathfrak{V}'$ be the cover whose only element is Y itself; clearly $\mathfrak{V} \prec \mathfrak{V}'$, and the hypothesis on $\mathfrak{V}$ means that $\mathfrak{V}' \prec \mathfrak{V}$. It follows (no. 22) that $H^p(\mathfrak{V}, \mathcal{F}) = H^p(\mathfrak{V}', \mathcal{F}) = 0$ if $p > 0$.

The equation $H^p(\mathfrak{V}', \mathcal{F}) = 0$ for $p > 0$ follows from the Corollary at the end of no. 20.

PROPOSITION 4. *Suppose that the cover $\mathfrak{V}$ is finer that the cover $\mathfrak{U}$. Then $\iota'' : H^n(\mathfrak{V}, \mathcal{F}) \to H^n(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$ is bijective for all $n \geq 0$. Moreover, the homomorphism $\iota' \circ \iota''^{-1} : H^n(\mathfrak{U}, \mathcal{F}) \to H^n(\mathfrak{V}, \mathcal{F})$ coincides with the homomorphism $\sigma(\mathfrak{V}, \mathcal{U})$ of no. 21.*

It seems that there is a typo here: the homomorphism in question is $\iota''^{-1} \circ \iota' : H^n(\mathfrak{U}, \mathcal{F}) \to H^n(\mathfrak{V}, \mathcal{F})$.

On applying Lemma 1 with $\mathfrak{V} = \mathfrak{U}_{s'}$ and $Y = V_{s'}$, we see that $H^p(\mathfrak{U}_{s'}, \mathcal{F}) = 0$ for all $p > 0$, and Proposition 3 then implies that

$$\iota'' : H^n(\mathfrak{V}, \mathcal{F}) \to H^n(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$$

is bijective for each $n \geq 0$.

Let $\tau : J \to I$ be a function such that $V_j \subset U_{\tau j}$; in order to prove the second part of the Proposition, we need to show that, if f is an n-cocycle of $C(\mathfrak{U}, \mathcal{F})$, the cocycles $\iota(f)$ and $\iota''(\tau f)$ are cohomologous in $C(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$.

For each integer p, $0 \leq p \leq n-1$, define $g^p \in C^{p,n-p-1}(\mathfrak{U}, \mathfrak{V}; \mathcal{F})$ by the following formula:

$$g^p_{i_0 \cdots i_p, j_0 \cdots j_{n-p-1}} = \rho_p(f_{i_0 \cdots i_p \tau j_0 \cdots \tau j_{n-p-1}}),$$

where ρ_p is the restriction homomorphism defined by the inclusion

$$U_{i_0 \cdots i_p} \cap V_{j_0 \cdots j_{n-p-1}} \quad \text{in} \quad U_{i_0 \cdots i_p \tau j_0 \cdots \tau j_{n-p-1}}.$$

One verifies by a direct calculation (taking into account that f is a cocycle) that we have:

$$d''(g^0) = \iota''(\tau f), \cdots, d''(g^p) = d'(g^{p-1}), \cdots, d'(g^{n-1}) = (-1)^n \iota'(f)$$

whence $d(g^0 - g^1 + \cdots + (-1)^{n-1} g^{n-1}) = \iota''(\tau f) - \iota'(f)$, which shows that $\iota''(\tau f)$ and $\iota'(f)$ are cohomologous.

Here is the calculation. To be a little bit clearer we let ρ_{i_k} (instead of ρ_k) denote the restriction operator that restricts to the intersection of the domain with U_{i_k}, and we let ρ_{j_h} be the restriction to the intersection of the domain with V_{j_h}. From the definitions we have

$$\begin{aligned}(d' g^{p-1})_{i_0 \cdots i_p, j_0 \cdots j_{n-p-1}} &= \sum_{k=0}^{p} (-1)^k \rho_{i_k} (g^{p-1}_{i_0 \cdots \hat{i_k} \cdots i_p, j_0 \cdots j_{n-p-1}}) \\ &= \sum_{k=0}^{p} (-1)^k \rho_{i_k} \rho_{p-1} (f_{i_0 \cdots \hat{i_k} \cdots i_p, \tau j_0 \cdots \tau j_{n-p-1}})\end{aligned}$$

and

$$\begin{aligned}(d'' g^p)_{i_0 \cdots i_p, j_0 \cdots j_{n-p-1}} &= \sum_{h=0}^{n-p-1} (-1)^{p+h} \rho_{j_h} (g^p_{i_0 \cdots i_p, j_0 \cdots \hat{j_h} \cdots j_{n-p-1}}) \\ &= \sum_{h=0}^{n-p-1} (-1)^{p+h} \rho_{j_h} \rho_p (f_{i_0 \cdots i_p, \tau j_0 \cdots \tau \hat{j_h} \cdots \tau j_{n-p-1}}).\end{aligned}$$

In view of the given condition

$$0 = (df)_{i_0 \cdots i_n} = \sum_{j=0}^{n} (-1)^j \rho_j (f_{i_0 \cdots \hat{i_j} \cdots i_n})$$

we now see that

$$\begin{aligned}(d' g^{p-1})_{i_0 \cdots i_p, j_0 \cdots j_{n-p-1}} &- (d'' g^p)_{i_0 \cdots i_p, j_0 \cdots j_{n-p-1}} \\ &= \rho_p((df)_{i_0 \cdots i_p, \tau j_0 \cdots \tau j_{n-p-1}}) = 0.\end{aligned}$$

Specializing the first of the equations above to $p = n$ and the second to $p = 0$ gives

$$(d' g^{n-1})_{i_0 \cdots i_{n-1}, j_0} = \sum_{k=0}^{n-1} (-1)^k \rho_{i_k} \rho_{n-1} (f_{i_0 \cdots \hat{i_k} \cdots i_{n-1}, \tau j_0})$$

$$= -(-1)^n \rho_n(f_{i_0\cdots i_{n-1},\tau j_0}) = -(-1)^n(\iota' f)_{i_0\cdots i_{n-1},j_0}$$

and

$$(d'' g^0)_{i_0,j_0\cdots j_{n-1}} = \sum_{h=0}^{n-1}(-1)^h \rho_{j_h}\rho_0(f_{i_0,\tau j_0\cdots\tau\hat{j}_h\cdots\tau j_{n-1}})$$
$$= \rho_0(f_{i_0,\tau j_0\cdots\tau j_{n-1}}) = (\iota''(\tau f))_{i_0,j_0\cdots j_{n-1}}.$$

We now complete the calculation:

$$d\Big(\sum_{p=0}^{n-1}(-1)^p g^p\Big) = \sum_{p=0}^{n-1}(-1)^p[d'(g^p) + d''(g^p)]$$
$$= d'(g^0) + d''(g^0) + \sum_{p=1}^{n-1}(-1)^p[d'(g^{p-1}) + d'(g^p)]$$
$$= d''(g^0) + (-1)^{n-1}d'(g^{n-1}) = \iota''(\tau f) - \iota' f.$$

PROPOSITION 5. *Suppose that $\mathfrak{V}$ is finer that $\mathfrak{U}$, and that $H^q(\mathfrak{V}_s,\mathcal{F}) = 0$ for all s and all $q > 0$. Then the homomorphism $\sigma(\mathfrak{V},\mathfrak{U}) : H^n(\mathfrak{U},\mathcal{F}) \to H^n(\mathfrak{V},\mathcal{F})$ is bijective for all $n \geq 0$.*

If we apply Proposition 3, with the roles of $\mathfrak{U}$ and $\mathfrak{V}$ reversed, we see that $\iota' : H^n(\mathfrak{V},\mathcal{F}) \to H^n(\mathfrak{U},\mathfrak{V};\mathcal{F})$ is bijective. The Proposition then follows directly from Proposition 4.

THEOREM 1. *Let X be a topological space, $\mathfrak{U} = \{U_i\}_{i\in I}$ a cover of X, and $\mathcal{F}$ a sheaf on X. Suppose that there exist a family $\mathfrak{V}^\alpha$, $\alpha \in A$, of covers of X satisfying the following two conditions:*

(a) For any cover $\mathfrak{W}$ of X, there is some $\alpha \in A$ such that $\mathfrak{V}^\alpha \prec \mathfrak{W}$.

(b) $H^q(\mathfrak{V}^\alpha_s,\mathcal{F}) = 0$ for all $\alpha \in A$, all simplices $s \in S(I)$,and all $q > 0$.

Then $\sigma(\mathfrak{U}) : H^n(\mathfrak{U},\mathcal{F}) \to H^n(X,\mathcal{F})$ is bijective for all $n \geq 0$.

Since the $\mathfrak{V}^\alpha$ are arbitrarily fine, we may suppose that they are finer than $\mathfrak{U}$. In that case the homomorphism

$$\sigma(\mathfrak{V}^\alpha,\mathfrak{U}) : H^n(\mathfrak{U},\mathcal{F}) \to H^n(\mathfrak{V}^\alpha,\mathcal{F})$$

is bijective for all $n \geq 0$, from Proposition 5. Since the $\mathfrak{V}^\alpha$ are arbitrarily fine, $H^n(X,\mathcal{F})$ is the inductive limit of the $H^n(\mathfrak{V}^\alpha,\mathcal{F})$, and the theorem follows immediately from that.

REMARKS. (1) It is likely that Theorem 1 remains true when one replaces condition (b) with the following condition:

(b′) $\lim_\alpha H^q(B^\alpha_s,\mathcal{F}) = 0$ for all simplices s in $S(I)$ and all $q > 0$.

(2) Theorem 1 is analogous to a theorem of Leray on acyclic covers. Cf. [10], and also [4], exposé XVII-7.

Chapter II. Algebraic Varieties—Coherent algebraic sheaves on affine varieties

Throughout the remainder of this article K is an algebraically closed field, of arbitrary characteristic.

§1. Algebraic Varieties

30. Spaces satisfying condition (A). Let X be a topological space. Condition (A) is:

- (A)—*Every decreasing sequence of closed subsets of X is stationary.*

Put another way, if one has $F_1 \supset F_2 \supset F_3 \supset \cdots$ with the F_i closed in X, then there is an m such that $F_m = F_n$ for all $n \geq m$. Or again:

- (A′)—*The collection of closed subsets of X, ordered by inclusion, satisfies the minimal condition.*

In terminology that developed subsequently, a topological space satisfying condition (A) is said to be *Noetherian*, and a sequence this is eventually unchanging is said to *stabilize*. By the "minimal condition" Serre presumably means the following: a partially ordered set satisfies the *minimal condition* if every subset has a minimal element.

Examples. Endow a set with the topology in which the closed subsets are the finite subsets and X itself; condition (A) is evidently satisfied. More generally, every algebraic variety, endowed with the Zariski topology, satisfies (A) (cf. no. 34).

The Zariski topology (in two senses) is discussed at some length in Section A9.

Proposition 1.

(a) *If X satisfies condition (A), X is quasi-compact.*

(b) *If X satisfies condition (A), then all subspaces also satisfy it.*

(c) *If X is a finite union of subspaces satisfying condition (A), then X satisfies condition (A).*

If F_i is a decreasing filtration of closed subsets of X, and if X satisfies (A′), then there is some F_i that is contained in all the others; if $\bigcap F_i = \emptyset$, then there is some i such that $F_i = \emptyset$, which establishes (a).

Let $G_1 \supset G_2 \supset G_3 \supset \cdots$ be a decreasing sequence of closed subsets of a subspace Y of X; if X satisfies (A), there is an n such that $\overline{G}_m = \overline{G}_n$ for all $m \geq n$, whence $G_m = Y \cap \overline{G}_m = Y \cap \overline{G}_n = G_n$, which proves (b).

Let $F_1 \supset F_2 \supset F_3 \supset \cdots$ be a decreasing sequence of closed subsets of a space X satisfying the hypothesis of (c); since each Y_i satisfies (A) there exists for each i an n_i such that $F_m \cap Y_i = F_{n_i} \cap Y_i$ for $m \geq n_i$; if $n = \mathrm{Sup}(n_i)$, then $F_m = F_n$ if $m \geq n$, so (c) holds.

A space X is said to be *irreducible* if it is not the union of two closed subspaces, neither of which is X itself; it amounts to the same thing to say that any two nonempty open subsets of X have a nonempty intersection. All finite families of nonempty open subsets of X have a nonempty intersection, and each open subset of X is also irreducible.

Proposition 2. *Every space X satisfying condition (A) is a finite union of irreducible closed subspaces Y_i. If we suppose that Y_i is not contained in Y_j for each pair (i, j), $i \neq j$, the set of Y_i is uniquely determined by X; these Y_i are called the irreducible components of X.*

Suppose X is not a finite union of irreducible closed sets. It must not be irreducible itself, so X is a union $X = Y_1 \cup Z_1$ of two closed proper subsets. In turn at least one of these, say Z_1, is not a finite union of irreducible closed subsets, so $Z_1 = Y_2 \cup Z_2$ is a union of two closed proper subsets, and we may suppose that Z_2 is a not a union of finitely many irreducible closed subsets. Continuing indefinitely in this manner yields a decreasing sequence $Z_1 \supset Z_2 \supset Z_3 \supset \cdots$ of closed subsets that does not stabilize. Thus:

The existence of a decomposition $X = \bigcup Y_i$ is evidently a consequence of (A). If Z_k is another decomposition of X, then $Y_i = \bigcup Y_i \cap Z_k$, and, since Y_i is irreducible, that implies the existence of an index k such that $Z_k \supset Y_i$. Interchanging the roles of Y_i and Z_k, in the same way there exists an index i' such that $Z_k \subset Y_{i'}$, whence $Y_i \subset Z_k \subset Y_{i'}$. In view of the hypothesis made on the Y_i this entails that $Y_i = Z_k$, and thus the uniqueness of the decomposition.

Proposition 3. *Let X be a topological space that is a finite union of nonempty open subsets V_i. For X to be irreducible, it is necessary and sufficient that the V_i are irreducible and that $V_i \cap V_j \neq \emptyset$ for each pair (i, j).*

That these conditions are necessary was pointed out above; let's show that they are sufficient. If $X = Y \cup Z$, where Y and Z are closed, then $V_i = (V_i \cap Y) \cup (V_i \cap Z)$, which implies that each V_i is contained in either Y or Z. If Y and Z are different from X, one can find indices i, j such that V_i is not contained in Y and V_j is not contained in Z, so we have $V_i \subset Z$ and $V_j \subset Y$. Let $T = V_j \setminus V_i \cap V_j$; T is closed in V_j, and we have $V_j = T \cup (Z \cap V_j)$; since V_j is irreducible, this implies either that $T = V_j$, which is to say that $V_i \cap V_j = \emptyset$, or that $Z \cap V_j = V_j$, i.e., $V_j \subset Z$, and a contradiction results in either case, qed.

31. Locally closed subspaces of affine space. Let r be an integer ≥ 0, and let $X = K^r$ be the *affine space* of dimension r for the field K. We endow X with the *Zariski topology*; recall that a subset of X is closed for this topology if it is the set of common zeros of a family of polynomials $P^\alpha \in K[X_1, \cdots, X_r]$. Since the ring of polynomials is Noetherian, X satisfies condition (A) of the preceeding no.; moreover, one can easily show that X is irreducible.

That that ring of polynomials is Noetherian follows from the Hilbert basis theorem (Theorem A4.12). Section A9 contains more expansive discussion of the basic material Serre is reviewing here.

If $x = (x_1, \ldots, x_r)$ is a point in X, we denote the *local ring* of x by $\mathcal{O}_x$; recall that this is the subring of the field $K(X_1, \ldots, X_r)$ formed by rational functions R that can be put in the form:

$R = P/Q$, where P and Q are polynomials, and $Q(x) \neq 0$. Such a rational function is said to be *regular* at x; at every point x where $Q(x) \neq 0$, the function $x \mapsto P(x)/Q(x)$ is a continuous function with values in K (K is endowed with the Zariski topology) that can be identified with R, the field K being infinite. The $\mathcal{O}_x$, $x \in X$, form a subsheaf $\mathcal{O}$ of the sheaf $\mathcal{F}(X)$ of germs of functions on X with values in K (cf. no. 3); the sheaf $\mathcal{O}$ is a sheaf of rings.

We are going to extend the preceding to subspaces that are locally closed in X (we say that a subset of a space is *locally closed* in X if it is the intersection of an open subset and a closed subset of X). Let Y be such a subspace, and let $\mathcal{F}(Y)$ be the sheaf of germs of functions on Y with values in K; if x is a point in Y, the operation of restriction defines a canonical homomorphism

$$\varepsilon_x : \mathcal{F}(X)_x \to \mathcal{F}(Y)_x.$$

The image of $\mathcal{O}_x$ under ε_x is a subring of $\mathcal{F}(Y)_x$, which we denote by $\mathcal{O}_{x,Y}$; the $\mathcal{O}_{x,Y}$ form a subsheaf $\mathcal{O}_Y$ of $\mathcal{F}(Y)$, that we call the *sheaf of local rings* of Y. A section of $\mathcal{O}_Y$ on an open $V \subset Y$ is thus, by definition, a function $f : V \to K$ that is equal, in a neighborhood of each point $x \in V$, to the restriction to V of a rational function that is regular at x; such a function is said to be *regular* on V; it is continuous when we endow V with the topology induced by the topology of X, and K with the Zariski topology. The set of functions that are regular at each point of V is a ring, the ring $\Gamma(V, \mathcal{O}_Y)$; note that if $f \in \Gamma(V, \mathcal{O}_Y)$ and $f(x) \neq 0$ for all $x \in V$, then $1/f$ is also in $\Gamma(V, \mathcal{O}_Y)$.

There is an alternative characterization of the sheaf $\mathcal{O}_Y$:

Proposition 4. *Let U (resp. F) be an open (resp. closed) subspace of X, and let $Y = U \cap F$. Let $I(F)$ be the ideal of $K[X_1, \ldots, X_r]$ consisting of the polynomials that vanish on F. If x is a point of Y, the kernel of the surjection $\varepsilon_x : \mathcal{O}_x \to \mathcal{O}_{x,Y}$ is equal to the ideal $I(F) \cdot \mathcal{O}_x$ of $\mathcal{O}_x$.*

It is clear that every element of $I(F) \cdot \mathcal{O}_x$ is contained in the kernel of ε_x. Conversely, let $R = P/Q$ be an element of the kernel, P and Q being polynomials with $Q(x) \neq 0$. By hypothesis there is an open neighborhood W of x such that $P(y) = 0$ for all $y \in W \cap F$; let F' be the complement of W, which is closed in X; since $x \notin F'$, there exists, by definition of the Zariski topology, a polynomial P_1, vanishing on F' and not at x; the polynomial $P \cdot P_1$ is evidently in $I(F)$, and we can write $R = P \cdot P_1 / Q \cdot P_1$, thereby showing that $R \in I(F) \cdot \mathcal{O}_x$.

Corollary. *The ring $\mathcal{O}_{x,Y}$ is isomorphic to the ring of fractions of $K[X_1, \ldots, X_r]/I(F)$ relative to the maximal ideal defined by the point x.*

This follows immediately from the construction of the ring of fractions of a quotient ring (cf. for example [8], Chap. XV, §5, th. XI).

Proposition A5.3 handles the formalities of this intuitively obvious point.

32. Regular functions. Let U (resp. V) be a locally closed subspace of K^r (resp. K^s). A function $\varphi : U \to V$ is said to be *regular* on U (or simply regular) if:

(a) φ is continuous.

(b) If $x \in U$, and if $f \in \mathcal{O}_{\varphi(x),V}$, then $f \circ \varphi \in \mathcal{O}_{x,U}$.

Denote the coordinates of the point $\varphi(x)$ by $\varphi_i(x)$, $1 \leq i \leq s$. Then:

Proposition 5. *In order for $\varphi : U \to V$ to be regular on U it is necessary and sufficient that the $\varphi_i : U \to K$ are regular on U for all i, $1 \leq i \leq s$.*

Since the coordinate functions are regular on V, the condition in necessary. Conversely, suppose that it is the case that $\varphi_i \in \Gamma(U, \mathcal{O}_U)$ for all i; if $P(X_1, \ldots, X_s)$ is a polynomial, the function $P(\varphi_1, \ldots, \varphi_s)$ is contained in $\Gamma(U, \mathcal{O}_U)$ because $\Gamma(U, \mathcal{O}_U)$ is a ring; it follows that it is a continuous function on U, so that the set of points mapped to zero is closed, which proves the continuity of φ. If we have an $x \in U$ and $f \in \mathcal{O}_{\varphi(x),V}$, locally we can write f in the form $f = P/Q$, where P and Q are polynomials and $Q(\varphi(x)) \neq 0$. The function $f \circ \varphi$ is then equal to $P \circ \varphi / Q \circ \varphi$ is a neighborhood of x; since $Q \circ \varphi(x) \neq 0$, it follows that $f \circ \varphi$ is regular in a neighborhood of x, qed.

The composition of two regular functions in regular. A bijection $\varphi : U \to V$ is called a *biregular isomorphism* (or simply an isomorphism) if φ and φ^{-1} are regular functions; in other words, φ is a homeomorphism from U to V that transforms the sheaf $\mathcal{O}_U$ into the sheaf $\mathcal{O}_V$.

33. Products. If r and r' are two integers ≥ 0, we identify the affine space $K^{r+r'}$ with $K^r \times K^{r'}$. The Zariski topology of $K^{r+r'}$ is finer than the product of the Zariski topologies of K^r and $K^{r'}$; it is strictly finer if r and r' are > 0. It follows that if U and U' are locally closed subspaces of K^r and $K^{r'}$, $U \times U'$ is a locally closed subspace of $K^{r+r'}$ and the sheaf $\mathcal{O}_{U \times U'}$ is well defined.

Let W be another locally closed subspace of K^t, $t \geq 0$, and let $\varphi : W \to U$ and $\varphi' : W \to U'$ be two functions. It follows immediately from Proposition 5 that we have:

Proposition 6. *In order for a function $x \to (\varphi(x), \varphi'(x))$ to be a regular function, it is necessary and sufficient that φ and φ' are regular.*

Since all constant functions are regular, the preceeding Proposition shows that each *section* $x \to (x, x'_0)$, $x'_0 \in U'$, is a regular function from U to $U \times U'$; moreover, the *projections* $U \times U' \to U$ and $U \times U' \to U'$ are evidently regular.

Let V and V' be locally closed subspaces of K^s and $K^{s'}$, and let $\psi : U \to V$ and $\psi' : U' \to V'$ be two functions. The preceeding remarks, together with Proposition 6, show that we now have (cf. [1], Chap. IV):

Proposition 7. *In order for $\psi \times \psi' : U \times U' \to V \times V'$ to be regular, it is necessary and sufficient that ψ and ψ' are regular.*

Whence:

Corollary. *In order for $\psi \times \psi' : U \times U' \to V \times V'$ to be a regular isomorphism, it is necessary and sufficient that ψ and ψ' are regular isomorphisms.*

34. Definition of the structure of an algebraic variety. An algebraic variety over K (or simply an algebraic variety) is a set X endowed with:

(1°) a topology,

(2°) a subsheaf of the sheaf $F(X)$ of germs of functions on X with values in K,

that satisfy the axioms (VA$_I$) and (VA$_{II}$) given below.

First of all we note that if X and Y are endowed with two structures of the preceeding type, there is a notion of *isomorphism* between X and Y; this is a homeomorphism from X to Y that induces an isomorphism of $\mathcal{O}_X$ and $\mathcal{O}_Y$. Also, if X' is an open subset of X, we can endow X' with the induced topology and the induced sheaf: there is a notion of the *induced structure* on an open set. With these points clarified, we can state the axiom (VA$_I$):

(VA$_I$)— *There is a finite open cover $\mathfrak{V} = \{V_i\}_{i \in I}$ of the space X such that each V_i, endowed with the induced structure, is isomorphic to a locally closed subspace U_i of an affine space, endowed with the sheaf $\mathcal{O}_{U_i}$ defined in no. 31.*

To simplify the language, we say that a space X endowed with a sheaf satisfying (VA$_I$) is a *prealgebraic variety—(.* An isomorphism $\varphi_i : V_i \to U_i$ will be called a *chart* on the open set V_i; condition (VA$_I$) indicates that it is possible to construct X using finitely many of these charts. Proposition 1 of no. 30 shows that X satisfies condition (A), hence is quasicompact, as do all its subspaces.

The topology of X will be called the “Zariski topology” of X, and the sheaves $\mathcal{O}_X$ will be called the *sheaf of local rings* of X.

Proposition 8. *Let X be a set that is a finite union of subsets X_j, $j \in J$. Suppose that each X_j is endowed with the structure of a prealgebraic variety, and that the following conditions are satisfied:*

•

(a) $X_i \cap X_j$ *is open in* X_i *for all* $i, j \in J$;

(b) the structures induced by X_i *and by* X_j *on* $X_i \cap X_j$ *are the same for all* $i, j \in J$.

Then there is a unique structure of a prealgebraic variety on X *such that the* X_j *are open subsets of* X *and the structure induced on each* X_j *is the given structure.*

The existence and uniqueness of the topology of X and the sheaf $\mathcal{O}_X$ is immediate; it remains to verify that this topology and sheaf satisfy (VA$_I$), and this follows from the fact that the X_j are finite in number and each satisfies (VA$_I$).

Corollary. *Let* X *and* X' *be two prealgebraic varieties. Then* $X \times X'$ *has a unique structure of a prealgebraic variety satisfying the following condition: If* $\varphi : V \to U$ *and* $\varphi' : V' \to U'$ *are charts (with* V *open in* X *and* V' *open in* X'*) then* $V \times V'$ *is open in* $X \times X'$ *and* $\varphi \times \varphi' : V \times V' \to U \times U'$ *is a chart.*

Cover X with a finite number of open sets V_i with charts $\varphi_i : V_i \to U_i$, and let (V'_j, U'_j, φ'_j) be an analogous system for X'. The set $X \times X'$ is the union of the $V_i \times V'_j$; endow each $V_i \times V'_j$ with the prealgebraic variety structure that is the image of that of $U_i \times U'_i$ induced by $\varphi_i^{-1} \times \varphi_j^{-1}$; the hypotheses (a) and (b) of Proposition 8 are applicable to this covering of $X \times X'$, by virtue of the corollary of Proposition 7. We obtain in this way a prealgebraic variety structure on $X \times X'$ that satisfies the desired conditions.

One can apply the preceeding corollary to the particular case $X' = X$; then $X \times X$ is endowed with the structure of a prealgebraic variety, and in particular with a topology. We can now state the axiom (VA$_{II}$):

(VA$_{II}$)—*The diagonal* Δ *of* $X \times X$ *is closed in* $X \times X$.

Let's suppose that X is a prealgebriac variety, constructed using the "gluing" procedure of Proposition 8; in order for the condition (VA$_{II}$) to be satisfied it is necessary and sufficient that each $X_{ij} = \Delta \cap X_i \times X_j$ is closed in $X_i \times X_j$. Now X_{ij} is the set of (x, x) for $x \in X_i \cap X_j$. Suppose also that there are charts $\varphi_i : X_i \to U_i$ and $\varphi_j : X_j \to U_j$, and let $T_{ij} = \varphi_i \times \varphi_j(X_{ij})$; T_{ij} is the set of $(\varphi_i(x), \varphi_j(x))$ for x in $X_i \cap X_j$. Axiom (VA$_{II}$) now takes the following form:

(VA$'_{II}$)—*For each pair* (i, j), T_{ij} *is closed in* $U_i \times U_j$.

prealgebraic variety—)

From this form one recovers axiom (A) of Weil (cf. [16], p. 167), taking into account that Weil considers only irreducible varieties.

Examples of algebraic varieties: Each locally closed subspace U of an affine space, endowed with the induced topology and the sheaf $\mathcal{O}_U$ defined in no. 31, is an algebraic variety. Every projective variety is an algebraic variety (cf. no. 51). Every algebraic fiber space (cf. [17]) whose base and fiber are algebraic varieties is an algebraic variety.

In each case the verification of (VA$_{II}$) is simply a matter of observing that the diagonal is defined by equations of the form $x_i = x'_i$, which are algebraic of course.

Remarks. (1) We note the analogy between the condition (VA$_{II}$) and the condition of *separation* imposed on topological spaces and differentiable and analytic manifolds.

If X is a topological space and $X \times X$ has the product topology, then X is a Hausdorff space if and only if Δ is closed. But an algebraic variety is never Hausdorff, unless it is finite. Also, the diagonal is typically not closed in the product topology when X has the Zariski topology, so (VA$_{II}$) has substance, and is thus a potential surrogate for the Hausdorff condition. The hope (which will eventually be realized) is that separation will entail consequences that are similar to the implications of the Hausdorff condition.

(2) Simple examples show that (VA$_{II}$) *is not* a consequence of (VA$_I$).

The line with two origins works here.

35. Regular functions, induced structures, products. Let X and Y be two algebraic varieties, φ a function from X to Y. We say that φ is *regular* if

(a) φ is continuous,

(b) if $x \in X$, and if $f \in \mathcal{O}_{\varphi(x),Y}$, then $f \circ \varphi \in \mathcal{O}_{x,X}$.

As we saw in no. 32, the composition of two regular functions is regular, and, in order for a bijection $\varphi : X \to Y$ to be an isomorphism, it is necessary and sufficient that φ and φ^{-1} are regular. The regular functions form a family of *morphisms* for the structure of algebraic varieties, in the sense of [1], Chap. IV.

Of course Serre is saying that algebraic varieties and regular functions constitute a category, but categorical language was poorly developed back then.

Let X be an algebraic variety, and X' a closed subset of X. Endow X' with the topology induced by the topology of X and the sheaf $\mathcal{O}_X$ induced by $\mathcal{O}_X$ (more precisely, for each $x \in X'$ we define $\mathcal{O}_{x,X}$ to be the image of $\mathcal{O}_{x,X}$ under the canonical homomorphism: $\mathcal{F}(X)_x \to \mathcal{F}(X')_x$).

Clearly there are typos here, insofar as one of the symbols $\mathcal{O}_X$ should be $\mathcal{O}_{X'}$ and one of the symbols $\mathcal{O}_{x,X}$ should be $\mathcal{O}_{x,X'}$. It would have been easy enough to correct them, but leaving them in illustrates one of the problems that journal articles (not exclusively, but more so than books) present.

Axiom (VA_I) holds: if $\varphi_i : V_i \to U_i$ is a system of charts such that $X = \bigcup V_i$, we can set $V'_i = X' \cap V_i$ and $U'_i = \varphi_i(V'_i)$, and $\varphi_i : V'_i \to U'_i$ is a system of charts such that $X' = \bigcup V'_i$. Axiom (VA_{II}) is satisfied because the topology of $X' \times X'$ is induced by that of $X \times X$ (one can also use (VA'_{II})). In this way we define the structure of an algebraic variety on X', which is called the structure *induced* by that of X; we also say that X' is a *subvariety* of X (in Weil [16], the term "subvariety" is reserved for those that herein are called closed irreducible subvarieties). If ι denotes the injection of X' in X, ι is a regular function; moreover, if φ is a function from an algebraic variety Y to X', in order for φ to be regular, it is necessary and sufficient that $\iota \circ \varphi$ is regular (it is this fact that justifies the term "induced structure," cf. [1], loc. cit.).

If X and X' are two algebraic varieties, $X \times X'$ is an algebraic variety, called the *product variety*; it suffices to show that axiom (VA_{II}) is satisfied. In other words, if $\varphi : V_i \to U_i$ and $\varphi'_{i'} : V'_{i'} \to U'_{i'}$ are systems of charts with $X = \bigcup V_i$ and $X' = \bigcup V'_{i'}$, the set $T_{ij} \times T'_{i'j'}$ is then closed in $U_i \times U_j \times U'_{i'} \times U'_{j'}$ (the notation is from no. 34); this follows immediately from the fact that T_{ij} and $T'_{i'j'}$ are closed in $U_i \times U_j$ and $U'_{i'} \times U'_{j'}$ respectively.

It's not quite as immediate as it would be with the product topology. Here we need to recognize that the Zariski topology of $U_i \times U_j \times U'_{i'} \times U'_{j'}$ is finer than the product topology derived from the Zariski topologies of $U_i \times U_j$ and $U'_{i'} \times U'_{j'}$.

Propositions 6 and 7 are valid without any modification for algebraic varieties.

If $\varphi : X \to Y$ is a regular function, the graph Φ of φ is *closed* in $X \times Y$ because it is the inverse image of the diagonal of $Y \times Y$ under the function $\varphi \times 1 : X \times Y \to Y \times Y$; moreover, the function $\psi : X \to \Phi$ defined by $\psi(x) = (x, \varphi(x))$ is an isomorphism; in effect, ψ is a regular function, as is ψ^{-1} (because it is the restriction of the projection $X \times Y \to X$).

36. Field of rational functions on an irreducible variety. We first prove two lemmas of a purely topological nature:

LEMMA 1. *Let X be a connected space, G an abelian group, and $\mathcal{G}$ the constant sheaf on X, isomorphic to G. The canonical function $G \to \Gamma(X, \mathcal{G})$ is bijective.*

An element of $\Gamma(X, \mathcal{G})$ is nothing other than a continuous function from X to G, endowed with the discrete topology. Since X is connected, such a function is constant, which implies the lemma.

We say that a sheaf $\mathcal{F}$ on a space X is *locally constant* if each point in X has a neighborhood U such that $\mathcal{F}(U)$ is constant on U.

LEMMA 2. *A locally constant sheaf on an irreducible space is constant.*

Let $\mathcal{F}$ be the sheaf, X the space, and set $F = \Gamma(X, \mathcal{F})$; it suffices to show that the canonical homomorphism $\rho_x : F \to \mathcal{F}_x$ is bijective for each x, since we obtain in this way an isomorphism between the constant sheaf of F and the given sheaf $\mathcal{F}$.

If $f \in F$, the locus of points $x \in X$ such that $f(x) = 0$ is closed (from the general properties of sheaves) and open (because $\mathcal{F}$ is locally constant); since an irreducible space is connected, this locus is either $\emptyset$ or X, which shows that ρ_x is injective.

Now let $m \in \mathcal{F}_x$, and let s be a section of $\mathcal{F}$ on a neighborhood U of x such that $s(x) = m$; cover X with nonempty open sets U_i such that $\mathcal{F}(U_i)$ is constant on U_i; since X is irreducible, we have $U \cap U_i \neq \emptyset$; choosing a point $x_i \in U \cap U_i$; evidently there is a section of $\mathcal{F}$ on U_i such that $s_i(x_i) = s(x_i)$, and since the sections s and s_i coincide at x_i, they coincide everywhere in $U_i \cap U$; the same s_i and s_j coincide on $U_i \cap U_j$, since they coincide on $U \cap U_i \cap U_j \neq \emptyset$; now the sections s_i define a unique section s of $\mathcal{F}$ above X, and we have $\rho_x(s) = m$, which completes the proof.

Now let X be an irreducible algebraic variety. If U is a nonempty open subset of X, set $\mathcal{A}_U = \Gamma(U, \mathcal{O}_X)$; $\mathcal{A}_U$ is an integral domain: in effect, supposing that we have $f \cdot g = 0$, f and g are regular functions from U to K; if F (resp. G) is the locus of points $x \in U$ such that $f(x) = 0$ (resp. $g(x) = 0$), we have $U = F \cup G$, and F and G are closed in U, since f and g are continuous; since U is irreducible, this entails that $F = U$ or $G = U$, so that either f or g vanishes on U. We can therefore speak of the field of quotients of $\mathcal{A}_U$, which we denote by $\mathcal{K}_U$; if $U \subset V$, the homomorphism $\rho_U^V : \mathcal{A}_V \to \mathcal{A}_U$ is injective since U is dense in V, and we have a well defined isomorphism φ_U^V between $\mathcal{K}_V$ and $\mathcal{K}_U$; the system of $\{\mathcal{K}_U, \varphi_U^V\}$ define a *sheaf of fields* $\mathcal{K}$; moreover, $\mathcal{K}_x$ is canonically isomorphic to the field of quotients of $\mathcal{O}_{x,X}$.

PROPOSITION 9. *For every irreducible algebraic variety X, the sheaf $\mathcal{K}$ defined as above is a constant sheaf.*

In view of Lemma 2, it suffices to prove the Proposition when X is a locally closed subvariety of the affine space K^r; let F be the closure of X in K^r, and let $I(F)$ be the ideal of $K[X_1, \ldots, X_r]$ consisting of the polynomials that vanish on F (or on X, which amounts to the same thing). If we set $A = K[X_1, \ldots, X_r]/I(F)$, the ring A is an integral domain since X is irreducible; let $K(A)$ be the field of quotients of A. The corollary of Proposition 4 allows us to identify $\mathcal{O}_{x,X}$ with the ring of fractions of A relative to the maximal ideal defined by x; in this way we obtain an isomorphism of the field $K(A)$ and the field of fractions of $\mathcal{O}_{x,X}$, and it is easy to verify that this defines an isomorphism between the constant sheaf equal to $K(A)$ and the sheaf $\mathcal{F}$, which proves the proposition.

From Lemma 1, the sections of the sheaf $\mathcal{K}$ form a field, isomorphic to $\mathcal{K}_x$, for each $x \in X$, that we denote by $K(X)$. We call this the *field of rational functions* on X; it is an extension of finite type of the field K, and its transcendence degree over K is the *dimension* of X (we extend this definition to reducible algebraic varieties by setting $\dim X = \operatorname{Sup} \dim Y_i$, if X is a union of irreducible subvarieties Y_i). In general one can identify the field $K(X)$ with the field $\mathcal{K}_x$; since we have $\mathcal{O}_{x,X} \subset \mathcal{K}_x$, we can identify $\mathcal{O}_{x,X}$ with a subring of $K(X)$ (it is the ring of specialization of the point x in $K(x)$, in the sense of Weil, [16], p. 77). If U is open in X, $\Gamma(U, \mathcal{O}_x)$ is then the intersection in $K(X)$ of the rings $\mathcal{O}_{x,X}$ for x contained in U.

If Y is a subvariety of X, we have $\dim Y \leq \dim X$; if in addition Y is closed, and does not contain an irreducible component of X, we have $\dim Y < \dim X$, because this can be reduced to the case of a subvariety of K^r (cf. for example [8], Chap. X, §5, th. II).

This depends on background material in field theory that can be found in Chapter 8 of Milne's notes on field theory. The claim reduces to a claim about each of the irreducible components of Y, so we may assume that Y is irreducible. Since Y is contained in one of the irreducible components of X, we may assume that X is irreducible. In the affine case we may assume that Y corresponds to a prime ideal $\mathfrak{p}$ in $K[X]$, so that $K[Y] = K[X]/\mathfrak{p}$. Let $d = \dim Y$. Suppose that $K[Y] = K[f_1, \dots, f_n]$ where $f_1, \dots, f_d$ are a transcendence basis for $K(Y)$. It suffices to show that if $0 \neq g \in \mathfrak{p}$, then $f_1, \dots, f_d, g$ are algebraically independent in $K(X)$. If not, there is an algebraic relation

$$c_m(f_1, \dots, f_d)g^m + \cdots + c_1(f_1, \dots, f_d)g + c_0(f_1, \dots, f_d) = 0$$

where $c_i(f_1, \dots, f_d) \in K[f_1, \dots, f_d]$. Since X is irreducible, $K[f_1, \dots, f_d, g]$ is an integral domain, so we can cancel powers of g to make $c_0(f_1, \dots, f_d) \neq 0$. Restricting this equation to Y gives $c_0(f_1, \dots, f_d) = 0$, which contradicts the algebraic independence of $f_1, \dots, f_d$.

§2. Coherent Algebraic Sheaves

37. The sheaf of local rings of an algebraic variety. We return to the notation of no. 31: let $X = K^r$, and let $\mathcal{O}$ be a sheaf of local rings of X. Then:

Lemma 1. *The sheaf $\mathcal{O}$ is a coherent sheaf of rings, in the sense of no. 15.*

Fix $x \in X$, a neighborhood U of x, and sections $f_1, \dots, f_p$ of $\mathcal{O}$ on U, which is to say rational functions that are regular at each point of U; we need to show that the sheaf of relations between $f_1, \dots, f_p$ is a sheaf of finite type over $\mathcal{O}$. Replacing U with a smaller neighborhood if need be, we may suppose that the f_i can be written as $f_i = P_i/Q$, where the P_i and Q are polynomials and Q does not vanish anywhere in U. Now suppose that $y \in U$ and $g_i \in \mathcal{O}_y$

are such that $\sum_{i=1}^{p} g_i f_i$ vanishes in a neighborhood of y. Again, we can write the g_i in the form $g_i = R_i/S$, where the R_i and S are polynomials, and S does not vanish at y. The relation "$\sum_{i=1}^{p} g_i f_i = 0$ in a neighborhood of y" is equivalent to the relation "$\sum_{i=1}^{p} R_i P_i = 0$ in a neighborhood of y," which is in turn equivalent to $\sum_{i=1}^{p} R_i P_i = 0$. Since the module of relations between the polynomials P_i is a module of finite type (since the ring of polynomials is Noetherian), it follows that the sheaf of relations between the f_i is of finite type.

> Lemma A4.6 asserts that if R is a Noetherian ring and M is a finitely generated R-module, then M is Noetherian. The ring of polynomials in r variables with coefficients in K is Noetherian by the Hilbert basis theorem, so its p-fold cartesian product is also Noetherian (Corollary A4.5) and any submodule of a Noetherian R-module is Noetherian.

Now let V be a closed subvariety of $X = K^r$; for each $x \in X$ let $\mathcal{I}_x(V)$ be the ideal of $\mathcal{O}_x$ consisting of those elements $f \in \mathcal{O}_x$ whose restrictions to V vanish in a neighborhood of x. (We have $\mathcal{I}_x(V) = \mathcal{O}_x$ if $x \notin V$). The $\mathcal{I}_x(V)$ form a subsheaf $\mathcal{I}(V)$ of the sheaf $\mathcal{O}$.

LEMMA 2. *The sheaf $\mathcal{I}(V)$ is a coherent sheaf of $\mathcal{O}$-modules.*

Let $I(V)$ be the ideal of $K[X_1, \dots, X_r]$ consisting of the polynomials that vanish on V. From Proposition 4 of no. 31, $\mathcal{I}_x(V)$ is equal to $I(V) \cdot \mathcal{O}_x$ for all $x \in V$, and one sees immediately that that formula holds also for $x \notin V$. The ideal $I(V)$ is generated by a finite number of elements, and it follows that the sheaf $\mathcal{I}(V)$ is of finite type, hence coherent by virtue of Lemma 1 and Proposition 8 of no. 15.

We can now extend Lemma 1 to an arbitrary algebraic variety:

PROPOSITION 1. *If V is an algebraic variety, the sheaf $\mathcal{O}_V$ is a coherent sheaf of rings on V.*

The question being local, we can suppose that V is a closed subvariety of the affine space K^r. From Lemma 2, the sheaf $\mathcal{I}(V)$ is a coherent sheaf of ideals, whence the sheaf $\mathcal{O}/\mathcal{I}(V)$ is a coherent sheaf of rings on X, as per Theorem 3 of no. 16. This sheaf of rings vanishes outside of V, and its restriction to V is none other than $\mathcal{O}_V$ (no. 31); hence the sheaf $\mathcal{O}_V$ is a coherent sheaf of rings on V (no. 17, corollary to Proposition 11).

REMARK It is clear that Proposition 1 holds, more generally, for all prealgebraic varieties.

38. Coherent algebraic sheaves. If V is an algebraic variety with sheaf of local rings $\mathcal{O}_V$, an *algebraic sheaf* on V is a sheaf of $\mathcal{O}_V$-modules, in the sense of no. 6; if $\mathcal{F}$ and $\mathcal{G}$ are two algebraic sheaves, we say that $\varphi : \mathcal{F} \to \mathcal{G}$ is an *algebraic homomorphism* (or simply a homomorphism) if it is an $\mathcal{O}_V$-homomorphism; recall that this means that each $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is $\mathcal{O}_{x,V}$-linear and that φ maps each local section of $\mathcal{F}$ to a local section of $\mathcal{G}$.

If $\mathcal{F}$ is an algebraic sheaf on V, the cohomology groups $H^q(V, \mathcal{F})$ are modules on $\Gamma(V, \mathcal{O}_V)$, cf. no. 23; in particular, they are vector spaces over K.

An algebraic sheaf $\mathcal{F}$ on V is said to be *coherent* if it is a coherent sheaf of $\mathcal{O}_V$-modules, in the sense of no. 12; in view of Proposition 7 of no. 15 and Proposition 1 above, such a sheaf is characterized by the fact that it is locally isomorphic to the kernel of an algebraic homomorphism $\varphi : \mathcal{O}_V^q \to \mathcal{O}_V^p$.

We will give several examples of coherent algebraic sheaves, and will see others later, notably cf. no. 48, 57.

39. Sheaves of ideals defined on a closed subvariety.

Let W be a closed subvariety of an algebraic variety V. For each $x \in V$, let $\mathcal{I}_x(W)$ be the ideal of $\mathcal{O}_{x,V}$ consisting of the elements f whose restriction to W is null in a neighborhood of x; let $\mathcal{I}(W)$ be the subsheaf of $\mathcal{O}_V$ defined by the $\mathcal{I}_x(W)$. The following Proposition generalizes Lemma 2:

Proposition 2. *The sheaf $\mathcal{I}(W)$ is a coherent algebraic sheaf.*

Since the issue is local, we can assume that V (and therefore also W) is a closed subvariety of the affine space K^r. It follows from Lemma 2, applied to W, that the ideal defined by W in K^r is of finite type; consequently $\mathcal{I}(W)$, which is the image of the canonical homomorphism $\mathcal{O} \to \mathcal{O}_V$, is also of finite type, and is coherent due to Proposition 8 of no. 15 and Proposition 1 of no. 37.

Let $\mathcal{O}_W$ be the sheaf of local rings of W, and let $\mathcal{O}_W^V$ be the sheaf on V obtained by prolonging $\mathcal{O}_W$ by 0 elsewhere (cf. no. 5); this sheaf is canonically isomorphic to $\mathcal{O}_V/\mathcal{I}(W)$, which is to say that there is an exact sequence:

$$0 \to \mathcal{I}(W) \to \mathcal{O}_V \to \mathcal{O}_W^V \to 0.$$

Now let $\mathcal{F}$ be an algebraic sheaf on W, and let $\mathcal{F}^V$ be the sheaf obtained by prolonging $\mathcal{F}$ by 0 outside of W; we can think of $\mathcal{F}^V$ as a sheaf of $\mathcal{O}_W^V$-modules, and also as a sheaf of $\mathcal{O}_V$-modules whose annihilator contains $\mathcal{I}(W)$. We have:

Proposition 3. *If $\mathcal{F}$ is a coherent algebraic sheaf on W, $\mathcal{F}^V$ is a coherent algebraic sheaf on V. Conversely, if $\mathcal{G}$ is a coherent algebraic sheaf on V whose annihilator contains $\mathcal{I}(W)$, the restriction of $\mathcal{G}$ to W is a coherent algebraic sheaf on W.*

If $\mathcal{F}$ is a coherent algebraic sheaf on W, $\mathcal{F}^V$ is a coherent sheaf of $\mathcal{O}_W^V$-modules (no. 17, Proposition 11), hence a coherent sheaf of $\mathcal{O}_V$-modules (no. 16, Theorem 3). Conversely, if $\mathcal{G}$ is a coherent algebraic sheaf on V, whose annihilator contains $\mathcal{I}(W)$, $\mathcal{G}$ can be regarded as a sheaf of $\mathcal{O}_V/\mathcal{I}(W)$-modules, and is therefore a coherent sheaf (no. 16, Theorem 3); the restriction of $\mathcal{G}$ to W is then a coherent sheaf of $\mathcal{O}_W$-modules (no. 17, Proposition 11).

Thus any coherent algebraic sheaf on W can be identified with a coherent algebraic sheaf on V (and this identification does not change the cohomology groups, as per Proposition 8 of no. 26). In particular, any coherent algebraic sheaf on an affine (resp. projective) variety can be thought of as a coherent algebraic sheaf on affine (resp. projective) space; we will often make use of this possibility in what follows.

REMARK: Let $\mathcal{G}$ be a coherent algebraic sheaf on V, which vanishes outside of W; the annihilator of $\mathcal{G}$ *does not necessarily contain* $\mathcal{I}(W)$ (in other words, $\mathcal{G}$ cannot always be regarded as a coherent algebraic sheaf on W); the only thing one can say for sure is that it contains a *power* of $\mathcal{I}(W)$.

40. Sheaves of fractional ideals.

Let V be an irreducible algebraic variety, and let $K(V)$ be the constant sheaf of rational functions on V (cf. no. 36); $K(V)$ is an algebraic sheaf, which is not coherent if $\dim V > 0$. A algebraic subsheaf $\mathcal{F}$ of $K(V)$ could be called a "sheaf of fractional ideals," because each $\mathcal{F}_x$ is a fractional ideal of $\mathcal{O}_{x,V}$.

PROPOSITION 4. *In order for an algebraic subsheaf $\mathcal{F}$ of $K(V)$ to be coherent, it is necessary and sufficient that it be of finite type.*

The necessity is trivial. To demonstrate sufficiency it suffices to prove that $K(V)$ satisfies condition (b) of definition 2 of no. 12, which is to say that if $f_1, \ldots, f_p$ are rational fractions, the sheaf $\mathcal{R}(f_1, \ldots, f_p)$ is of finite type. If x is a point of V, one can find the functions g_i and h such that $f_i = g_i/h$, g_i and h are regular in a neighborhood of x, and h does not vanish anywhere in U; the sheaf $\mathcal{R}(f_1, \ldots, f_p)$ is then equal to the sheaf $\mathcal{R}(g_1, \ldots, g_p)$ which is of finite type, because $\mathcal{O}_V$ is a coherent sheaf of rings.

41. Sheaves associated with a fiber bundle with vector space fiber. Let E be an algebraic fiber space, with r-dimensional fiber, and with an algebraic variety V as its base; by definition, the fiber type of E is a the vector space K^r, and the structural group is the linear group $GL(r, K)$ operating on K^r in the usual fashion (for the definition of an algebraic fiber space, cf. [17]; see also [15], no. 4 for *analytic* fiber spaces with vector space fibers).

Serre is writing from the perspective of the early days of the theory of fiber bundles. (He was an important contributor.) As time went on the theory solidified its place within mathematics, and established certain terminology that renders Serre's discussion a bit obsolete. Briefly, here are the main definitions in the modern system of terminology. A *fiber bundle* is a continuous function $p : E \to B$, where E and B are topological spaces, such that for a third topological space F, called the *fiber* of the bundle, there is a covering $\{U_\alpha\}_{\alpha \in A}$ of B with open sets such that for each α there is a homeomorphism $\phi_\alpha : U_\alpha \times F \to p^{-1}(U_\alpha)$ such that $p(\phi(x, f)) = x$ for all $(x, f) \in U_\alpha \times F$. Sometimes a fiber bundle is described as a "twisted product," reflecting the fact that locally it has the structure of a cartesian product, but more complicated things may be happening at the global level.

One may impose structure by requiring that the spaces and maps belong to some nice category—smooth, analytic, algebraic. In addition, for any $\alpha, \beta \in A$ and $x \in U_\alpha \cap U_\beta$ there is a homeomorphism $\psi^x_{\alpha,\beta} : F \to F$ defined implicitly by the equation $\phi_\beta^{-1}(\phi_\alpha(x, f)) = (x, \psi^x_{\alpha,\beta}(f))$, and additional structure is imposed by requiring that these homeomorphisms lie in a subgroup of the group of homeomorphisms between F and itself called the *structural group*. In topological literature the structural group might be unmentioned if the intent is to impose no restriction.

In contemporary terminology a fibre bundle is said to be a *vector bundle* if its fibre is a finite dimensional vector space and the structural group is the general linear group of the space. Thus the object Serre is introducing might be called an "algebraic vector bundle" in contemporary literature.

If U is an open subset of V, let $\mathcal{S}(E)_U$ be the set of regular sections on U; if $V \supset U$, there is the restriction homomorphism $\varphi_U^V : \mathcal{S}(E)_V \to \mathcal{S}(E)_U$; thus this is a sheaf, called the *sheaf of germs of sections* of E. Since E is a fiber bundle whose fiber is a vector space, each $\mathcal{S}(E)_U$ is a $\Gamma(U, \mathcal{O}_V)$-module, and it follows that $\mathcal{S}(E)$ is an algebraic sheaf on V. If we identify E locally with $V \times K^r$, we see that:

PROPOSITION 5. *The sheaf $\mathcal{S}(E)$ is locally isomorphic to $\mathcal{O}_V^r$; in particular, it is a coherent algebraic sheaf.*

Conversely, it is easy to see that each algebraic sheaf on V, locally isomorphic to $\mathcal{O}_V^r$, is isomorphic to a sheaf $\mathcal{S}(E)$, where E is unique up to isomorphism.

If V is a variety without singularities, one can take for E the fiber space of p-covectors tangent to V (p is an integer ≥ 0); let Ω^p be the corresponding sheaf $\mathcal{S}(E)$; an element of Ω^p_x, $x \in V$, is none other than a *differential form* of degree p on V, regular at x. If we set $h^{p,q} = \dim_K H^q(V, \Omega^p)$, we know that, in the classical case (and if V is projective) $h^{p,q}$ is equal to the dimension of the space of harmonic forms of type (p, q) (theorem of Dolbeault[5]), and if B_n denotes the n^{th} Betti number of V, we have $B_n = \sum_{p+q=n} h^{p,q}$. In the general case, one can take the preceding formula as the *definition* of the Betti numbers of a projective variety without singularities (we will see in effect in no. 66 that the $h^{p,q}$ are finite). It would be interesting to study their properties and notably to see whether they coincide with those that occur in the conjectures of Weil on varieties over finite fields.[6] Here we point out only that they satisfy the "duality of Poincaré" $B_n = B_{2m-n}$ when V is irreducible and of dimension m.

The cohomology groups $H^q(V, \mathcal{S}(E))$ are also pertinent to other questions, notably the theorem of Riemann-Roch, likewise in the classification of algebraic fiber spaces with base V and structural group the affine group $x \to ax+b$ (cf. [17] §4, which treats the case where $\dim V = 1$).

§3. Coherent algebraic sheaves on affine varieties

42. Affine varieties. An algebraic variety V is said to be *affine* if it is isomorphic to a closed subvariety of an affine space. The product of two affine

[5]P. Dolbeault. *Sur la cohomology des varietés analytique complexes.* C. R. Paris, 236, 1953, p. 175-177.
[6] Bulletin Amer. Math. Soc., 55, 1949, p. 507.

varieties is an affine variety; every closed subvariety of an affine variety is an affine variety.

An open subset U of an an algebraic variety X is said to be *affine* if, endowed with the structure of an algebraic variety induced by that of X, it is an affine variety.

PROPOSITION 1. *Let U and V be two open subsets of an algebraic variety X. If U and V are affine, $U \cap V$ is affine.*

Let Δ be the diagonal of $X \times X$; from no. 35, the function $x \to (x, x)$ is a biregular isomorphism between $U \cap V$ and $\Delta \cap U \times V$. Since U and V are affine varieties, $U \times V$ is also an affine variety; on the other hand, Δ is closed in $X \times X$ from Axiom (VA_{II}), so $\Delta \cap U \times V$ is closed in $U \times V$, and is consequently an affine variety, qed.

(It is easy to see that the Proposition is false for prealgebraic varieties: Axiom (VA_{II}) plays an essential role here.)

We now introduce a notation that will be used throughout the rest of this paragraph: if V is an algebraic variety, and f is a regular function on V, we denote by V_f the open subset of V consisting of the points $x \in V$ such that $f(x) \neq 0$.

PROPOSITION 2. *If V is an affine algebraic variety, and f is a regular function on V, the open set V_f is an open affine subvariety.*

Let W be the subset of $V \times K$ consisting of the pairs (x, λ) such that $\lambda \cdot f(x) = 1$; it is clear that W is closed in $V \times K$, and thus is an affine variety. For each $(x, \lambda) \in W$ set $\pi(x, \lambda) = x$; the function π is a regular function from W to V_f. Inversely, for all $x \in V_f$, let $\omega(x) = (x, 1/f(x))$; the function $\omega : V_f \to W$ is regular, and we have $\pi \circ \omega = 1$, $\omega \circ \pi = 1$, so V_f and W are isomorphic, qed.

Does this really have to be so roundabout? Couldn't we just say that $V \setminus V_f$ is closed in V, so V_f is open? Axioms (VA_I) and (VA_{II}) seem to be automatic?

PROPOSITION 3. *Let V be a closed subvariety of K^r, F a closed subset of V, and $U = V \setminus F$. The open sets V_P, with P running over the set of polynomials that vanish on F, form a base of the topology of U.*

Let $U' = V \setminus F'$ be an open subset of U, and let $x \in U'$; we need to show that there is a P such that $V_P \subset U'$ and $x \in V_P$; in other words, P vanishes on F' but not at x; the existence of such a polynomial is a simple consequence of the definition of the topology of K^r.

THEOREM 1. *The open affine subvarieties of an algebraic variety X form a base of open sets for the topology of X.*

The question being local, we may suppose that X is a locally closed subvariety of an affine space K^r; in this case, the theorem follows immediately from Propositions 2 and 3.

COROLLARY. *The coverings of X formed of open affine subsets are arbitrarily fine.*

We note that if $\mathfrak{U} = \{U_i\}_{i \in I}$ is such a covering, the $U_{i_0 \cdots i_p}$ are all open affines, by Proposition 1.

43. Some preliminary properties of irreducible varieties. Let V be a closed subvariety of K^r, and let $I(V)$ be the ideal of $K[X_1, \ldots, X_r]$ consisting of the polynomials that vanish on V; let A be the quotient ring $k[X_1, \ldots, X_r]/I(V)$; there is a canonical homomorphism

$$\iota : A \to \Gamma(V, \mathcal{O}_V)$$

that is injective by virtue of the definition of $I(V)$.

PROPOSITION 4. *If V is irreducible, $\iota : A \to \Gamma(V, \mathcal{O}_V)$ is bijective.*

(In fact, this holds for *all* closed subvarieties of K^r, as we will show in the next no.)

Let $K(V)$ be the field of fractions of A; from no. 36, we can identify $\mathcal{O}_{x,V}$ with the ring of fractions of A relative to the maximal ideal $\mathfrak{m}_x$ of polynomials vanishing at x, and we have $\Gamma(V, \mathcal{O}_V) = A = \bigcap_{x \in V} \mathcal{O}_{x,V}$ (all the $\mathcal{O}_{x,V}$ are regarded as subrings of $K(V)$). But every maximal ideal of A is equal to one of the $\mathfrak{m}_x$, since K is algebraically closed (theorem of zeros of Hilbert); from this it follows immediately (cf. [8], Chap. XV, §5, th. X) that $A = \bigcap_{x \in V} \mathcal{O}_{x,V} = \Gamma(V, \mathcal{O}_V)$, qed.

As a matter of definition, and no. 36, we have

$$\Gamma(V, \mathcal{O}_V) = \bigcap_{x \in V} \mathcal{O}_{x,V} = \bigcap_{x \in V} A_{\mathfrak{m}_x}.$$

Hilbert's nullstellensatz (Theorem A9.1) implies that every maximal ideal of A is $\mathfrak{m}_x$ for some x. Now $\bigcap_{x \in V} A_{\mathfrak{m}_x} = A$ follows from the fact that any integral domain A is the intersection, inside its field of fractions, of the subrings $A_{\mathfrak{m}}$ for maximal ideals $\mathfrak{m}$. To show this suppose that f is an element of the intersection, but not an element of A itself, and let $I = \{ a \in A : af \in A \}$. Then $1 \notin I$, so I is a proper ideal and is consequently contained in some maximal ideal $\mathfrak{m}$. Since $A \setminus \mathfrak{m} \subset A \setminus I$, f has no representation of the form a/s with $s \notin \mathfrak{m}$, which contradicts the assumption that $f \in A_{\mathfrak{m}}$.

PROPOSITION 5. *Let X be an irreducible algebraic variety, Q a regular function on X, and P a regular function on X_Q. Then, for all sufficiently large n, the rational function $Q^n P$ is regular on all of X.*

In view of the quasi-compactness of X, the question is local; from Theorem 1, we can suppose that X is a closed subvariety of K^r. The preceeding Proposition shows then that Q is an element of $A = K[X_1, \ldots, X_r]/I(X)$. The hypothesis on P means that, for each point $x \in X_Q$, we can write $P = P_x/Q_x$, with P_x and Q_x in A, and $Q_x(x) \neq 0$; if $\mathfrak{a}$ denotes the ideal of A generated by the Q_x, the variety of zeros of $\mathfrak{a}$ is contained in the variety of zeros of Q; by virtue of the theorem of zeros of Hilbert, this implies that $Q^n \in \mathfrak{a}$ for n sufficiently large, whence $Q^n = \sum R_x \cdot Q_x$ and $Q^n P = \sum R_x \cdot P_x$ with $R_x \in A$, which shows that $Q^n P$ is regular on X.

(Equally, one can use the fact that X_Q is affine if X is, and apply Proposition 4 to X_Q.)

Proposition 6. *Let X be an irreducible algebraic variety, Q a regular function on X, $\mathcal{F}$ a coherent algebraic sheaf on X, and s a section of $\mathcal{F}$ above X whose restriction to X_Q vanishes. Then, for all sufficiently large n, the section $Q^n s$ vanishes on all of X.*

The question being local, we may suppose that:

(a) X is a closed subvariety of K^r,

(b) $\mathcal{F}$ is isomorphic to the cokernel of the isomorphism $\varphi : \mathcal{O}_X^p \to \mathcal{O}_X^q$,

(c) s is the image of a section σ of $\mathcal{O}_X^q$.

(In effect, all these conditions hold locally.)

Let $A = \Gamma(X, \mathcal{O}_X) = K[X_1, \dots, X_r]/I(X)$. The section σ can be identified with a system of q elements of A. On the other hand, let

$$t_1 = \varphi(1, 0, \dots, 0),\ \dots\ , t_p = \varphi(0, \dots, 0, 1);$$

the t_i, $1 \le i \le p$, are sections of $\mathcal{O}_X^q$ above X, so they can be identified with some systems of q elements of A. The hypothesis on s means that, for each $x \in X_Q$, we have $\sigma(x) \in \varphi(\mathcal{O}_{x,X}^p)$, which is to say that σ can be written in the form $\sigma = \sum_{i=a}^p f_i \cdot t_i$, with $f_i \in \mathcal{O}_{x,X}$; or, after clearing denominators, that there exist $Q_x \in A$, $Q_x(x) \neq 0$, such that $Q_x \cdot \sigma = \sum_{i=1}^p R_i \cdot t_i$, with $R_i \in A$. The reasoning above then shows that, for all sufficiently large n, Q^n belongs to the ideal generated by the Q_x, so $Q^n \sigma(x) \in \varphi(\mathcal{O}_{x,X}^p)$ for all $x \in X$, which means that Q^n vanishes on all of X.

44. Vanishing of certain cohomology groups.

Proposition 7. *Let X be an irreducible affine variety, let Q_i be a finite family of regular functions on X, which do not vanish simultaneously, and let $\mathfrak{U}$ be the open cover consisting of the $X_{Q_i} = U_i$. If $\mathcal{F}$ is a coherent algebraic subsheaf of $\mathcal{O}_X^p$, then $H^q(\mathfrak{U}, \mathcal{F}) = 0$ for all $q > 0$.*

After replacing $\mathfrak{U}$ with an equivalent open cover, we may suppose that none of the functions Q_i vanishes identically, so that $U_i \neq \emptyset$ for all i.

Let $f = (f_{i_0 \cdots i_q})$ be a q-cocycle of $\mathfrak{U}$ with values in $\mathcal{F}$. Each $f_{i_0 \cdots i_q}$ is a section of $\mathcal{F}$ on $U_{i_0 \cdots i_q}$, which may be identified with a system of p regular functions on $U_{i_0 \cdots i_q}$; applying Proposition 5 to $Q = Q_{i_0} \cdots Q_{i_q}$, we see that, for large enough n, $g_{i_0 \cdots i_q} = (Q_{i_0} \cdots Q_{i_q})^n f_{i_0 \cdots i_q}$ is a system of p regular functions on all of X, so it is a section of $\mathcal{O}^p$ above X. Choose an integer n such that this holds for all the systems $i_0, \dots, i_q$; this is possible because the number of such systems is finite. Consider the image of $g_{i_0 \cdots i_q}$ in the coherent sheaf $\mathcal{O}_X^p/\mathcal{F}$; this is a section that vanishes on $U_{i_0 \cdots i_q}$; applying Proposition 6, we see that, for all sufficiently large m, the product of this section with $(Q_{i_0} \cdots Q_{i_q})^m$ vanishes on all of X, which means that $(Q_{i_0} \cdots Q_{i_q})^m g_{i_0 \cdots i_q}$ is a section of $\mathcal{F}$ on all of X. On setting $N = m + n$, we see that we have constructed sections $h_{i_0 \cdots i_q}$ of $\mathcal{F}$ above X, which coincide with $(Q_{i_0} \cdots Q_{i_q})^N f_{i_0 \cdots i_q}$ on $U_{i_0 \cdots i_q}$.

Since the Q_i^N do not vanish simultaneously, there exist functions

$$R_i \in \Gamma(X, \mathcal{O}_X)$$

such that $\sum R_i \cdot Q_i^N = 1$. For each system $i_1, \dots, i_{q-1}$ let

$$k_{i_0 \cdots i_{q-1}} = \sum_i R_i \cdot h_{i i_0 \cdots i_{q-1}} / (Q_{i_0} \cdots Q_{i_{q-1}})^N,$$

which makes sense because $Q_{i_0}, \dots, Q_{i_{q-1}}$ are different from 0 on $U_{i_0 \cdots i_{q-1}}$.

In this way we define a cochain $k \in C^{q-1}(\mathfrak{U}, \mathcal{F})$. I claim that $f = dk$, which establishes the proposition.

It must be verified that $(dk)_{i_0 \cdots i_q} = f_{i_0 \cdots i_q}$; it suffices to show that these two sections coincide on $U = \bigcap U_i$, because they then coincide everywhere because they are systems of p rational functions and $U \neq \emptyset$. Thus, above U, we can write

$$k_{i_0 \cdots i_{q-1}} = \sum_i R_i \cdot Q_i^N \cdot f_{i i_0 \cdots i_{q-1}},$$

whence

$$(dk)_{i_0 \cdots i_q} = \sum_{j=0}^{q} (-1)^q \sum_i R_i \cdot Q_i^N \cdot f_{i i_0 \cdots \hat{i_j} \cdots i_q},$$

and, taking into account that f is a cocycle,

specifically, $0 = (df)_{i i_0 \cdots i_q} = f_{i_0 \cdots i_q} + \sum_{j=0}^{q} (-1)^{j+1} f_{i i_0 \cdots \hat{i_j} \cdots i_q}$,

$$(dk)_{i_0 \cdots i_q} = \sum_i R_i \cdot Q_i^N \cdot f_{i_0 \cdots i_q} = f_{i_0 \cdots i_q}, \qquad \text{qed.}$$

Corollary 1. *$H^q(X, \mathcal{F}) = 0$ for all $q > 0$.*

In effect Proposition 3 shows that the covers of the type utilized in Proposition 7 are arbitrarily fine.

Corollary 2. *The homomorphism $\Gamma(X, \mathcal{O}_X^p) \to \Gamma(X, \mathcal{O}_X^p/\mathcal{F})$ is surjective.*

The follows from Corollary 1 above and Corollary 2 of Proposition 6 of no. 24.

Corollary 3. *Let V be a closed subvariety of K^r, and let*

$$A = K[X_1, \dots, X_r]/I(V).$$

The homomorphism $\iota : A \to \Gamma(V, \mathcal{O}_V)$ is bijective.

We apply Corollary 2 above with $X = K^r$, $p = 1$, $\mathcal{F} = \mathcal{I}(V)$ the sheaf of ideals defined by V; we find that each element of $\Gamma(V, \mathcal{O}_V)$ is the restriction of a section of $\mathcal{O}$ on X, which is to say a polynomial, from Proposition 4 applied to X.

45. Sections of a coherent algebraic sheaf on an affine variety.

THEOREM 2. *Let $\mathcal{F}$ be a coherent algebraic sheaf on an affine variety X. For all $x \in X$, the $\mathcal{O}_{x,X}$-module $\mathcal{F}_x$ is generated by the elements of $\Gamma(X, \mathcal{F})$.*

Since X is affine, it can be embedded as a closed subvariety in an affine space K^r; on extending the sheaf $\mathcal{F}$ by 0 outside X, we obtain a coherent algebraic sheaf on K^r (cf. no. 39), and the claim for the given sheaf follows if we can prove it for this new sheaf. In other words, we may suppose that $X = K^r$.

In view of the definition of a coherent sheaf, there is a covering of X consisting of open sets on which $\mathcal{F}$ is isomorphic to a quotient of a sheaf $\mathcal{O}^p$. Utilizing Proposition 3, we see that there is a finite number of polynomials Q_i, which do not vanish simultaneously, such that above each $U_i = X_{Q_i}$ there is a surjective homomorphism $\varphi_x : \mathcal{O}^{p_i} \to \mathcal{F}$; moreover, we can assume that none of these polynomials are identically zero.

The point x is in one of the U_i, say U_0; it is clear that $\mathcal{F}_x$ is generated by the sections of $\mathcal{F}$ on U_0; since Q_0 is invertible in $\mathcal{O}_x$ it suffices to prove the following lemma:

LEMMA 1. *If s_0 is a section of $\mathcal{F}$ above U_0, there is an integer N and a section s of $\mathcal{F}$ above X such that $s = Q_0^N \cdot s_0$ above U_0.*

From Proposition 2, $U_i \cap U_0$ is an affine variety, which is evidently irreducible; on applying Corollary 2 of Proposition 7 to that variety and $\varphi_i : \mathcal{O}^{p_i} \to \mathcal{F}$, we see that there is a section σ_{0i} of $\mathcal{O}^{p_i}$ on $U_i \cap U_0$ such that $\varphi_i(\sigma_{0i}) = s_0$ on $U_i \cap U_0$; since $U_i \cap U_0$ is the set of points of U_i where Q_0 does not vanish, we can apply Proposition 5 to $X = U_i$, $Q = Q_0$, finding that there exist, for sufficiently large n, a section σ_i of $\mathcal{O}^{p_i}$ above U_i that coincides with $Q_0^n \cdot \sigma_{0i}$ above $U_i \cap U_0$; on setting $s'_i = \varphi_i(\sigma_i)$, we obtain a section of $\mathcal{F}$ above U_i that coincides with $Q_0^n \cdot s_0$ above $U_i \cap U_0$. The sections s'_i and s'_j coincide on $U_i \cap U_j \cap U_0$; on applying Proposition 6 to $s'_i - s'_j$ we see that, for m sufficiently large, we have $Q_0^m \cdot (s'_i - s'_j) = 0$ on all of $U_i \cap U_j$. The $Q_0^m \cdot s'_i$ then define a unique section s of $\mathcal{F}$ on X, and we have $s = Q_0^{n+m} s_0$ on U_0, which establishes the lemma, and achieves the proof of Theorem 2.

COROLLARY 1. *The sheaf $\mathcal{F}$ is isomorphic to a quotient sheaf of a sheaf $\mathcal{O}_X^p$.*

Since $\mathcal{F}_x$ is an $\mathcal{O}_{x,X}$-module of finite type, it follows from the theorem above that there is a finite number of sections of $\mathcal{F}$ that generate $\mathcal{F}_x$; from Proposition 1 of no. 12, these sections also generate $\mathcal{F}_y$ for y close to x. The space X is quasi-compact, so we conclude that there is a finite number of sections $s_1, \ldots, s_p$ of $\mathcal{F}$ that generate $\mathcal{F}_x$ for all $x \in X$, which means that $\mathcal{F}$ is isomorphic to a quotient sheaf of $\mathcal{O}_X^p$.

Corollary 2. *Let $\mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}$ be an exact sequence of coherent algebraic sheaves on an affine variety X. Then the sequence*

$$\Gamma(X, \mathcal{A}) \xrightarrow{\alpha} \Gamma(X, \mathcal{B}) \xrightarrow{\beta} \Gamma(X, \mathcal{C})$$

is exact.

We may suppose, as in the proof of Theorem 2, that X is the affine space K^r, which is irreducible. Let $\mathcal{I} = \mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$; the claim comes down to showing that $\alpha : \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{I})$ is surjective. But, from Corollary 1, we can find a surjective homomorphism $\varphi : \mathcal{O}_X^p \to \mathcal{A}$, and, from Corollary 2 of Proposition 7, $\alpha \circ \varphi : \Gamma(X, \mathcal{O}_X^p) \to \Gamma(X, \mathcal{I})$ is surjective, so a fortiori $\alpha : \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{I})$ is also surjective, qed.

46. Cohomology groups of an affine variety with values in a coherent algebraic sheaf.

Theorem 3. *Let X be an affine variety, Q_i a finite family of regular functions on X, which do not all vanish simultaneously, and $\mathfrak{U}$ the open cover of X consisting of the $X_{Q_i} = U_i$. If $\mathcal{F}$ is a coherent algebraic sheaf on X, we have $H^q(\mathfrak{U}, \mathcal{F}) = 0$ for all $q > 0$.*

First suppose that X is irreducible. From Corollary 1 of Theorem 2, one can find an exact sequence

$$0 \to \mathcal{R} \to \mathcal{O}_X^p \to \mathcal{F} \to 0.$$

The sequence of complexes: $0 \to C(\mathfrak{U}, \mathcal{R}) \to C(\mathfrak{U}, \mathcal{O}_X^p) \to C(\mathfrak{U}, \mathcal{F}) \to 0$ is *exact*; in effect, this amounts to each section of $\mathcal{F}$ on a $U_{i_0 \cdots i_q}$ being the image of a section of $\mathcal{O}_X^p$ on $U_{i_0 \cdots i_q}$, which follows from Corollary 2 of Proposition 7, applied to the irreducible variety $U_{i_0 \cdots i_q}$. This exact sequence of complexes gives rise to an exact sequence of cohomology groups:

$$\cdots \to H^q(\mathfrak{U}, \mathcal{O}_X^p) \to H^q(\mathfrak{U}, \mathcal{F}) \to H^{q+1}(\mathfrak{U}, \mathcal{R}) \to \cdots,$$

and since $H^q(\mathfrak{U}, \mathcal{O}_X^p) = H^{q+1}(\mathfrak{U}, \mathcal{R}) = 0$ for $q > 0$ by Proposition 7, we can conclude that $H^q(\mathfrak{U}, \mathcal{F}) = 0$.

We now pass to the general case. We can embed X as a closed subvariety of an affine space K^r; from Corollary 3 of Proposition 7, the functions Q_i are induced by polynomials P_i; let R_j be a finite system of generators of the ideal $I(X)$. The functions P_i, R_j do not vanish simultaneously on K^r, so they define an open cover $\mathfrak{U}'$ of K^r; let $\mathcal{F}'$ be the sheaf obtained by extending $\mathcal{F}$ by 0 outside of X; on applying the special case already established to the space K^r, the functions P_i, R_j, and the sheaf $\mathcal{F}'$, we see that $H^q(\mathfrak{U}', \mathcal{F}') = 0$ for

$q > 0$. One verifies immediately that the complex $C(\mathfrak{U}', \mathcal{F}')$ is isomorphic to the complex $C(\mathfrak{U}, \mathcal{F})$, so it follows that $H^q(\mathfrak{U}, \mathcal{F}) = 0$, qed.

COROLLARY 1. *If X is an affine variety, and $\mathcal{F}$ is a coherent algebraic sheaf on X, we have $H^q(X, \mathcal{F}) = 0$ for all $q > 0$.*

In effect the covers of the type utilized in the last theorem are arbitrarily fine.

COROLLARY 2. *Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves on an affine variety X. If the sheaf $\mathcal{A}$ is coherent algebraic, the homomorphism $\Gamma(X, \mathcal{B}) \to \Gamma(X, \mathcal{C})$ is surjective.*

This follows from Corollary 1, when we put $q = 1$.

47. Coverings of algebraic varieties by open affine subsets.

PROPOSITION 8. *Let X be an affine variety, and let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a covering of X by finitely many open affine subsets. If $\mathcal{F}$ is a coherent algebraic sheaf on X, we have $H^q(\mathfrak{U}, \mathcal{F}) = 0$ for all $q > 0$.*

From Proposition 3 there are regular functions P_j on X such that the covering $\mathfrak{V} = \{X_{P_j}\}$ is finer than $\mathfrak{U}$. For all $(i_0, \dots, i_p)$, the covering $\mathfrak{V}_{i_0 \cdots i_p}$ induced by $\mathfrak{V}$ on $U_{i_0 \cdots i_p}$ is defined by the restrictions of the P_j to $U_{i_0 \cdots i_p}$; since $U_{i_0 \cdots i_p}$ is an affine variety, by Proposition 1, we can apply Theorem 3 to it, concluding that $H^q(\mathfrak{V}_{i_0 \cdots i_p}, \mathcal{F}) = 0$ for all $q > 0$. Then applying Proposition 5 of no. 29, we see that

$$H^q(\mathfrak{U}, \mathcal{F}) = H^q(\mathfrak{V}, \mathcal{F}),$$

and, since $H^q(\mathfrak{V}, \mathcal{F}) = 0$ for all $q > 0$ from Theorem 3, the Proposition is proved.

THEOREM 4. *Let X be an algebraic variety, $\mathcal{F}$ a coherent algebraic sheaf on X, and $\mathfrak{U} = \{U_i\}_{i \in I}$ a finite open cover of X by affine open subsets. The homomorphism $\sigma(\mathfrak{U}) : H^n(\mathfrak{U}, \mathcal{F}) \to H^n(X, \mathcal{F})$ is bijective for all $n \geq 0$.*

Consider the family $\mathfrak{V}^\alpha$ of finite covers of X by open affine subsets. From the Corollary to Theorem 1, these covers are arbitrarily fine. On the other hand for, for each system $(i_0, \dots, i_p)$, the cover $\mathfrak{V}^\alpha_{i_0 \cdots i_p}$ induced by $\mathfrak{V}^\alpha$ on $U_{i_0 \dots i_p}$ is a cover by open affine subsets, by Proposition 1; from Proposition 8, we have $H^q(\mathfrak{V}^\alpha_{i_0 \cdots i_p}, \mathcal{F}) = 0$ for $q > 0$. Conditions (a) and (b) of Theorem 1, no. 29, hold, and the claim follows.

THEOREM 5. *Let X be an algebraic variety, and $\mathfrak{U} = \{U_i\}_{i \in I}$ a finite covering of X by affine open subsets. Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves on X, with $\mathcal{A}$ being coherent algebraic. The canonical homomorphism $H^q_0(\mathfrak{U}, \mathcal{C}) \to H^q(\mathfrak{U}, \mathcal{C})$ (cf. no. 24) is bijective for all $q \geq 0$.*

It evidently suffices to show that $C_0(\mathfrak{U}, \mathcal{C}) = C(\mathfrak{U}, \mathcal{C})$, which is to say that each section of $\mathcal{C}$ above $U_{i_0 \cdots i_p}$ is the image of a section of $\mathcal{B}$ above $U_{i_0 \cdots i_p}$, which follows from Corollary 2 of Theorem 3.

Corollary 1. *Let X be an algebraic variety, and let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves on X, with $\mathcal{A}$ coherent algebraic. The canonical homomorphism $H_0^q(X, \mathcal{C}) \to H^q(X, \mathcal{C})$ is bijective for all $q \geq 0$.*

This is an immediate consequence of Theorems 1 and 5.

Corollary 2. *There is an exact sequence:*

$$\cdots \to H^q(X, \mathcal{B}) \to H^q(X, \mathcal{C}) \to H^{q+1}(X, \mathcal{A}) \to H^{q+1}(X, \mathcal{B}) \to \cdots$$

This is Proposition 5 of no. 24 with $H_0^q(X, \mathcal{C})$ replaced by $H^q(X, \mathcal{C})$.

§4. Correspondence between modules of finite type and coherent algebraic sheaves

49. Sheaves associated with a module. Let V be an affine variety, $\mathcal{O}$ the sheaf of local rings of V; the ring $A = \Gamma(V, \mathcal{O})$ is called the *coordinate ring* of V; it is a K-algebra that has no nilpotent elements aside from 0. If V is embedded as a closed subvariety of an affine space K^r, we know (cf. no. 44) that A is identified with the quotient algebra of $K[X_1, \ldots, X_r]$ by the ideal of polynomials that vanish on V; it follows that the algebra is generated by a finite number of elements.

Conversely, one can easily verify that, if A is a commutative K-algebra without nilpotent elements (other than 0) and is generated by a finite number of elements, there is an affine variety V such that A is isomorphic to $\Gamma(V, \mathcal{O})$; moreover, V is determined up to isomorphism by that property (one can identify V with the set of characters of A, endowed with the usual topology).

I am not sure about the usage of the word 'character' ('caractère') here. As for the assertion, if A is generated by $f_1, \ldots, f_r$, then A can be identified with $K[X_1, \ldots, X_r]/I$ where

$$I = \{\, P \in K[X_1, \ldots, X_r] : P(f_1, \ldots, f_r) = 0 \,\}.$$

Since A has no nilpotents, I is a radical ideal, and Hilbert's nullstellensatz implies that I is the set of polynomials that vanish on the variety $V(I) \subset K^r$. The claim follows from Corollary 3 of Proposition 7 in no. 44.

Let M be an A-module; M defines a constant sheaf on V, which is also denoted by M; in this way A itself defines a constant sheaf, and the sheaf M can be thought of as a sheaf of A-modules. Setting $\mathcal{A}(M) = \mathcal{O} \otimes_A M$, with the sheaf $\mathcal{O}$ regarded as a sheaf of A-modules; it is clear that $\mathcal{A}(M)$ is an algebraic sheaf on V. Moreover, if $\varphi : M \to M'$ is an A-homomorphism, there is a homomorphism $\mathcal{A}(\varphi) = 1 \otimes \varphi : \mathcal{A}(M) \to \mathcal{A}(M')$; that is, $\mathcal{A}(M)$ is a covariant functor of the module M.

Proposition 1. *The functor $\mathcal{A}(M)$ is exact.*

Of course nowadays we say "the functor $\mathcal{A}$" rather than "the functor $\mathcal{A}(M)$."

More substantively, the concept introduced here, of a sheaf given by an $\mathcal{O}$-module, can be thought of a precursor of the notion of a quasicoherent sheaf, which is (roughly speaking) a sheaf that is locally of this form.

Let $M \to M' \to M''$ be an exact sequence of A-modules. We need to show that the sequence $\mathcal{A}(M) \to \mathcal{A}(M') \to \mathcal{A}(M'')$ is exact, which is to say that for each $x \in V$, the sequence:

$$\mathcal{O}_x \otimes_A M \to \mathcal{O}_x \otimes_A M' \to \mathcal{O}_x \otimes_A M''$$

is exact.

Now $\mathcal{O}_x$ is none other than the ring of fractions A_S where S is the set of $f \in A$ such that $f(x) \neq 0$ (for the definition of the ring of fraction, cf. [8], [12] or [13]). Proposition 1 is consequently a particular case of the following result:

LEMMA 1. *Let A be a ring, S a subset of A containing all products of its elements, but not containing* 0, *and A_S the ring of fractions of A relative to S. If $M \to M' \to M''$ is an exact sequence of A-modules, the sequence $A_S \otimes_A M \to A_S \otimes_A M' \to A_S \otimes_A M''$ is exact.*

This is Proposition A5.1.

Let M_S be the set of fractions m/s, with $m \in M$, $s \in S$, and two fractions m/s and m'/s' identified if there is $s'' \in S$ such that $s''(s' \cdot m - s \cdot m') = 0$; it is easy to see that M_S is an A_S-module, and the function

$$a/s \otimes m \to a \cdot m/s$$

is an isomorphism of $A_S \otimes_A M$ and M_S; the claim boils down to the sequence

$$M_S \to M'_S \to M''_S$$

being exact, which is immediate.

PROPOSITION 2. $\mathcal{A}(M) = 0$ *implies* $M = 0$.

Let m be an element of M; if $\mathcal{A}(M) = 0$, we have $1 \otimes m = 0$ in $\mathcal{O}_x \otimes_A M$ for all $x \in V$. In view of the above, $1 \otimes m = 0$ is equivalent to the existence, for each $x \in V$, of an element $s \in A$ with $s(x) \neq 0$ such that $s \cdot m = 0$; consequently the annihilator of m in M is not contained in any maximal ideal of A, which implies that it is equal to A, whence $m = 0$.

PROPOSITION 3. *If M is an A-module of finite type, $\mathcal{A}(M)$ is a coherent algebraic sheaf on V.*

Since M is of finite type and A is Noetherian, M is isomorphic to the cokernel of a homomorphism $\varphi : A^q \to A^p$, and $\mathcal{A}(M)$ is isomorphic to the cokernel of $\mathcal{A}(\varphi) : \mathcal{A}(A^q) \to \mathcal{A}(A^p)$. Since $\mathcal{A}(A^p) = \mathcal{O}^p$ and $\mathcal{A}(A^q) = \mathcal{O}^q$, it follows that $\mathcal{A}(M)$ is coherent.

49. Modules associated with an algebraic sheaf. Let $\mathcal{F}$ be an algebraic sheaf on V, and let $\Gamma(\mathcal{F}) = \Gamma(V, \mathcal{F})$; since $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, $\Gamma(\mathcal{F})$ is endowed with a natural structure of an A-module. Each algebraic homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ defines an A-homomorphism $\Gamma(\varphi) : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{G})$. If one has an exact sequence of coherent algebraic sheafs $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$, the sequence

$$\Gamma(\mathcal{F}) \to \Gamma(\mathcal{G}) \to \Gamma(\mathcal{H})$$

is exact (no. 45); on applying this to the exact sequence $\mathcal{O}^p \to \mathcal{F} \to 0$, we see that $\Gamma(\mathcal{F})$ is an A-module of finite type if $\mathcal{F}$ is coherent.

Corollary 1 in no. 45 implies that there is an exact sequence $\mathcal{O}^p \to \mathcal{F} \to 0$.

The functors $\mathcal{A}(M)$ and $\Gamma(\mathcal{F})$ are "reciprocals" of each other.

Theorem 1. *(a) If M is an A-module of finite type, $\Gamma(\mathcal{A}(M))$ is canonically isomorphic to M.*

(b) If $\mathcal{F}$ is a coherent algebraic sheaf on V, $\mathcal{A}(\Gamma(\mathcal{F}))$ is canonically isomorphic to $\mathcal{F}$.

Let's first show (a). Each element $m \in M$ defines a section $\alpha(m)$ of $\mathcal{A}(M)$ by the formula $\alpha(m)(x) = 1 \otimes m \in \mathcal{O}_x \otimes_A M$; in this way we have a homomorphism $\alpha : M \to \Gamma(\mathcal{A}(M))$. When M is a free module of finite type, α is bijective (it suffices to see this when $M = A$, after which the general case is evident); if M is a module of finite type, there is an exact sequence $L^1 \to L^0 \to M \to 0$ where L^0 and L^1 are free of finite type; the sequence $\mathcal{A}(L^1) \to \mathcal{A}(L^0) \to \mathcal{A}(M) \to 0$ is exact, whence the sequence $\Gamma(\mathcal{A}(L^1)) \to \Gamma(\mathcal{A}(L^0)) \to \Gamma(\mathcal{A}(M)) \to 0$ is exact. The commutative diagram:

$$\begin{array}{ccccccc}
L^1 & \longrightarrow & L^0 & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\
\Gamma(\mathcal{A}(L^1)) & \longrightarrow & \Gamma(\mathcal{A}(L^0)) & \longrightarrow & \Gamma(\mathcal{A}(M)) & \longrightarrow & 0
\end{array}$$

then shows that $\alpha : M \to \Gamma(\mathcal{A}(M))$ is bijective, which establishes (a).

Add another pair of zeros on the right and apply the five lemma.

Now let $\mathcal{F}$ be a coherent algebraic sheaf on V. If we associate to each $s \in \Gamma(\mathcal{F})$ the element $s(x) \in \mathcal{F}_x$, we obtain an A-homomorphism: $\Gamma(\mathcal{F}) \to \mathcal{F}_x$ which extends to an $\mathcal{O}_x$-homomorphism $\beta_x : \mathcal{O}_x \otimes_A \Gamma(\mathcal{F}) \to \mathcal{F}_x$; one easily verifies that the β_x form a homomorphism of sheaves $\beta : \mathcal{A}(\Gamma(\mathcal{F})) \to \mathcal{F}$. When $\mathcal{F} = \mathcal{O}^p$, the homomorphism is bijective; it follows, by the same reasoning as above, that β is bijective for all coherent algebraic sheaves, which proves (b).

We can now give an example of a sheaf that is not of the form $\mathcal{A}(M)$. Let $V = K$, understood as 1-dimensional affine space. Let $\mathcal{F}$ be the sheaf on V such that for each open $U \subset V$, the space of sections $\Gamma(U, \mathcal{F})$ is $\mathcal{O}_V(U)$ if $0 \notin U$ and 0 if $0 \in U$. Any global section of $\mathcal{F}$ has to agree with the zero section in a neighborhood of 0, and is consequently identically zero. If $\mathcal{F} = \mathcal{A}(M)$, then by (a) we would have $M = 0$, and of course $\mathcal{F}$ is not $\mathcal{A}(0)$.

Remarks. (1) One can also deduce (b) from (a); cf. no. 65, proof of Proposition 6.

(2) We will see in Chapter III how one needs to modify the preceeding correspondence when one studies coherent sheaves on projective space.

50. Projective modules and fiber spaces with vector space fiber. Recall ([6]. Chap. I, th. 2.2) that an A-module is said to be *projective* if it is a direct factor of a free A-module.

Proposition 4. *Let M be an A-module of finite type. In order for M to be projective, it is necessary and sufficient that the $\mathcal{O}_x$-module $\mathcal{O}_x \otimes_A M$ is free for all $x \in V$.*

If M is projective, $\mathcal{O}_x \otimes_A M$ is projective, hence $\mathcal{O}_x$-free since $\mathcal{O}_x$ is a local ring (cf. [6], Chap. VIII, th. 6.1′).

Suddenly there is a flurry of homological algebra, and in general, from this point on, the expected background in that subject will be much higher. Proposition B7.3 asserts that if M is a projective R-module and S is an R-algebra, then $S \otimes_R M$ is a projective S-module, which is why $\mathcal{O}_x \otimes_A M$ is a projective $\mathcal{O}_x$-module. CE's Chap. VIII, th. 6.1′ is Theorem D5.3. It asserts that if R is a Noetherian local ring, $\mathfrak{m}$ is its maximal ideal, and $k = R/\mathfrak{m}$ is its residue field, then every finitely generated R-module M such that $\mathrm{Tor}_1^R(M, k) = 0$ is free, and every finite set of generators contains a base. Of course $\mathrm{Tor}_1^{\mathcal{O}_x}(\mathcal{O}_X \otimes_A M, K) = 0$ because $\mathcal{O}_x \otimes_A M$ is projective.

Reciprocally, if all the $\mathcal{O}_x \otimes_A M$ are free, we have

$$\dim(M) = \sup \dim_{x \in V}(\mathcal{O}_x \otimes_A M) = 0 \qquad \text{(cf. [6], Chap. VII, Exer. 11)},$$

which implies that M is projective ([6], Chap. VI, §2).

CE's Chap. VII, Exer. 11 is Theorem D5.8. Among other things, it asserts that if R is Noetherian and M is a finitely generated R-module, then

$$\dim_R M = \sup \dim_{R_\mathfrak{m}} M_\mathfrak{m}$$

where the supremum is over all maximal ideals $\mathfrak{m} \subset R$. By the definition of projective dimension, an R-module is projective if and only if its dimension is zero. Thus the logic here is that if each $\mathcal{O}_x \otimes_A M$ is free, then it is projective, hence zero dimensional, after which it follows that M is zero dimensional, hence projective.

Note that, if $\mathcal{F}$ is a coherent algebraic sheaf on V, and if $\mathcal{F}_x$ is isomorphic to $\mathcal{O}_x^p$, $\mathcal{F}$ is isomorphic to $\mathcal{O}^p$ above a neighborhood of x; if this property is satisfied by every $x \in V$, the sheaf $\mathcal{F}$ is then locally isomorphic to the sheaf $\mathcal{O}^p$, with the integer p constant on each connected component of V. On applying this to the sheaf $\mathcal{A}(M)$, we obtain:

Corollary. *Let $\mathcal{F}$ be a coherent algebraic sheaf on a connected affine variety V. The following three properties are equivalent:*

(i) $\Gamma(\mathcal{F})$ *is a projective A-module.*

(ii) $\mathcal{F}$ *is locally isomorphic to a sheaf $\mathcal{O}^p$.*

(iii) $\mathcal{F}$ *is isomorphic to the sheaf of germs of sections of a fiber space with vector space fiber and base V.*

Moreover, the function $E \to \Gamma(\mathcal{S}(E))$ (E denotes a fiber space with vector space fiber) puts in one-to-one correspondence the classes of fiber spaces and the classes of A-modules of finite type; in this correspondence, a *trivial* fiber space corresponds to a *free* module, and reciprocally.

Note that, when $V = K^r$ (in which case $A = K[X_1, \dots, X_r]$), it is not known whether there exist projective A-modules of finite type that are not free, or, what amounts to the same thing, whether there are fiber spaces with vector space fiber, and base K^r, that are nontrivial.

This question became known as Serre's conjecture. (There is another "Serre conjecture" in number theory that was proved in 2006 by Chandrashekhar Khare and Jean-Pierre Wintenberger.) In 1976 Daniel Quillen (who was awarded the Fields Medal in 1978) and Andrei Suslin independently proved the following generalization: if R is a principal ideal domain, then any projective $R[X_1, \dots, X_n]$-module is free. A simpler proof due to Leonid Vaserstein can be found in Lang (1993).

Chapter III. Coherent Algebraic Sheaves on Projective Varieties

§1. Projective Varieties

51. Notation. (The notation introduced below will be used without reference throughout the remainder of the chapter.)

Let r be an integer ≥ 0, and let $Y = K^{r+1} \setminus \{0\}$; the multiplicative group K^* of elements $\neq 0$ operates on Y by the formula

$$\lambda(\mu_0, \dots, \mu_r) = (\lambda\mu_0, \dots, \lambda\mu_r).$$

Two points y and y' we be said to be equivalent if there is a $\lambda \in K^*$ such that $y' = \lambda y$; the quotient space of Y by this equivalence relation is denoted by $\mathbf{P}_r(K)$, or simply X; it is the *projective space of dimension r on K*; the canonical projection of Y on X will be denoted by π.

Let $I = \{0, 1, \ldots, r\}$; for each $i \in I$, we denote by t_i the i^{th} coordinate function on K^{r+1}, defined by the formula:

$$t_i(\mu_0, \ldots, \mu_r) = \mu_i.$$

We denote by V_i the open subset of K^{r+1} consisting of the points where t_i is $\neq 0$, and by U_i the image of V_i under π; the $\{U_i\}_{i\in I}$ form an open cover of X. If $i \in I$ and $j \in I$, the function t_j/t_i is regular on V_i, and invariant under K^*, so it defines a function on U_i which we also denote by t_j/t_i; for fixed i, the functions t_j/t_i, $j \neq i$, define bijections $\psi_i : U_i \to K^r$.

We endow K^{r+1} with its structure as an algebraic variety, and Y with the induced structure. Similarly, we endow X with the quotient topology induced by that of Y: a closed subset of X is thus the image under π of a closed cone of K^{r+1}. If U is open in X, we set $A_U = \Gamma(\pi^{-1}(U), \mathcal{O}_Y)$; this is the ring of regular functions on $\pi^{-1}(U)$. Let A_U^0 be the subring of A_U consisting of the elements that are invariant under K^* (that is, the functions that are homogeneous of degree 0). When $V \supset U$, there is a restriction homomorphism $\varphi_U^V : A_V^0 \to A_U^0$, and the system of the (A_U^0, φ_U^V) define a sheaf that may be regarded as a subsheaf of the sheaf $\mathcal{F}(X)$ of germs of functions on X. For such a function f, defined in a neighborhood of x, to be in $\mathcal{O}_{x,X}$, it is necessary and sufficient that it coincide with a function of the form P/Q, where P and Q are two homogeneous polynomials of the same degree in $t_0, \ldots, t_r$, with $Q(y) \neq 0$ for $y \in \pi^{-1}(x)$ (which we write more briefly as $Q(x) \neq 0$).

Proposition 1. *The projective space $X = \mathbf{P}_r(K)$, endowed with the preceeding topology and sheaf, is an algebraic variety.*

The U_i, $i \in I$, are open subsets of X, and one verifies right away that the bijections $\psi_i : U_i \to K^r$ defined above are biregular isomorphisms, which shows that the axiom (VA_I) is satisfied. To demonstrate that (VA_{II}) is as well, we need to see that the subset of K^r consisting of the pairs $(\psi_i(x), \psi_j(x))$ for $x \in U_i \cap U_j$, is closed, which does not present any difficulties.

In the following, X will always be endowed with the structure of an algebraic variety which we have defined; the sheaf $\mathcal{O}_X$ will be denoted simply by $\mathcal{O}$. An algebraic variety V is said to be *projective* if it is isomorphic to a closed subvariety of a projective space. The study of coherent algebraic sheaves on projective varieties reduces to the study of coherent algebraic sheaves on the $\mathbf{P}_r(K)$, cf. no. 39.

52. Cohomology of subvarieties of projective space. Let's apply Theorem 4 of no. 47 to the covering $\mathfrak{U} = \{U_i\}_{i\in I}$, defined in the preceeding no.: this is possible because each of the U_i is isomorphic to K^r. In this way we obtain:

Proposition 2. *If $\mathcal{F}$ is a coherent algebraic sheaf on $X = \mathbf{P}_r(K)$, the homomorphism $\sigma(\mathfrak{U}) : H^n(\mathfrak{U}, \mathcal{F}) \to H^n(X, \mathcal{F})$ is bijective for all $n \geq 0$.*

Because $\mathfrak{U}$ consists of $r+1$ open sets, we have (cf. no. 20, Corollary to Proposition 2):

COROLLARY. *$H^n(X, \mathcal{F}) = 0$ for all $n > r$.*

The last result can be generalized in the following manner:

PROPOSITION 3. *Let V be an algebraic variety, isomorphic to a locally closed subvariety of a projective space X. Let $\mathcal{F}$ be a coherent algebraic sheaf on V, and let W be a subvariety of V such that $\mathcal{F}$ is null outside of W. Then $H^n(V, \mathcal{F}) = 0$ for $n > \dim W$.*

In particular, taking $W = V$, we see that:

COROLLARY. *$H^n(V, \mathcal{F}) = 0$ for $n > \dim V$.*

We identify V with a locally closed subvariety of $X = \mathbf{P}_r(K)$; there is an open subset U of X such that V is closed in U. We suppose that W is closed in V, which is evidently legitimate; then W is closed in U.

The hypotheses are preserved if we replace W with its closure in V because taking the closure does not increase dimension. Any irreducible component of the closure of W contains a point $x \in W$. (The union of those irreducible components of the closure of W that have a nonempty intersection with W is a closed set containing W, so it is the closure of W.) Therefore the dimension of this component is the transcendence degree of the field of fractions of $\mathcal{O}_{x,W}$ (cf. no. 36) which is not greater than the dimension of W.

Let $F = X \setminus U$. Before proving Proposition 3 we establish two lemmas.

LEMMA 1. *Let $k = \dim W$; there exist $k+1$ homogeneous polynomials $P_i(t_0, \ldots, t_r)$, of degree > 0, that vanish on F but do not vanish simultaneously on W.*

(By abuse of language, we say that a homogeneous polynomial P vanishes at a point x of $\mathbf{P}_r(K)$ if it vanishes on $\pi^{-1}(x)$.)

We argue by recurrence on k, the case where $k = -1$ being trivial. Choose a point in each irreducible component of W, and let P_1 be a homogeneous polynomial that vanishes on F, of degree > 0, and does not vanish at any of these points (the existence of P_1 follows from the fact that F is closed, taking into account the definition of the topology of $\mathbf{P}_r(K)$).

In somewhat pedantic detail, the existence of such a polynomial for each point follows from this argument, and because K is infinite some linear combination of these polynomials does not vanish at any of the points.

Let W' be the subvariety of W consisting of the points $x \in W$ such that $P_1(x) = 0$; in view of the construction of P_1, each irreducible component of W is not contained in W', and it follows (cf. no. 36) that $\dim W' < k$. On applying the recurrence hypothesis to W', we see that there exist k polynomials

$P_2, \ldots, P_{k+1}$ that vanish on F but do not vanish simultaneously on W'; it is clear that the polynomials $P_1, \ldots, P_{k+1}$ satisfy the desired condition.

LEMMA 2. *Let $P(t_0, \ldots, t_r)$ be a homogeneous polynomial of degree $n > 0$. The set X_P of points $x \in X$ such that $P(x) \neq 0$ is an affine open subset of X.*

If, to each point $y \in (\mu_0, \ldots, \mu_r) \in Y$, one associates the obvious point in a space K^N whose coordinates are all the monomials $\mu_0^{m_0} \cdots \mu_r^{m_r}$, $m_0 + \cdots + m_r = n$, one obtains, on passage to quotients, a function $\varphi_n : X \to \mathbf{P}_{N-1}(K)$. It is a classical fact, and quite easy to verify, that φ_n is a biregular isomorphism between X and a closed subvariety of $\mathbf{P}_{N-1}(K)$ ("variety of Veronese"); now φ_n transforms the open set X_P to the set of points of $\varphi_n(X)$ that are not situated on a certain hyperplane of $\mathbf{P}_{N-1}(K)$; as the complement of a hyperplane is isomorphic to an affine space, we can conclude that X_P is indeed isomorphic to a closed subvariety of an affine space.

We now prove Proposition 3. Prolong the sheaf $\mathcal{F}$ by 0 on $U \setminus V$; we obtain a coherent algebraic sheaf on U, that we also denote by $\mathcal{F}$, and we know (cf. no. 26) that $H^n(U, \mathcal{F}) = H^n(V, \mathcal{F})$. Let $P_1, \ldots, P_{k+1}$ be homogeneous polynomials satisfying the conditions of Lemma 1; let $P_{k+2}, \ldots, P_h$ be polynomials of degree > 0, vanishing on $W \cup F$ and not vanishing simultaneously at any point of $U \setminus W$ (to obtain such polynomials, it suffices to take a system of homogeneous generators of the ideal defined by $W \cup F$ in $K[t_0, \ldots, t_r]$). For each i, $1 \leq i \leq h$, let V_i be the set of points $x \in X$ such that $P_i(x) \neq 0$; then $V_i \subset U$, and the hypotheses made above show that $\mathfrak{V} = \{V_i\}$ is an open cover of U; in addition, Lemma 2 shows that the V_i are open affines, so $H^n(\mathfrak{V}, \mathcal{F}) = H^n(U, \mathcal{F}) = H^n(V, \mathcal{F})$ for all $n \geq 0$. If $n > k$, and the indices $i_0, \ldots, i_n$ are distinct, one of these indices is $> k + 1$, and $V_{i_0 \cdots i_q}$ does not intersect W, so we conclude that the group of alternating cochains $C'^n(\mathfrak{V}, \mathcal{F})$ is null if $n > k$, and this implies that $H^n(\mathfrak{V}, \mathcal{F}) = 0$, from Proposition 2 of no. 20.

53. Cohomology of irreducible algebraic curves. If V is an irreducible algebraic variety of dimension 1, the closed subvarities of V, distinct from V, are the *finite* subsets. If F is a finite subset of V, and x is a point in F, we set $V_x^F = (V \setminus F) \cup \{x\}$; the V_x^F, $x \in F$, form a finite open cover $\mathfrak{V}^F$ of V.

LEMMA 3. *The open covers of the form $\mathfrak{V}^F$ are arbitrarily fine.*

Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of V, that we may suppose is finite, since V is quasi-compact. We may also suppose that $U_i \neq \emptyset$ for all $i \in I$. If we set $F_i = V \setminus U_i$, F_i is finite, as is $F = \bigcup_{i \in I} F_i$. We will show that $\mathfrak{V}^F \prec \mathfrak{U}$, which proves the lemma. Let x be an element of F; there is $i \in I$ such that $x \notin F_i$, since the U_i cover V; we then have $F \setminus \{x\} \supset F_i$, since $F \supset F_i$, which means that $V_x^F \subset U_i$, so indeed $\mathfrak{V}^F \prec \mathfrak{U}$.

LEMMA 4. *Let $\mathcal{F}$ be a sheaf on V, and F a finite subset of V. Then*

$$H^n(\mathfrak{V}^F, \mathcal{F}) = 0$$

for $n \geq 2$.

Put $W = V \setminus F$; it is clear that $V_{x_0}^F \cap \cdots \cap V_{x_n}^F = W$ if the $x_0, \ldots, x_n$ are distinct, and if $n \geq 1$. If we set $G = \Gamma(W, \mathcal{F})$, it follows that the alternating complex $C'(\mathfrak{V}^F, \mathcal{F})$ is isomorphic, in dimensions ≥ 1, to $C'(S(F), G)$, where $S(F)$ denotes the simplex which has F as its set of points. It follows from this that

$$H^n(\mathfrak{V}^F, \mathcal{F}) = H^n(S(F), G) = 0 \quad \text{for } n \geq 2,$$

the cohomology of a simplex being trivial.

Lemmas 3 and 4 evidently imply:

PROPOSITION 4. *If V is an irreducible algebraic curve, and $\mathcal{F}$ is a sheaf on V, then $H^n(V, \mathcal{F}) = 0$ for $n \geq 2$.*

REMARK. I don't know if an analogous result holds for varieties of arbitrary dimension.

§2. Graded modules and coherent algebraic sheafs on projective space

54. The operation $\mathcal{F}(n)$. Let $\mathcal{F}$ be an algebraic sheaf on $X = \mathbf{P}_r(K)$. Let $\mathcal{F}_i = \mathcal{F}(U_i)$ be the restriction of $\mathcal{F}$ to U_i (cf. no. 51); with n designating an arbitrary integer, let $\theta_{ij}(n)$ be the isomorphism from $\mathcal{F}_j(U_i \cap U_j)$ to $\mathcal{F}_i(U_i \cap U_j)$ defined by multiplication by the function t_j^n/t_i^n; this makes sense, since t_j/t_i is a regular function on $U_i \cap U_j$ with values in K^*. We have $\theta_{ij}(n) \circ \theta_{jk}(n) = \theta_{ik}(n)$ at each point of $U_i \cap U_j \cap U_k$; consequently we can apply Proposition 4 of no. 4, and thereby obtain an algebraic sheaf, denoted by $\mathcal{F}(n)$, defined by gluing of the sheafs $\mathcal{F}_i = \mathcal{F}(U_i)$ by means of the isomorphisms $\theta_{ij}(n)$.

There are canonical isomorphisms: $\mathcal{F}(0) \approx \mathcal{F}$, $\mathcal{F}(n)(m) \approx \mathcal{F}(n+m)$. Moreover, $\mathcal{F}(n)$ is locally isomorphic to $\mathcal{F}$, hence coherent if $\mathcal{F}$ is; in the same way it follows that each exact sequence $\mathcal{F} \to \mathcal{F}' \to \mathcal{F}''$ of algebraic sheaves gives rise to an exact sequence $\mathcal{F}(n) \to \mathcal{F}'(n) \to \mathcal{F}''(n)$ for all $n \in \mathbb{Z}$.

We can apply the construction above to the sheaf $\mathcal{F} = \mathcal{O}$, and thus obtain the sheafs $\mathcal{O}(n)$, $n \in \mathbb{Z}$. We will give another description of these sheaves: if U is open in X, let A_U^n be the subset of $A_U = \Gamma(\pi^{-1}(U), \mathcal{O}_Y)$ consisting of the functions that are homogeneous of degree n (that is to say, satisfying the condition $f(\lambda y) = \lambda^n f(y)$ for $\lambda \in K^*$, and $y \in \pi^{-1}(U)$); the A_U^n are A_U^0-modules, which give rise to a algebraic sheaf, that we denote by $\mathcal{O}'(n)$. An element of $\mathcal{O}'(n)_x$, $x \in X$, can be identified with a rational fraction P/Q, P and Q being homogeneous polynomials such that $Q(x) \neq 0$ and $\deg P - \deg Q = n$.

What's going on here? Lets start with $\mathcal{O}$. A section of $\mathcal{O}$ over an open set is just a (highly structured) K-valued function on that set, and in this sense we can identify $\mathcal{O}$ with the cartesian product $X \times K$. We have attached a copy of K to each point of X.

In the topological case we know this can be done in multiple ways. For example, for the circle $S^1 = \{\, (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \,\}$, in addition to $S^1 \times \mathbb{R}$, there is the Moëbius strip. In general this type of construct, in which a vector space of a given dimension is attached to each point of a "base space," is called a *vector bundle*. These come in continuous, C^∞, real analytic, and complex analytic flavors, among others. The tangent bundle and cotangent bundle of a smooth manifold are obvious, and obviously quite important, examples.

In the "floppy" C^∞ case the way we attach copies of the vector space near one point of the base space has no necessary relation to the way we attach copies near some other point, so local information cannot be used to distinguish different C^∞ vector bundles with the same base. Instead we have to use some global topological information, e.g., $S^1 \times \mathbb{R}$ has a nowhere vanishing global section and the Moëbius strip doesn't. The general development of these concepts is a substantial and very important topic in topology; Milnor and Stasheff (1974) is a classic, highly readable treatment.

In algebraic geometry a 1-dimensional fiber bundle is called an *invertible sheaf*. Roughly, for any sufficiently well behaved sheaf $\mathcal{F}$ there is a dual sheaf $\mathcal{F}^\vee$, which is formed by taking the duals of each of the vector spaces that $\mathcal{F}$ attaches to the various points of X. For a vector space V over K, $V \otimes_K V^\vee$ (here $V^\vee$ is the dual of V) is isomorphic to K if V is 1-dimensional, and otherwise it is some higher dimensional vector space. The terminology of algebraic geometry uses this property to *diagnose* 1-dimensionality: if $\mathcal{F} \otimes_K \mathcal{F}^\vee = \mathcal{O}$, then $\mathcal{F}$ is *invertible* (because $\mathcal{F}^\vee$ is the "inverse" of $\mathcal{F}$) hence 1-dimensional, and otherwise it isn't. It turns out that the isomorphism classes of invertible sheafs constitute an abelian group, with tensor product as the group operation and the isomorphism class of $\mathcal{O}$ as the identity element. This is called the *Picard group*.

This is an explanation, but it isn't an excuse. 'Invertible sheaf' is an horrific piece of terminology that reeks of insiderism. (Actually the insiders aren't that fond of it either. When they are talking to each other informally, algebraic geometers are quite likely to say "line bundle.")

Vector bundles are sometimes described as "twisted products." Accordingly, $\mathcal{O}(1)$ has come to be known as the *twisting sheaf of Serre*. As Serre explains more generally below, its global sections are different from the global sections of $\mathcal{O}$ (that is, the constant functions) so the two bundles are not isomorphic. The invertible sheaf $\mathcal{O}(-1)$ is called the *tautological line bundle* because it can be understood as the sheaf that attaches to each $x \in \mathbf{P}_r(K)$ the point x itself, when x is understood as a 1-dimensional linear subspace of K^{r+1}.

Above we saw that $\mathcal{F}(n)(m) = \mathcal{F}(m+n)$. Below it will emerge that $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(n)$. (One might think of tensoring with $\mathcal{O}(n)$ as a method of "interrogating" $\mathcal{F}$.) Applied to $\mathcal{O}(n)$ itself, these results imply that for positive n, $\mathcal{O}(n)$ is the n-fold tensor product of the twisting sheaf of Serre, and $\mathcal{O}(-n)$ is the n-fold tensor product of the tautological line bundle. The isomorphism classes of these bundles evidently constitute a subgroup of the Picard group, and in fact this is the entire Picard group of $\mathbf{P}_r(K)$.

PROPOSITION 1. *The sheafs $\mathcal{O}(n)$ and $\mathcal{O}'(n)$ are canonically isomorphic.*

By definition, a section of $\mathcal{O}(n)$ on an open set $U \subset X$ is a system (f_i) of sections of $\mathcal{O}$ on the $U \cap U_i$, with $f_i = (t_j^n/t_i^n)f_j$ on $U \cap U_i \cap U_j$; the f_j can be identified with the homogeneous regular functions of degree 0 on the $\pi^{-1}(U) \cap \pi^{-1}(U_i)$; set $g_i = t_i^n \cdot f_i$; we then have $g_i = g_j$ at each point of $\pi^{-1}(U) \cap \pi^{-1}(U_i) \cap \pi^{-1}(U_j)$, so the g_i are the restrictions of a unique function g, regular on $\pi^{-1}(U)$, and homogeneous of degree n. Inversely, such a function g defines a system f_i on setting $f_i = g/t_i^n$. The function $(f_i) \to g$ is then an isomorphism between $\mathcal{O}(n)$ and $\mathcal{O}'(n)$.

In the following, we frequently identify $\mathcal{O}(n)$ and $\mathcal{O}'(n)$ by means of the preceeding isomorphism. Observe that a section of $\mathcal{O}'(n)$ above X is none other than a regular function on Y that is homogeneous of degree n. If $r \geq 1$, such a function is identically 0 for $n < 0$, and it is a homogeneous *polynomial* of degree n for $n \geq 0$.

Proposition 2. *For an algebraic sheaf $\mathcal{F}$, the sheafs $\mathcal{F}(n)$ and $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(n)$ are canonically isomorphic.*

Since $\mathcal{O}(n)$ is obtained from the $\mathcal{O}_i$ by gluing using the $\theta_{ij}(n)$, $\mathcal{F} \otimes \mathcal{O}(n)$ is obtained from the $\mathcal{F}_i \otimes \mathcal{O}_i$ by gluing using the isomorphisms $1 \otimes \theta_{ij}(n)$; on identifying $\mathcal{F}_i \otimes \mathcal{O}_i$ with $\mathcal{F}_i$, we recover the definition of $\mathcal{F}(n)$.

In the following, we will often identify $\mathcal{F}(n)$ with $\mathcal{F} \otimes \mathcal{O}(n)$.

55. Sections of $\mathcal{F}(n)$. We first prove a lemma on affine varieties, which is analogous to the Lemma of no. 45.

Lemma 1. *Let V be an affine variety, Q a regular function on V, and V_Q the set of points $x \in V$ such that $Q(x) \neq 0$. Let $\mathcal{F}$ be a coherent algebraic sheaf on V, and let s be a section of $\mathcal{F}$ above V_Q. Then, for all sufficiently large n, there is a section s' of $\mathcal{F}$ above all of V, such that $s' = Q^n s$ above V_Q.*

By embedding V in an affine space, and extending $\mathcal{F}$ by 0 outside of V, we reduce to the case in which V itself is an affine space, and in particular is irreducible. From Corollary 1 of Theorem 2 of no. 45, there is a surjective homomorphism $\varphi : \mathcal{O}_V^p \to \mathcal{F}$; from Proposition 2 of no. 42, V_Q is an open affine, and consequently (no. 44, Corollary 2 to Proposition 7) there is a section σ of $\mathcal{O}_V^p$ above V_Q such that $\varphi(\sigma) = s$.

In this application of Corollary 2, the $\mathcal{F}$ of that result is the kernel of φ.

We can identify σ with a system of p regular functions on V_Q; applying Proposition 5 of no. 43 to each of these functions, we see that there is a section σ' of $\mathcal{O}_V^p$ on V such that $\sigma' = Q^n\sigma$ on V_Q, provided that n is large enough. On setting $s' = \varphi(\sigma')$, we obtain a section of $\mathcal{F}$ on V such that $s' = Q^n s$ on V_Q.

Theorem 1. *Let $\mathcal{F}$ be a coherent algebraic sheaf on $X = \mathbf{P}_r(K)$. There is an integer $n(\mathcal{F})$ such that, for all $n \geq n(\mathcal{F})$, and all $x \in X$, the $\mathcal{O}_x$-module $\mathcal{F}(n)_x$ is generated by the elements of $\Gamma(X, \mathcal{F}(n))$.*

By the definition of $\mathcal{F}(n)$, a section s of $\mathcal{F}(n)$ on X is a system (s_i) of sections of $\mathcal{F}$ on U_i, satisfying the conditions of coherence:

$$s_i = (t_j^n/t_i^n) \cdot s_j \quad \text{on } U_i \cap U_j;$$

we say that s_i is *the i^{th} component of s.*

On the other hand, since U_i is isomorphic to K^r, there is a finite number of sections s_i^α of $\mathcal{F}$ that generate $\mathcal{F}_x$ for all $x \in U_i$ (no. 45, Corollary 1 to Theorem 2); if, for a certain integer n, we can find sections s^α of $\mathcal{F}(n)$ whose i^{th} components are the s_i^α, it is evident that $\Gamma(X, \mathcal{F}(n))$ generates $\mathcal{F}(n)_x$ for all $x \in U_i$. Therefore the theorem will follow once we prove the following lemma:

LEMMA 2. *Let s_i be a section of $\mathcal{F}$ above U_i. For all sufficiently large n, there is a section s of $\mathcal{F}(n)$ whose i^{th} component is equal to s_i.*

We apply Lemma 1 to the affine variety $V = U_i$, to the function $Q = t_i/t_j$, and to the section s_i restricted to $U_i \cap U_j$; this is possible because t_i/t_j is a regular function on U_j, and on this set the locus of zeros of the function is $U_i \setminus U_i \cap U_j$. In this way we obtain an integer p and a section s'_j of $\mathcal{F}$ on U_j such that $s'_j = (t_i^p/t_j^p) \cdot s_i$ on $U_i \cap U_j$; for $j = i$ this implies that $s'_i = s_i$, which allows us to write the preceeding formula as $s'_j = (t_i^p/t_j^p) \cdot s'_i$.

The s'_j being defined for all indices j (with the same exponent p), consider $s'_j - (t_k^p/t_j^p) \cdot s'_i$; this is a section of $\mathcal{F}$ on $U_j \cap U_k$ whose restriction to $U_i \cap U_j \cap U_k$ vanishes; applying Proposition 6 of no. 43 to it, we see that, for all sufficiently large integers q, we have $(t_i^q/t_j^q)(s'_j - (t_k^p/t_j^p) \cdot s'_k) = 0$ on $U_j \cap U_k$; if we then set $s_j = (t_i^q/t_j^q) \cdot s'_j$, and $n = p+q$, the preceeding formula becomes $s_j = (t_k^n/t_j^n) \cdot s_k$, and the system $s = (s_j)$ is indeed a section of $\mathcal{F}(n)$ whose i^{th} component is equal to s_i, qed.

COROLLARY. *Each coherent algebraic sheaf $\mathcal{F}$ on $X = \mathbf{P}_r(K)$ is isomorphic to a quotient sheaf of a sheaf $\mathcal{O}(n)^p$, where n and p are suitable integers.*

From the preceeding theorem, there is an integer n such that $\mathcal{F}(-n)_x$ is generated by $\Gamma(X, \mathcal{F}(-n))$ for all $x \in X$; in view of the quasi-compactness of X, it is equivalent to say that $\mathcal{F}(-n)$ is isomorphic to a quotient sheaf of the sheaf $\mathcal{O}^p$, p being a suitable integer ≥ 0. It then follows that $\mathcal{F} \approx \mathcal{F}(-n)(n)$ is isomorphic to a quotient sheaf of $\mathcal{O}(n)^p \approx \mathcal{O}^p(n)$.

56. Graded modules. Let $S = K[t_0, \ldots, t_r]$ be the algebra of the polynomials in $t_0, \ldots, t_r$; for each integer $n \geq 0$, let S_n be the vector subspace of S consisting of the polynomials that are homogeneous of degree n; the algebra S is the direct sum of the S_n, $n \in \mathbb{Z}$, and $S_p S_q \subset S_{p+q}$; in other words, S is a *graded algebra*.

Recall that an S-module M is said to be *graded* if there is a given decomposition of M as a direct sum: $M = \sum_{n \in \mathbb{Z}} M_n$, where the M_n are subgroups of M such that $S_p M_q \subset M_{p+q}$, for all pairs of integers (p, q). An element of M_n is said to be *homogeneous of degree* n; a sub-module N of M is said to be *homogeneous* if it is the direct sum of the $N \cap M_n$, in which case it is itself a graded S-module. If M and M' are two graded S-modules, an S-homomorphism

$$\varphi : M \to M'$$

is said to be *homogeneous of degree* s if $\varphi(M_n) \subset M'_{n+s}$ for all $n \in \mathbb{Z}$. An S-homomorphism that is homogeneous of degree 0 will simply be called a *homomorphism*.

If M is a graded S-module, and n is an integer, we denote by $M(n)$ the graded S-module:

$$M(n) = \sum_{p \in \mathbb{Z}} M(n)_p, \quad \text{with } M(n)_p = M_{n+p}.$$

Then $M(n) = M$ as an S-module, but an element of $M(n)$ that is homogeneous of degree p is homogeneous of degree $n+p$ in M; in other words, $M(n)$ lowers the degrees of M by n units.

We denote by $\mathcal{C}$ the class of graded S-modules M such that $M_n = 0$ for sufficiently large n. If $A \to B \to C$ is an exact sequence of homomorphisms of graded S-modules, the relations $A \in \mathcal{C}$ and $C \in \mathcal{C}$ evidently imply that $B \in \mathcal{C}$; that is, $\mathcal{C}$ is indeed a class, in the sense of [14], Chap. I. In a general fashion, we use the terminology introduced in that article; in particular, a homomorphism $\varphi : A \to B$ will be said to be $\mathcal{C}$-*injective* (resp. $\mathcal{C}$-*surjective*) if $\mathrm{Ker}(\varphi) \in \mathcal{C}$ (resp. if $\mathrm{Coker}(\varphi) \in \mathcal{C}$), and $\mathcal{C}$-*bijective* if it is both $\mathcal{C}$-injective and $\mathcal{C}$-surjective.

A graded S-module M is said to be *of finite type* if it is generated by a finite number of elements; we say that M *satisfies the condition* (TF) if there exists an integer p such that the submodule $\sum_{n \geq p} M_n$ of M is of finite type; it amounts to the same thing to say that M is $\mathcal{C}$-isomorphic to a module of finite type. The modules satisfying (TF) form a class containing $\mathcal{C}$.

A graded S-module M is said to be *free* (resp. *free of finite type*) if it admits a basis (resp. a finite basis) consisting of homogeneous elements, which is to say that it is isomorphic to a direct sum (resp. a finite direct sum) of modules $S(n_i)$.

57. Algebraic sheaves associated with a graded S-module. If U is a nonempty subset of X, $S(U)$ denotes the subset of $S = K[t_0, \ldots, t_r]$ consisting of the homogeneous polynomials Q such that $Q(x) \neq 0$ for all $x \in U$; $S(U)$ is a multiplicatively stable subset of S, that does not contain 0. For $U = X$ we write $S(x)$ in place of $S(\{x\})$.

There are typos in the original here: presumably "For $U = X$" above, and "For $U = x$" a few lines below, should both be "For $U = \{x\}$."

Let M be a graded S-module. We denote by M_U the set of fractions m/Q, with $m \in M$, $Q \in S(U)$, and m and Q homogeneous of the same degree; we identify two fractions m/Q and m'/Q' if there exist $Q'' \in S(U)$ such that

$$Q''(Q' \cdot m - Q \cdot m') = 0;$$

it is clear that this defines an equivalence relation between pairs (m, Q). For $U = x$, we write M_x in place of $M_{\{x\}}$.

Applying this with $M = S$, we find that S_U is the ring of fractions P/Q, where P and Q are homogeneous polynomials of the same degree and $Q \in S(U)$; if M is a general graded S-module, we can endow M_U with the structure of an S_U-module by setting:

$$m/Q + m'/Q' = (Q'm + Qm')/QQ'$$

$$(P/Q) \cdot (m/Q') = Pm/QQ'.$$

If $U \subset V$, then $S(V) \subset S(U)$, so there are canonical homomorphisms

$$\varphi_U^V : M_V \to M_U;$$

the system (M_U, φ_U^V), where U and V vary over the nonempty open subsets of X, define a sheaf that we denote by $\mathcal{A}(M)$; it is immediate that

$$\lim_{x \in U} M_U = M_x,$$

which is to say that $\mathcal{A}(M)_x = M_x$. In particular, we have $\mathcal{A}(S) = \mathcal{O}$, and since the M_U are S_U-modules, it follows that $\mathcal{A}(M)$ is a sheaf of $\mathcal{A}(S)$-modules, which is to say an *algebraic sheaf* on X. All homomorphisms $\varphi : M \to M'$ define in a natural fashion the S_U-linear homomorphisms $\varphi_U : M_U \to M'_U$, from which we obtain a homomorphism of sheaves $\mathcal{A}(\varphi) : \mathcal{A}(M) \to \mathcal{A}(M')$, that we often denote by φ. Evidently

$$\mathcal{A}(\varphi + \psi) = \mathcal{A}(\varphi) + \mathcal{A}(\psi), \quad \mathcal{A}(1) = 1, \quad \mathcal{A}(\varphi \circ \psi) = \mathcal{A}(\varphi) \circ \mathcal{A}(\psi).$$

The operation $\mathcal{A}(M)$ is thus an *additive covariant functor*, defined on the caegory of graded S-modules, and with values in the category of algebraic sheaves on X.

(The definitions above are all analogues of those of §4 of Chap. II; at the same time there is a difference insofar as S_U *is not* the ring of fractions of S relative to $S(U)$, but only its homogeneous component of degree 0.)

58. First properties of the functor $\mathcal{A}(M)$.

PROPOSITION 3. *The functor $\mathcal{A}(M)$ is an exact functor.*

Let $M \xrightarrow{\alpha} M' \xrightarrow{\beta} M''$ be an exact sequence of graded S-modules; we will show that $M_x \xrightarrow{\alpha} M'_x \xrightarrow{\beta} M''_x$ is also exact. If m'/Q is an element of the kernel of β, from the definition of M''_x there is $R \in S(x)$ such that $R\beta(m') = 0$; but then there exists $m \in M$ such that $\alpha(m) = Rm'$, and we have $\alpha(m/RQ) = m'/Q$, qed. (Compare with no. 48, Lemma 1.)

PROPOSITION 4. *If M is a graded S-module, and if n is an integer, $\mathcal{A}(M(n))$ is canonically isomorphic to $\mathcal{A}(M)(n)$.*

Consider $i \in I$, $x \in U_i$, and $m/Q \in M(n)_x$, with $m \in M(n)_p$, $Q \in S(x)$, $\deg Q = p$. Set

$$\eta_{i,x}(m/Q) = m/t_i^n Q \in M_x,$$

which is sensible because $m \in M_{n+p}$ and $t_i^n Q \in S(x)$. We see immediately that $\eta_{i,x} : M(n)_x \to M_x$ is bijective for all $x \in U_i$. In addition, $\eta_i \circ \eta_j^{-1} = \theta_{ij}(n)$ above $U_i \cap U_j$. From the definition of the operation $\mathcal{F}(n)$, and Proposition 4 of no. 4, this shows that $\mathcal{A}(M(n))$ is indeed isomorphic to $\mathcal{A}(M)(n)$.

COROLLARY. *$\mathcal{A}(S(n))$ is canonically isomorphic to $\mathcal{O}(n)$.*

In effect, we have already said that $\mathcal{A}(S)$ is isomorphic to $\mathcal{O}$.

(It is also directly evident that $\mathcal{A}(S(n))$ is isomorphic to $\mathcal{O}'(n)$, since $\mathcal{O}'(n)_x$ is precisely the set of fractions P/Q, such that $\deg P - \deg Q = n$, and $Q \in S(x)$.)

PROPOSITION 5. *Let M be a graded S-module satisfying the condition (TF). The algebraic sheaf $\mathcal{A}(M)$ is then a coherent sheaf, and, in order for $\mathcal{A}(M) = 0$ it is necessary and sufficient that $M \in \mathcal{C}$.*

If $M \in \mathcal{C}$, for all $m \in M$ and all $x \in X$, there exist $Q \in S(x)$ such that $Qm = 0$: it suffices to take Q of sufficiently large degree; we then have $M_x = 0$, whence $\mathcal{A}(M) = 0$. Now let M be a graded S-module satisfying the condition (TF); there is a homogeneous submodule N of M, of finite type, such that $M/N \in \mathcal{C}$; on applying the preceeding, and Proposition 3, one sees that $\mathcal{A}(N) \to \mathcal{A}(M)$ is bijective, and it suffices to prove that $\mathcal{A}(N)$ is coherent. Since N is of finite type, there is an exact sequence $L^1 \to L^0 \to N \to 0$, where L^0 and L^1 are *free* modules of finite type. From Proposition 3, the sequence $\mathcal{A}(L^1) \to \mathcal{A}(L^0) \to \mathcal{A}(N) \to 0$ is exact. But, from the corollary to Proposition 4, $\mathcal{A}(L^0)$ and $\mathcal{A}(L^1)$ are isomorphic to finite direct sums of sheafs $\mathcal{O}(n_i)$, so they are coherent. It follows that $\mathcal{A}(N)$ is coherent.

Finally let M be a graded S-module verifying (TF), and such that $\mathcal{A}(M) = 0$; in view of the preceeding, we may suppose that M is of finite type.

In detail, (TF) means that there is a homogeneous submodule $N \subset M$ of finite type such that $M/N \in \mathcal{C}$. We have $\mathcal{A}(N) = 0$ because $0 \to \mathcal{A}(N) \to \mathcal{A}(M)$ is exact, and if $N \in \mathcal{C}$, then $M \in \mathcal{C}$, so it suffices to prove the assertion with N in place of M.

If m is a homogeneous element of M, let $\mathfrak{a}_m$ be the annihilator of m, which is to say the set of polynomials $Q \in S$ such that $Q \cdot m = 0$; it is clear that $\mathfrak{a}_m$ is a homogeneous ideal. In addition, the hypothesis, $M_x = 0$ for all $x \in X$ entails that the variety of zeros of $\mathfrak{a}_m$ in K^{r+1} is empty or $\{0\}$;

Concretely, for the image of m in M_x to be 0 means that there is some $Q \in \mathfrak{a}_m$ with $Q(x) \neq 0$.

the theorem of zeros of Hilbert then shows that all homogeneous polynomials of sufficiently large degree appear in $\mathfrak{a}_m$. Applying this to a finite system of generators of M, one immediately concludes that $M_p = 0$ for sufficiently large p, which achieves the demonstration.

On combining Propositions 3 and 5 we obtain:

Proposition 6. *Let M and M' be two graded S-modules satisfying (TF), and let $\varphi : M \to M'$ be a homomorphism from M to M'. In order for*

$$\mathcal{A}(\varphi) : \mathcal{A}(M) \to \mathcal{A}(M')$$

to be injective (resp. surjective, bijective), it is necessary and sufficient that φ be $\mathcal{C}$-injective (resp. $\mathcal{C}$-surjective, $\mathcal{C}$-bijective).

59. The graded S-module associated with an algebraic sheaf. Let $\mathcal{F}$ be an algebraic sheaf on X, and let:

$$\Gamma(\mathcal{F}) = \sum_{n \in \mathbb{Z}} \Gamma(\mathcal{F})_n, \quad \text{with} \quad \Gamma(\mathcal{F})_n = \Gamma(X, \mathcal{F}(n)).$$

The group $\Gamma(\mathcal{F})$ is a graded group; we will endow it with an S-module structure. Consider $s \in \Gamma(X, \mathcal{F}(q))$ and $P \in S_p$; we can identify P with a section of $\mathcal{O}(p)$ (cf. no. 54), so $P \otimes s$ is a section of $\mathcal{O}(p) \otimes \mathcal{F}(q) = \mathcal{F}(q)(p) = \mathcal{F}(p+q)$, where we are using the isomorphisms of no. 54; we have in this way defined a section of $\mathcal{F}(p+q)$ that we denote by $P \cdot s$ in place of $P \otimes s$. The function $(P, s) \to P \cdot s$ endows $\Gamma(\mathcal{F})$ with an S-module structure that is compatible with its gradation.

On can also define $P \cdot s$ by means of its components on the U_i; if the components of s are $s_i \in \Gamma(U_i, \mathcal{F})$, with $s_i = (t_j^q / t_i^q) \cdot s_j$ on $U_i \cap U_j$, then $(P \cdot s)_i = (P/t_i^p) \cdot s_i$, which makes sense because P/t_i^p is a regular function on U_i.

In order to compare the functors $\mathcal{A}(M)$ and $\Gamma(\mathcal{F})$ we define two canonical homomorphisms:

$$\alpha : M \to \Gamma(\mathcal{A}(M)) \quad \text{and} \quad \beta : \mathcal{A}(\Gamma(\mathcal{F})) \to \mathcal{F}.$$

Definition of α. Let M be a graded S-module, and let $m \in M_0$ be a homogeneous element of degree 0 of M. The element $m/1$ is a well defined element of M_x, and varies continuous with $x \in X$; thus m defines a section $\alpha(m)$ of $\mathcal{A}(M)$. If m is now homogeneous of degree n, m is homogeneous of degree 0 in $M(n)$ and therefore defines a section of $\mathcal{A}(M(n)) = \mathcal{A}(M)(n)$ (cf. Proposition 4). From the definition of $\alpha : M \to \Gamma(\mathcal{A}(M))$, it is immediate that it is a homomorphism.

Definition of β. Let $\mathcal{F}$ be an algebraic sheaf on X, and let s/Q be an element of $\Gamma(\mathcal{F})_x$, with $s \in \Gamma(X, \mathcal{F}(n))$, $Q \in S_n$, and $Q(x) \neq 0$. The function $1/Q$ is homogeneous of degree $-n$, and regular at x, so s/Q is a section of $\mathcal{O}(-n) \otimes \mathcal{F}(n) = \mathcal{F}$ in a neighborhood of x and defines an element of $\mathcal{F}_x$, which we denote by $\beta_x(s/Q)$, since it depends only on s/Q. We can equally define β_x in terms of the local representatives s_i of s: if $x \in U_i$, $\beta_x(s/Q) = (t_i^n/Q) \cdot s_i(x)$. The collection of homomorphisms β_x define the homomorphism $\beta : \mathcal{A}(\Gamma(\mathcal{F})) \to \mathcal{F}$.

The homomorphisms α and β are related by the following Propositions, which are proved by direct calculation:

Proposition 7. *Let M be a graded S-module. The compositions of the homomorphisms $\mathcal{A}(M) \to \mathcal{A}(\Gamma(\mathcal{A}(M))) \to \mathcal{A}(M)$ is the identity.*

(The first homomorphism is defined by $\alpha : M \to \Gamma(\mathcal{A}(M))$, and the second is β, applied to $\mathcal{F} = \mathcal{A}(M)$.)

> Why Serre thought the verification was not worth writing out will be apparent, but we do it anyway. The sheaf map $\mathcal{A}(M) \to \mathcal{A}(\Gamma(\mathcal{A}(M)))$ can be defined by specifying the maps $\mathcal{A}(M)_x \to \mathcal{A}(\Gamma(\mathcal{A}(M)))_x$. An element of $\mathcal{A}(M)_x$ has the form m/Q where $Q(x) \neq 0$ and m and Q have the same degree, say n. Then m defines a global section $\alpha(m) \in \mathcal{A}(M)(n)$, and $\alpha(m)/Q$ is an element of $\mathcal{A}(\Gamma(\mathcal{A}(M)))_x$. But clearly $\beta_x(\alpha(m)/Q) = m/Q$.

Proposition 8. *Let $\mathcal{F}$ be an algebraic sheaf on X. The composition of the homomorphisms $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{A}(\Gamma(\mathcal{F}))) \to \Gamma(\mathcal{F})$ is the identity.*

(The first homomorphism is α, applied to $M = \Gamma(\mathcal{F})$, while the second is defined by $\beta : \mathcal{A}(\Gamma(\mathcal{F})) \to \mathcal{F}$.)

We will show in no. 65 that $\beta : \mathcal{A}(\Gamma(\mathcal{F})) \to \mathcal{F}$ is bijective is $\mathcal{F}$ is coherent, and that $\alpha : M \to \Gamma(\mathcal{A}(M))$ is $\mathcal{C}$-bijective if M satisfies condition (TF).

60. The case of coherent algebraic sheaves. We first establish a preliminary result:

Proposition 9. *Let $\mathcal{L}$ be an algebraic sheaf on X that is a finite sum of sheafs $\mathcal{O}(n_i)$. Then $\Gamma(\mathcal{L})$ satisfies (TF), and $\beta : \mathcal{A}(\Gamma(\mathcal{L})) \to \mathcal{L}$ is bijective.*

This reduces right away to the case $\mathcal{L} = \mathcal{O}(n)$, then to $\mathcal{L} = \mathcal{O}$. In this case, we know that $\Gamma(\mathcal{O}(p)) = S_p$ for all $p \geq 0$, so we have $S \subset \Gamma(\mathcal{O})$, with the quotient module belonging to $\mathcal{C}$. It now follows that $\Gamma(\mathcal{O})$ satisfies (TF), and then that $\mathcal{A}(\Gamma(\mathcal{O})) = \mathcal{A}(S) = \mathcal{O}$, qed.

(We will see that $\Gamma(\mathcal{O}) = S$ if $r \geq 1$; in contrast, if $r = 0$, $\Gamma(\mathcal{O})$ is not an S-module of finite type.)

Theorem 2. *For any coherent algebraic sheaf $\mathcal{F}$ on X, there is a graded S-module M, satisfying (TF), such that $\mathcal{A}(M)$ is isomorphic to $\mathcal{F}$.*

From the corollary to Theorem 1, there is an exact sequence of algebraic sheaves:

$$\mathcal{L}^1 \xrightarrow{\varphi} \mathcal{L}^0 \to \mathcal{F} \to 0,$$

where $\mathcal{L}^1$ and $\mathcal{L}^0$ satisfy the hypotheses of the preceeding proposition. Let M be the cokernel of the homomorphism $\Gamma(\varphi) : \Gamma(\mathcal{L}^1) \to \Gamma(\mathcal{L}^0)$; from Proposition 9, M satisfies condition (TF). On applying the functor $\mathcal{A}$ to the exact sequence:

$$\Gamma(\mathcal{L}^1) \to \Gamma(\mathcal{L}^0) \to M \to 0,$$

we obtain an exact sequence:

$$\mathcal{A}(\Gamma(\mathcal{L}^1)) \to \mathcal{A}(\Gamma(\mathcal{L}^0)) \to \mathcal{A}(M) \to 0.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{A}(\Gamma(\mathcal{L}^1)) & \to & \mathcal{A}(\Gamma(\mathcal{L}^0)) & \to & \mathcal{A}(M) & \to & 0 \\
\beta\big\downarrow & & \beta\big\downarrow & & & & \\
\mathcal{L}^1 & \to & \mathcal{L}^0 & \to & \mathcal{F} & \to & 0.
\end{array}$$

From Proposition 9,the two vertical homomorphisms are bijective. It follows that $\mathcal{A}(M)$ is isomorphic to $\mathcal{F}$, qed.

We can define the image of $a \in \mathcal{A}(M)$ in $\mathcal{F}$ to be the image of $\beta(\lambda)$ where $\lambda \in \mathcal{L}^0$ maps to a. If λ' also maps to a, then exactness implies that $\lambda' - \lambda$ has a preimage in $\mathcal{A}(\Gamma(\mathcal{L}^1))$, so $\beta(\lambda' - \lambda)$ is in the image of $\mathcal{L}^1$ and thus in the kernel of $\mathcal{L}^0 \to \mathcal{F}$. Therefore the map $\mathcal{A}(M) \to \mathcal{F}$ is well defined, and (after adding another pair of zeros on the right) the five lemma implies that it is an isomorphism.

§3. Cohomology of projective space with values in a coherent algebraic sheaf

61. The complexes $C_k(M)$ and $C(M)$. We retain the notation of no. 51 and 56. In particular, I denotes the interval $\{0, 1, \dots, r\}$ and S is the graded algebra $K[t_0, \dots, t_r]$.

Let M be a graded S-module, and let k and q be two integers ≥ 0; we will define a group $C_k^q(M)$; an element of $C_k^q(M)$ is a function

$$(i_0, \dots, i_q) \to m\langle i_0 \cdots i_q\rangle$$

which assigns to each tuple $(i_0, \dots, i_q)$ of $q+1$ elements of I a homogeneous element of degree $k(q+1)$ of M, depending in an alternating fashion on $i_0, \dots, i_q$. In particular, we have $m\langle i_0 \cdots i_q\rangle = 0$ if two of the indices $i_0, \dots, i_q$ are equal. Addition in $C_k^q(M)$ is defined in the obvious manner, as is multiplication by an element $\lambda \in K$, and $C_k^q(M)$ is a *vector space over* K.

If m is an element of $C_k^q(M)$, define $dm \in C_k^{q+1}(M)$ by the formula:

$$(dm)\langle i_0, \dots, i_{q+1}\rangle = \sum_{j=0}^{q+1} (-1)^j t_{i_j}^k \cdot m\langle i_0 \cdots \hat{i}_j \cdots i_{q+1}\rangle.$$

A direct calculation verifies that $d \circ d = 0$; then, the direct sum $C_k(M) = \sum_{q=0}^{r} C_k^q(M)$, endowed with the coboundary operator d, is a *complex*, whose q^{th} cohomology group is denoted by $H_k^q(M)$.

(We point out, after [11], another interpretation of the elements of $C_k^q(M)$: introducing $r+1$ differential symbols $dx_0, \ldots, dx_r$, each $m \in C_k^q(M)$ corresponds to the "differential form" of degree $q+1$:

$$\omega_m = \sum_{i_0 < \cdots < i_q} m\langle i_0 \cdots i_q \rangle dx_{i_0} \wedge \cdots \wedge dx_{i_q}.$$

If we let $\alpha_k = \sum_{i=0}^{r} t_i^k \, dx_i$, we see that

$$\omega_{dm} = \alpha_k \wedge \omega_m,$$

so that the coboundary operator is exterior multiplication by the form α_k.)

If h is an integer $\geq k$, let $\rho_k^h : C_k^q(M) \to C_h^q(M)$ be the homomorphism defined by the formula:

$$\rho_k^h(m)\langle i_0 \cdots i_q \rangle = (t_{i_0} \cdots t_{i_q})^{h-k} m\langle i_0 \cdots i_q \rangle.$$

We have $\rho_k^h \circ d = d \circ \rho_k^h$, and $\rho_h^l \circ \rho_k^h = \rho_k^l$ if $k \leq h \leq l$. Therefore we can define the complex $C(M)$ to be the inductive limit of the system $(C_k(M), \rho_k^h)$ for $k \to +\infty$. The cohomology groups of this complex will be denoted by $H^q(M)$. Because cohomology commutes with inductive limits (cf. [6], Chap. V, Prop. 9.3*), we have:

$$H^q(M) = \lim_{k \to \infty} H_k^q(M).$$

The cited result is Proposition B3.1.

Each homomorphism $\varphi : M \to M'$ defines a homomorphism

$$\varphi : C_k(M) \to C_k(M')$$

by the formula: $\varphi(m)\langle i_0 \cdots i_q \rangle = \varphi(m\langle i_0 \cdots i_q \rangle)$, where, after passing to the limit, $\varphi : C(M) \to C(M')$; moreover, these homomorphisms commute with the coboundary operator, and consequently they define homomorphisms

$$\varphi : H_k^q(M) \to H_k^q(M') \quad \text{and} \quad \varphi : H^q(M) \to H^q(M').$$

Given an exact sequence $0 \to M \to M' \to M'' \to 0$, there is an exact sequence of complexes $0 \to C_k(M) \to C_k(M') \to C_k(M'') \to 0$, from which we obtain an exact sequence of cohomology:

$$\cdots \to H_k^q(M') \to H_k^q(M'') \to H_k^{q+1}(M) \to H_k^{q+1}(M') \to \cdots$$

The same results hold for $C(M)$ and the $H^q(M)$.

REMARK. Later (cf. no. 69) we will see that we can express the $H^q_k(M)$ using the Ext^q_S.

62. Calculation of $H^q_k(M)$ for certain modules M. Let M be a graded S-module, and let $m \in M$ be a homogeneous element of degree 0. The system $(t_i^k \cdot m)$ is a 0-cocycle of $C_k(M)$, that we denote by $\alpha^k(m)$, and that we identify with its cohomology class. In this way we obtain a K-linear homomorphism $\alpha^k : M_0 \to H^0_k(M)$; since $\alpha^h = \rho^h_k \circ \alpha^k$ if $h \geq k$, the α^k define by passage to the limit a homomorphism $\alpha : M_0 \to H^0(M)$.

We now introduce two notations:

If $(P_0, \ldots, P_h)$ are elements of S, we denote by $(P_0, \ldots, P_h)M$ the submodule of M consisting of the elements $\sum_{i=0}^h P_i \cdot m_i$, with $m_i \in M$; if the P_i are homogeneous, this submodule is homogeneous.

If P is an element of S, and N is a submodule of M, we denote by $N : P$ the submodule of M consisting of the elements $m \in M$ such that $P \cdot m \in N$; evidently $N : P \supset N$; if N and P are homogeneous, $N : P$ is homogeneous.

With these notations specified, we have:

PROPOSITION 1. *Let M be a graded S-module and k an integer ≥ 0. Suppose that, for each $i \in I$:*

$$(t_0^k, \ldots, t_{i-1}^k)M : t_i^k = (t_0^k, \ldots, t_{i-1}^k)M.$$

Then:

(a) $\alpha^k : M_0 \to H^0_k(M)$ is bijective (if $r \geq 1$).

(b) $H^q_k(M) = 0$ for $0 < q < r$.

(For $i = 0$, the hypothesis means that $t_0^k \cdot m = 0$ implies that $m = 0$.)

This Proposition is a particular case of a result due to de Rham [11] (the result of de Rham holds also when we do not suppose that the $m\langle i_0 \cdots i_q \rangle$ are homogeneous). See also [6], Chap. VIII, §4, which treats a particular case that suffices for our applications.

The cited result is Theorem G4.4; if you haven't done so already, this would be a very good time to read chapter G. In that context there are symbols $y_1, \ldots, y_n$, and for each $k = 0, \ldots, n$ we let $E_k(y_1, \ldots, y_n)$ be the $\mathbb{Z}$-module generated by those $y_{i_1} \wedge \cdots \wedge y_{i_k}$ with $i_1 < \cdots < i_k$. Let R be a ring, let M be an R-module (this would be M_0 in the current context) and for each k let

$$X_k = M \otimes_{\mathbb{Z}} E_k(y_1, \ldots, y_n).$$

There is a linear form

$$\omega = x_1 y_1 + \cdots + x_n y_n \in R \otimes_{\mathbb{Z}} E_1(y_1, \ldots, y_n)$$

given by some $x_1, \ldots, x_n \in R$, which gives rise to a cochain complex

$$0 \to X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n \to 0$$

with coboundary operator $\alpha \mapsto \omega \wedge \alpha$. Theorem G4.4 asserts that if

$$(x_1, \dots, x_{\ell-1})M : x_\ell = (x_1, \dots, x_{\ell-1})M \qquad (*)$$

for all $\ell = 1, \dots, n$, then this complex is exact at $X_1, \dots, X_{n-1}$.

In comparing this result with the current context, we have $n = r + 1$ and $C_k^q = X_{q+1}$. One needs to verify that $(*)$ for the M in the current context implies this condition with M_0 in place of M. Then the result implies that the truncated sequence $0 \to X_1 \to \cdots \to X_{r+1} \to 0$ is exact at $X_2, \dots, X_r$, which is (b), and its cohomology at X_1 is the image of $X_0 \to X_1$, which is what (a) asserts.

We apply Proposition 1 to the graded S-module $S(n)$:

Proposition 2. *Let k be an integer ≥ 0, n an arbitrary integer. Then:*

(a) $\alpha^k : S_n \to H_k^0(S(n))$ is bijective (if $r \geq 1$).

(b) $H_k^q(S(n)) = 0$ for $0 < q < r$.

(c) $H_k^r(S(n))$ admits as a base (over K) the cohomology classes of the monomials $t_0^{\alpha_0} \cdots t_r^{\alpha_r}$, with $0 \leq \alpha_i < k$ and $\sum_{i=0}^r \alpha_i = k(r+1) + n$.

It is clear that the S-module $S(n)$ satisfies the hypotheses of Proposition 1, which establishes (a) and (b). For the other part, for any graded S-module M, we have $H_k^r(M) = M_{k(r+1)}/(t_0^k, \dots, t_r^k)M_{kr}$; thus the monomials

$$t_0^{\alpha_0} \cdots t_r^{\alpha_r},\ \alpha_i \geq 0,\ \sum_{i=0}^r \alpha_i = k(r+1) + n,$$

form a base of $S(n)_{k(r+1)}$, and those of these monomials for which at least one of the α_i is $\geq k$ form a base of $(t_0^k, \dots, t_r^k)S(n)_{kr}$; hence (c).

It is convenient to write the exponents α_i in the form $k - \beta_i$. The conditions stated in (c) are now written as:

$$0 < \beta_i \leq k \quad \text{and} \quad \sum_{i=0}^r \beta_i = -n.$$

The second condition, together with $\beta_i > 0$, implies that $\beta_i \leq -n - r$; therefore if $k \geq -n - r$, the condition $\beta_i \leq k$ is a consequence of the preceeding two. Therefore:

Corollary 1. *For $k \geq -n - r$, $H_k^r(S(n))$ admits a base of the cohomology classes of the monomials $(t_0 \cdots t_r)^k / t_0^{\beta_0} \cdots t_r^{\beta_r}$, with $\beta_i > 0$ and $\sum_{i=0}^r \beta_i = -n$.*

In the same way:

Corollary 2. *If $h \geq k \geq -n - r$, the homomorphism*

$$\rho_k^h : H_k^q(S(n)) \to H_k^q(S(n))$$

is bijective for all $q \geq 0$.

For $q \neq r$, this follows from assertions (a) and (b) of Proposition 2. For $q = r$, this follows from Corollary 1, taking into account that ρ_k^h transforms

$$(t_0 \cdots t_r)^k / t_0^{\beta_0} \cdots t_r^{\beta_r} \quad \text{to} \quad (t_0 \cdots t_r)^h / t_0^{\beta_0} \cdots t_r^{\beta_r}.$$

COROLLARY 3. *The homomorphism* $\alpha : S_n \to H^0(S(n))$ *is bijective if* $r \geq 1$, *or if* $n \geq 0$. *We have* $H^q(S(n)) = 0$ *for* $0 < q < r$, *and* $H^r(S(n))$ *is a vector space of dimension* $\binom{-n-1}{r}$ *over* K.

The assertion relative to α follows from Proposition 2, (a), in the case where $r \geq 1$; it is immediate if $r = 0$ and $n \geq 0$. The rest of the Corollary is evidently a consequence of Corollaries 1 and 2 (if we agree that the binomial coefficient $\binom{a}{r}$ is zero if $a < r$).

The possible $(\beta_0, \ldots, \beta_r)$ are in one-to-one correspondence with the r-element subsets $\{\beta_0, \beta_0 + \beta_1, \ldots, \beta_0 + \cdots + \beta_{r-1}\} \subset \{1, \ldots, -n-1\}$.

63. General properties of the $H^q(M)$.

PROPOSITION 3. *Let* M *be a graded* S*-module satisfying condition (TF). Then:*

- *(a) There is an integer* $k(M)$ *such that* $\rho_k^h : H_k^q(M) \to H_h^q(M)$ *is bijective for* $h \geq k \geq k(M)$ *and any* q.
- *(b)* $H^q(M)$ *is a finite dimensional vector space over* K *for all* $q \geq 0$.
- *(c) There is an integer* $n(M)$ *such that, for* $n \geq n(M)$, $\alpha : M_n \to H^0(M(n))$ *is bijective, and* $H^q(M(n))$ *vanishes for all* $q > 0$.

We reduce right away to the case where M is of finite type. We then say that M is of *dimension* $\leq s$ (where s is an integer ≥ 0) if there is an exact sequence:

$$0 \to L^s \to L^{s-1} \to \cdots \to L^0 \to M \to 0,$$

where the L^i are free graded S-modules of finite type. From Hilbert's syzygies theorem (cf. [6], Chap. VIII, th. 6.5), that dimension is always $\leq r + 1$.

The cited result is Theorem I3.4.

We prove the Proposition by induction on the dimension of M. If it is 0, M is free of finite type, i.e. a direct sum of modules $S(n_i)$, and the Proposition follows from Corollaries 2 and 3 of Proposition 2. Now suppose that M has dimension $\leq s$, and let N be the kernel of $L^0 \to M$. The graded S-module N has dimension $\leq s - 1$, and there is an exact sequence

$$0 \to N \to L^0 \to M \to 0.$$

In view of the induction hypothesis, the Proposition holds for N and L^0. On applying the five lemma ([7], Chap. I, Lemma 4.3) to the commutative diagram:

$$\begin{array}{ccccccccc} H^q_k(N) & \to & H^q_k(L^0) & \to & H^q_k(M) & \to & H^{q+1}_k(N) & \to & H^{q+1}_k(L^0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^q_h(N) & \to & H^q_h(L^0) & \to & H^q_h(M) & \to & H^{q+1}_h(N) & \to & H^{q+1}_h(L^0), \end{array}$$

where $h \geq k \geq \mathrm{Sup}(k(N), k(L^0))$, we demonstrate (a), after which (b) is evident, since $H^q_k(M)$ is finite dimensional over K. On the other hand, the exact sequence

$$H^q(L^0(n)) \to H^q(M(n)) \to H^{q+1}(N(n))$$

shows that $H^0(M(n)) = 0$ for $n \geq \mathrm{Sup}(n(L^0), n(N))$. Finally consider the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & N_n \arrow[r] \arrow[d, "\alpha"'] & L_n \arrow[r] \arrow[d, "\alpha"'] & M_n \arrow[r] \arrow[d, "\alpha"'] & 0 \arrow[d] \\ 0 \arrow[r] & H^0(N(n)) \arrow[r] & H^0(L^0(n)) \arrow[r] & H^0(M(n)) \arrow[r] & H^1(N(n)); \end{tikzcd}$$

for $n \geq n(N)$, we have $H^1(N(n)) = 0$; from this we deduce that $\alpha : M_n \to H^0(M(n))$ is bijective for $n \geq \mathrm{Sup}(n(L^0), n(N))$, which completes the proof of the Proposition.

64. Comparison of the groups $H^q(M)$ and $H^q(X, \mathcal{A}(M))$. Let M be a graded S-module, and let $\mathcal{A}(M)$ be the algebraic sheaf on $X = \mathbf{P}_r(K)$ derived from M by the process of no. 57. We will compare $C(M)$ with $C'(\mathfrak{U}, \mathcal{A}(M))$, the complex of alternating cochains of the covering $\mathfrak{U} = \{U_i\}_{i \in I}$ with values in the sheaf $\mathcal{A}(M)$.

Fix $m \in C^q_k(M)$, and let $(i_0, \ldots, i_q)$ be a sequence of $q+1$ elements of I. The polynomial $(t_{i_0} \cdots t_{i_q})^k$ is apparently contained in $S(U_{i_0 \cdots i_q})$, with the notations of no. 57. Consequently $m\langle i_0 \cdots i_q \rangle / (t_{i_0} \cdots t_{i_q})^k$ is contained in M_U, where $U = U_{i_0 \cdots i_q}$, so it defines a section of $\mathcal{A}(M)$ above $U_{i_0 \cdots i_q}$. When $(i_0, \ldots, i_q)$ varies, the system formed by these sections is an alternating q-cochain of $\mathfrak{U}$, with values in $\mathcal{A}(M)$, that we denote by $\iota_k(m)$. We see right away that ι_k commutes with d, and that $\iota_k = \iota_h \circ \rho^h_k$ if $h \geq k$. By passage to the inductive limit, the ι_k define a homomorphism $\iota : C(M) \to C'(\mathfrak{U}, \mathcal{A}(M))$, that commutes with d.

Proposition 4. *If M satisfies (TF), $\iota : C(M) \to C'(\mathfrak{U}, \mathcal{A}(M))$ is bijective.*

If $M \in \mathcal{C}$, we have $M_n = 0$ for $n \geq n_0$, whence $C_k(M) = 0$ for $k \geq n_0$, and $C(M) = 0$. Since every S-module satisfying (TF) is $\mathcal{C}$-isomorphic to a module of finite type, this shows that the issue reduces to the case where M

is of finite type. One can then find an exact sequence $L^1 \to L^0 \to M \to 0$, where L^1 and L^0 are free of finite type. From Propositions 3 and 5 of no. 58, the sequence

$$\mathcal{A}(L^1) \to \mathcal{A}(L^0) \to \mathcal{A}(M) \to 0$$

is an exact sequence of coherent algebraic sheaves; since the $U_{i_0 \cdots i_q}$ are open affines, the sequence

$$C'(\mathfrak{U}, \mathcal{A}(L^1)) \to C'(\mathfrak{U}, \mathcal{A}(L^0)) \to C'(\mathfrak{U}, \mathcal{A}(M)) \to 0$$

is an exact sequence (cf. no. 45, Corollary 2 to Theorem 2). The commutative diagram

$$\begin{tikzcd}
C(L^1) \arrow[r] \arrow[d, "\iota"'] & C(L^0) \arrow[r] \arrow[d, "\iota"'] & C(M) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
C'(\mathfrak{U}, \mathcal{A}(L^1)) \arrow[r] & C'(\mathfrak{U}, \mathcal{A}(L^0)) \arrow[r] & C'(\mathfrak{U}, \mathcal{A}(M)) \arrow[r] & 0
\end{tikzcd}$$

then shows that, if the Proposition is true for the modules L^1 and L^0, it is also for M. We have now reduced to the particular case of a free module of finite type, and then, by decomposition of a direct sum, to the case where $M = S(n)$.

In this case, we have $\mathcal{A}(S(n)) = \mathcal{O}(n)$; a section $f_{i_0 \cdots i_q}$ of $\mathcal{O}(n)$ on $U_{i_0 \cdots i_q}$ is, by the definition of a sheaf, a regular function on $V_{i_0} \cap \cdots \cap V_{i_q}$ and homogeneous of degree n. Since $V_{i_0} \cap \cdots \cap V_{i_q}$ is the set of points of K^{r+1} where the function $t_{i_0} \cdots t_{i_q}$ is $\neq 0$, there is an integer k such that

$$f_{i_0 \cdots i_q} = P\langle i_0 \cdots i_q \rangle / (t_{i_0} \cdots t_{i_q})^k,$$

$P\langle i_0 \cdots i_q \rangle$ being a homogeneous polynomial of degree $n + (k(q+1)$, which is to say of degree $k(n+1)$ in $S(n)$. Therefore each alternating $f \in C'(\mathfrak{U}, \mathcal{O}(n))$ defines a system $P\langle i_0 \cdots i_q \rangle$ that is an element of $C_k(S(n))$; thus there is a homomorphism

$$\nu : C'(\mathfrak{U}, \mathcal{O}(n)) \to C(S(n)).$$

Since one verifies right away that $\iota \circ \nu = 1$ and $\nu \circ \iota = 1$, it follows that ι is bijective, which completes the proof.

Corollary. *ι defines an isomorphism from $H^q(M)$ to $H^q(X, \mathcal{A}(M))$ for all $q \geq 0$.*

In effect, we know that $H'^q(\mathfrak{U}, \mathcal{A}(M)) = H^q(\mathfrak{U}, \mathcal{A}(M))$ (no. 20, Proposition 2), and that $H^q(\mathfrak{U}, \mathcal{A}(M)) = H^q(X, \mathcal{A}(M))$ (no. 52, Proposition 2, which is applicable because $\mathcal{A}(M)$ is coherent).

Remark. It is easy to see that $\iota : C(M) \to C'(\mathfrak{U}, \mathcal{A}(M))$ is *injective*, even when M does not satisfy (TF).

65. Applications.

PROPOSITION 5. *If M is a graded S-module satisfying (TF), the homomorphism $\alpha : M \to \Gamma(X, \mathcal{A}(M))$, defined in no. 59, is $\mathcal{C}$-bijective.*

We need to show that $\alpha : M_n \to \Gamma(X, \mathcal{A}(M(n)))$ is bijective for sufficiently large n. Now, from Proposition 4, $\Gamma(X, \mathcal{A}(M(n)))$ can be identified with $H^0(M(n))$; thus the Proposition follows from (c) of Proposition 3, taking into account that the homomorphism α is transformed by the preceeding identification to the homomorphism defined at the beginning of no. 62, and also denoted by α.

PROPOSITION 6. *Let $\mathcal{F}$ be a coherent algebraic sheaf on X. The graded S-module $\Gamma(\mathcal{F})$ satisfies (TF), and the homomorphism $\beta : \mathcal{A}(\Gamma(\mathcal{F})) \to \mathcal{F}$, defined in no. 59, is bijective.*

From Theorem 2 of no. 60, we can suppose that $\mathcal{F} = \mathcal{A}(M)$, where M is a module satisfying (TF). From the preceeding Proposition, $\alpha : M \to \Gamma(\mathcal{A}(M))$ is $\mathcal{C}$-bijective; since M satisfies (TF), it follows that $\Gamma(\mathcal{A}(M))$ satisfies it as well. Applying Proposition 6 of no. 58, we see that $\alpha : \mathcal{A}(M) \to \mathcal{A}(\Gamma(\mathcal{A}(M)))$ is bijective. Since the composition $\mathcal{A}(M) \xrightarrow{\alpha} \mathcal{A}(\Gamma(\mathcal{A}(M))) \xrightarrow{\beta} \mathcal{A}(M)$ is the identity (no. 59, Proposition 7), it follows that β is bijective, qed.

PROPOSITION 7. *Let $\mathcal{F}$ be a coherent algebraic sheaf on X. The groups $H^q(X, \mathcal{F})$ are finite dimensional vector spaces over K for all $q \geq 0$, and we have $H^q(X, \mathcal{F}(n)) = 0$ for $q > 0$ and n sufficiently large.*

One may suppose, as above, that $\mathcal{F} = \mathcal{A}(M)$, where M is a module satisfying (TF). The Proposition then follows from Proposition 3 and the corollary to Proposition 4.

PROPOSITION 8. *We have $H^q(X, \mathcal{O}(n)) = 0$ for $0 < q < r$, and $H^r(X, \mathcal{O}(n))$ is a vector space of dimension $\binom{-n-1}{r}$ on K, admitting a base of the cohomology classes of alternating cocycles of $\mathfrak{U}$*

$$f_{01\cdots r} = 1/t_0^{\beta_0} \cdots t_r^{\beta_r}, \quad \textit{with } \beta_i > 0 \textit{ and } \sum_{i=0}^{r} \beta_i = -n.$$

We have $\mathcal{O}(n) = \mathcal{A}(S(n))$, where $H^q(X, \mathcal{O}(n)) = H^q(S(n))$, from the corollary to Proposition 4; the Proposition follows immediately from this and the corollaries to Proposition 2.

In particular, note that $H^r(X, \mathcal{O}(-r-1))$ is a one dimensional vector space over K, admitting a base consisting of the cohomology class of the cocycle $f_{01\cdots r} = 1/t_0 \cdots t_r$.

66. Coherent algebraic sheaves on projective varieties. Let V be a closed subvariety of a projective space $X = \mathbf{P}_r(K)$, and let $\mathcal{F}$ be a coherent algebraic sheaf on V. Prolonging $\mathcal{F}$ by 0 outside of V, we obtain a coherent algebraic sheaf on X (cf. no. 39), denoted by $\mathcal{F}^X$; we know that $H^q(X, \mathcal{F}^X) = H^q(V, \mathcal{F})$. Therefore the results of the preceeding no. apply to the groups $H^q(V, \mathcal{F})$. In this way (taking into account no. 52) we obtain:

THEOREM 1. *The groups $H^q(V, \mathcal{F})$ are finite dimensional vector spaces over K, vanishing for $q > \dim V$.*

In particular, for $q = 0$, we have:

COROLLARY. *$\Gamma(V, \mathcal{F})$ is a finite dimensional vector space over K.*

(It is natural to conjecture that the theorem above is true for all varieties that are *complete variety complete*, in the sense of Weil [16].)

Let $U'_i = U_i \cap V$; the U'_i form an open cover $\mathfrak{U}'$ of V. If $\mathcal{F}$ is an algebraic sheaf on V, let $\mathcal{F}_i = \mathcal{F}(U'_i)$, and let $\theta_{ij}(n)$ be the isomorphism from $\mathcal{F}_j(U'_i \cap U'_j)$ to $\mathcal{F}_i(U'_i \cap U'_j)$ defined by multiplication by $(t_j/t_i)^n$. We denote by $\mathcal{F}(n)$ the sheaf obtained from the $\mathcal{F}_i$, by reconnecting by means of the $\theta_{ij}(n)$. The operation $\mathcal{F}(n)$ enjoys the same properties as the one defined in no. 54, which it generalizes; in particular $\mathcal{F}(n)$ is canonically isomorphic to $\mathcal{F} \otimes \mathcal{O}_V(n)$.

We have $\mathcal{F}^X(n) = \mathcal{F}(n)^X$. Applying Theorem 1 of no. 55, then Proposition 7 of no. 65, we obtain:

THEOREM 2. *Let $\mathcal{F}$ be a coherent algebraic sheaf on V. There is an integer $m(\mathcal{F})$ such that, for all $n \geq m(\mathcal{F})$:*

(a) For all $x \in V$, the $\mathcal{O}_{x,V}$-module $\mathcal{F}(n)_x$ is generated by the elements of $\Gamma(V, \mathcal{F}(n))$,

(b) $H^q(V, \mathcal{F}(n)) = 0$ for all $q > 0$.

REMARK. It is essential to observe that the sheaf $\mathcal{F}(n)$ *does not depend only on $\mathcal{F}$ and n*, but also on the *embedding* of V in the projective space X. More precisely, let P be the principal fiber space $\pi^{-1}(V)$, with structural group the group K^*; n being an integer, make K^* operate on K by the formula:

$$(\lambda, \mu) \to \lambda^{-n}\mu \quad \text{if} \quad \lambda \in K^* \quad \text{and} \quad \mu \in K.$$

Let $E^n = P \times_{K^*} K$ be the fiber space associated with P and fiber K, endowed with the proceeding operations; let $\mathcal{S}(E^n)$ be the sheaf of germs of sections of E^n (cf. 41). Taking account of the fact that the t_i/t_j form a system of changes of charts for P, one immediately verifies that $\mathcal{S}(E^n)$ is canonically isomorphic to $\mathcal{O}_V(n)$. The formula $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{S}(E^n)$ then shows that the operation depends only *on the class of the fiber space P defined by the embedding $V \to X$.* In particular, if V is normal, $\mathcal{F}(n)$ depends only on the linear equivalence class of V in this embedding (cf. [17]).

67. A complement. If M is a graded S-module satisfying (TF), we denote by $M^\natural$ the graded S-module $\Gamma(\mathcal{A}(M))$. In no. 65 we saw that $\alpha : M \to M^\natural$ is $\mathcal{C}$-bijective. We will give conditions under which α is bijective.

PROPOSITION 9. *In order for $\alpha : M \to M^\natural$ to be bijective, it is necessary and sufficient that the following conditions hold:*

(i) If $m \in M$ is such that $t_i \cdot m = 0$ for all $i \in I$, then $m = 0$.

(ii) *If the elements $m_i \in M$, homogeneous and of the same degree, satisfy the relation $t_j \cdot m_i - t_i \cdot m_j = 0$ for all pairs (i, j), then there is an $m \in M$ such that $m_i = t_i \cdot m$.*

We first show that (i) and (ii) are satisfied by $M^\natural$, which proves their necessity. For (i), we may suppose that m is homogeneous, which is to say a section of $\mathcal{A}(M(n))$; in this case, the condition $t_i \cdot m = 0$ implies that m vanishes on U_i, and, since this holds for all i, that $m = 0$. For (ii), let n be the degree of the m_i; then we have $m_i \in \Gamma(\mathcal{A}(M(n)))$; since $1/t_i$ is a section of $\mathcal{O}(-1)$ on U_i, m_i/t_i is a section of $\mathcal{A}(M(n-1))$ on U_i, and the condition $t_j \cdot m_i - t_i \cdot m_j = 0$ shows that the various sections are the restrictions of a unique section m of $\mathcal{A}(M(n-1))$ on X; we now have to compare the sections $t_i \cdot m$ and m_i; to show that they coincide on U_j, it suffices to observe that $t_j(t_i \cdot m - m_i) = 0$ on U_j, which follows from the formula $t_j \cdot m_i = t_i \cdot m_j$ and the definition of m.

We now show that (i) implies that α is injective. For n large enough, we know that $\alpha : M_n \to M_n^\natural$ is bijective, and we can reason by descent on n; if $\alpha(m) = 0$, with $m \in M_n$, we will have $t_i \cdot \alpha(m) = \alpha(t_i \cdot m) = 0$, and the hypothesis of descent, which is applicable because $t_i \cdot m \in M_{n+1}$, shows that $m = 0$. Next, we show that (i) and (ii) imply that α is surjective. For the reasons given above, we can reason by descent on n. If $m' \in M_n^\natural$, the hypothesis of descent implies that there is a $m_i \in M_{n+1}$ such that $\alpha(m_i) = t_i \cdot m'$; we have $\alpha(t_j \cdot m_i - t_i \cdot m_j) = 0$, so that $t_j \cdot m_i - t_i \cdot m_j = 0$ because α is injective. Condition (ii) then implies the existence of $m \in M_n$ such that $t_i \cdot m = m_i$; we have $t_i(m' - \alpha(m)) = 0$, which shows that $m' = \alpha(m)$, completing the proof.

Remarks. (1) The proof shows that condition (i) is necessary and sufficient for α to be injective.

(2) We can express (i) and (ii) as follows: the homomorphism $\alpha^1 : M_n \to H_1^0(M(n))$ is bijective for all $n \in \mathbb{Z}$. Moreover, Proposition 4 implies that we may identify $M^\natural$ with the S-module $\sum_{n\in\mathbb{Z}} H^0(M(n))$, and it is easy to derive from this a purely algebraic proof of Proposition 9 (without using the sheaf $\mathcal{A}(M)$).

§4. Relation with the functors Ext_S^q

68. The functors Ext_S^q. We retain the notation of no. 56. If M and N are two graded S-modules, we denote by $\mathrm{Hom}_S(M, N)_n$ the group of S-homomorphisms from M to N that are homogeneous of degree n, and by $\mathrm{Hom}_S(M, N)$ the graded group $\sum_{n\in\mathbb{Z}} \mathrm{Hom}_S(M, N)_n$; this is a graded S-module; when M is of finite type it coincides with the S-module of all the S-homomorphisms from M to N.

The derived functors (cf. [6], Chap. V) of the functor $\mathrm{Hom}_S(M, N)$ are the functors $\mathrm{Ext}^q_S(M, N)$, $q = 0, 1, \ldots$. We briefly recall the definition:[7]

We choose a "resolution" of M, which is to say an exact sequence:

$$\cdots \to L^{q+1} \to L^q \to \cdots \to L^0 \to M \to 0,$$

where the L^q are free graded S-modules, and the functions are homomorphisms (that is, as usual, S-module homomorphisms that are homogeneous of degree 0). If we let $C^q = \mathrm{Hom}_S(L^q, N)$, the homomorphism $L^{q+1} \to L^q$ defines a homomorphism $d : C^q \to C^{q+1}$, "by transposition," that satisfies $d \circ d = 0$; in this way $C = \sum_{q \geq 0} C^q$ is found to be endowed with the structure of a complex, and the q^{th} cohomology group is, by definition, $\mathrm{Ext}^q_S(M, N)$; one can show that $\mathrm{Ext}^q_S(M, N)$ depends only on M and N, and not on the chosen resolutions. Since the C^q are graded S-modules, and $d : C^q \to C^{q+1}$ is homogeneous of degree 0, the S-modules $\mathrm{Ext}^q_S(M, N)$ are graded by subspaces $\mathrm{Ext}^q_S(M, N)_n$; the $\mathrm{Ext}^q_S(M, N)_n$ are the cohomology groups of the complex consisting of the $\mathrm{Hom}_S(M, N)_n$, which is to say the derived functors of the functor $\mathrm{Hom}_S(M, N)_n$.

Recall the principal properties of the Ext^q_S:

$\mathrm{Ext}^0_S(M, N) = \mathrm{Hom}_S(M, N)$; $\mathrm{Ext}^q_S(M, N) = 0$ for $q > r+1$ if M is of finite type (due to the theorem of syzygies of Hilbert, cf. [6], Chap. VIII, th. 6.5);

> The Hilbert syzygy theorem is Theorem I3.4. It implies that $\dim_S M \leq \dim S = \dim_S K = r + 1$, so there is a projective resolution $\cdots \to L^1 \to L^0 \to M \to 0$ with $L^{r+2} = 0$, and since this can be used to define the $\mathrm{Ext}^q_S(M, N)$, the claim follows. Unless I am mistaken, the assertion does not depend on M being of finite type; possibly Serre is concerned about the ambiguity of definition discussed in footnote 5.

$\mathrm{Ext}^q_S(M, N)$ is an S-module of finite type if M and N are of finite type (because we can choose a resolution where the L^q are of finite type);

> This follows from Lemma I2.2.

for all $n \in \mathbb{Z}$ there are two canonical homomorphisms:

$$\mathrm{Ext}^q_S(M(n), N) \approx \mathrm{Ext}^q_S(M, N(-n)) \approx \mathrm{Ext}^q_S(M, N)(-n).$$

> A resolution $\cdots \to L^1 \to L^0 \to M \to 0$ induces a resolution $\cdots \to L^1(n) \to L^0(n) \to M(n) \to 0$, after which this is a simple matter of applying the definitions, taking account of the gradings.

[7] When M is not a module of finite type, the $\mathrm{Ext}^q_S(M, N)$ defined below may differ from the $\mathrm{Ext}^q_S(M, N)$ defined in [6]; this is because the $\mathrm{Hom}_S(M, N)$ does not have the same sense in the two cases. Nevertheless, all the proofs of [6] are valid without change in the case considered here: all claims follows directly or from the chapter of [6].

The exact sequences

$$0 \to N \to N' \to N'' \to 0 \quad \text{and} \quad 0 \to M \to M' \to M'' \to 0$$

give rise to exact sequences:

$$\cdots \to \mathrm{Ext}^q_S(M, N) \to \mathrm{Ext}^q_S(M, N') \to \mathrm{Ext}^q_S(M, N'') \to \mathrm{Ext}^{q+1}_S(M, N) \to \cdots$$

$$\cdots \to \mathrm{Ext}^q_S(M'', N) \to \mathrm{Ext}^q_S(M', N) \to \mathrm{Ext}^q_S(M, N) \to \mathrm{Ext}^{q+1}_S(M'', N) \to \cdots$$

As with much of this paragraph, this comes from Theorem D3.2.

69. Interpretation of the $H^q_k(M)$ in terms of the Ext^q_S. Let M be a graded S-module, and let k be an integer ≥ 0. Put

$$B^q_k(M) = \sum_{n \in \mathbb{Z}} H^q_k(M(n)),$$

with the notations of no. 61.

In this way we obtain a graded group, isomorphic to the q^{th} cohomology group of the complex $\sum_{n \in \mathbb{Z}} C_k(M(n))$; this complex can be endowed with an S-module structure compatible with its grading by setting

$$(P \cdot m)\langle i_0 \cdots i_q \rangle = P \cdot m\langle i_0 \cdot i_q \rangle, \quad \text{if } P \in S_p, \text{ and } m\langle i_0 \cdot i_q \rangle \in C^q_k(M(n));$$

since the coboundary operator is an S-homomorphism that is homogeneous of degree 0, it follows that the $B^q_k(M)$ are themselves graded S-modules.

We set

$$B^q(M) = \lim_{k \to \infty} B^q_k(M) = \sum_{n \in \mathbb{Z}} H^q(M(n)).$$

The $B^q(M)$ are graded S-modules. For $q = 0$, we have

$$B^0(M) = \sum_{n \in \mathbb{Z}} H^0(M(n)),$$

and in this way we recover the module denoted $M^\natural$ in no. 67 (when M satisfies condition (TF)). For each $n \in \mathbb{Z}$, in no. 62 we defined a linear transformation $\alpha : M_n \to H^0(M(n))$; one verifies immediately that the sum of these transformations defines a homomorphism, that we also denote by α, from M to $B^0(M)$.

Proposition 1. *Let k be an integer ≥ 0, and let J_k be the ideal $(t_0^k, \ldots, t_r^k)$ of S. For each graded S-module M, the graded S-modules $B^q_k(M)$ and $\mathrm{Ext}^q_S(J_k, M)$ are isomorphic.*

Let L_k^q, $q = 0, \dots, r$, be the graded S-module admitting as a base the elements $e\langle i_0 \cdots i_q \rangle$, $0 \le i_0 < i_1 < \cdots < i_q \le r$, of degree $k(q+1)$; we define an operator $d : L_k^{q+1} \to L_k^q$ and an operator $\varepsilon : L_k^0 \to J_k$ by the formulas:

$$d(e\langle i_0 \cdots i_{q+1}\rangle) = \sum_{j=0}^{q+1} (-1)^j t_{i_j}^k \cdot e\langle i_0 \cdots \hat{i}_j \cdots i_{q+1}\rangle.$$

$$\varepsilon(e\langle i \rangle) = t_i^k.$$

Lemma 1. *The sequence of homomorphisms:*

$$0 \to L_k^r \xrightarrow{d} L_k^{r-1} \to \cdots \to L_k^0 \xrightarrow{\varepsilon} J_k \to 0$$

is an exact sequence.

For $k = 1$ this result is well known (cf. [6], Chap. VIII, §4); the general case is proved in the same way (it reduces to the special case); one can also use the theorem proved in [11].

For us this is the exactness of the complex $Y^{(n)}$ in Proposition G6.1.

Proposition 1 follows immediately from the Lemma, if one notices that the complex formed by the $\mathrm{Hom}_S(L_k^q, M)$ and the transpose of d is none other than the complex $\sum_{n \in \mathbb{Z}} C_k(M(n))$.

Corollary 1. *$H_k^q(M)$ is isomorphic to $\mathrm{Ext}_S^q(J_k, M)_0$.*

In effect these two groups are the components of degree 0 of the graded groups $B_k^q(M)$ and $\mathrm{Ext}_S^q(J_k, M)$.

Corollary 2. *$H^q(M)$ is isomorphic to $\lim_{k\to\infty} \mathrm{Ext}_S^q(J_k, M)_0$.*

It is easy to see that the homomorphism $\rho_k^h : H_k^q(M) \to H_k^q(M)$ of no. 61 is transformed by the isomorphism of Corollary 1 to the homomorphism from

$$\mathrm{Ext}_S^q(J_k, M)_0 \quad \text{to} \quad \mathrm{Ext}_S^q(J_h, M)_0$$

defined by the inclusion $J_h \to J_k$; this gives Corollary 2.

Remark. Let M be a graded S-module of finite type; M defines (cf. no. 48) a coherent algebraic sheaf $\mathcal{F}'$ on K^{r+1}, and thus on $Y = K^{r+1} \setminus \{0\}$, and one can verify that $H^q(Y, \mathcal{F})$ is isomorphic to $B^q(M)$.

70. Definition of the functors $T^q(M)$. We first define the notion of the *dual* module of a graded S-module. Let M be a graded S-module; for all $n \in \mathbb{Z}$, M_n is vector space over K, whose vectorial dual space we denote by $(M_n)'$. Let

$$M^* = \sum_{n \in \mathbb{Z}} M_n^* \quad \text{with} \quad M_n^* = (M_{-n})'.$$

We will endow M^* with an S-module structure that is compatible with its grading; for all $P \in S_p$, the function $m \to P \cdot m$ is a K-linear function from M_{-n-p} to M_{-n}, which defines by transposition a K-linear function from $(M_{-n})' = M_n^*$ to $(M_{-n-p})' = M_{n+p}^*$; this defines the S-module structure of M^*. Equivalently, we could have defined M^* to be $\mathrm{Hom}_S(M, K)$ after identifying K with the graded S-module $S/(t_0, \dots, t_r)$.

The graded S-module M^* is called the *dual* of M; we have $M^{**} = M$ if each of the M_n is finite dimensional over K, which is the case if $M = \Gamma(\mathcal{F})$, $\mathcal{F}$ being a coherent sheaf on X, or if M is of finite type. Each homomorphism defines by transposition a homomorphism from N^* to M^*. If the sequence $M \to N \to P$ is exact, the sequence $P^* \to N^* \to M^*$ is also exact; which is to say that $M \to M^*$ is a *contravariant* functor, and is *exact*. When I is a homogeneous ideal of S, the dual of S/I is none other than the "inverse system" of I, in the sense of Macaulay (cf. [9], no. 25).

Now let M be a graded S-module, and q an integer ≥ 0. In the preceeding no. we defined the graded S-module $B^q(M)$; *the dual module of $B^q(M)$ will be denoted by $T^q(M)$*. Thus we have, by definition:

$$T^q(M) = \sum_{n \in \mathbb{Z}} T^q(M)_n, \quad \text{with} \quad T^q(M)_n = (H_q(M(-n)))'.$$

Each homomorphism $\varphi : M \to N$ defines a homomorphism from $B^q(M)$ to $B^q(N)$, from which we obtain a homomorphism from $T^q(N)$ to $T^q(M)$; therefore the $T^q(M)$ are *contravariant* functors of M (we will see in no. 72 that they can be expressed very simply as functions of the Ext_S). Each exact sequence:

$$0 \to M \to N \to P \to 0$$

gives rise to an exact sequence:

$$\cdots \to B^q(M) \to B^q(N) \to B^q(P) \to B^{q+1}(M) \to \cdots,$$

from which we obtain the exact sequence

$$\cdots \to T^{q+1}(M) \to T^q(P) \to T^q(N) \to T^q(M) \to \cdots.$$

The homomorphism $\alpha : M \to B^0(M)$ defines by transposition a homomorphism $\alpha^* : T^0(M) \to M^*$.

Since $B^q(M) = 0$ for $q > r$, we have $T^q(M) = 0$ for $q > r$.

71. Determination of $T^r(M)$. (In this no., and in the next, we suppose that $r \geq 1$; the case $r = 0$ leads to conditions that are a bit different, and also trivial.)

The graded S-module $S(-r-1)$ is denoted by Ω; it is a free module, admitting for base an element of degree $r+1$. We saw in no. 62 that $H^r(\Omega) = H^r_k(\Omega)$ for sufficiently large k, and that $H^r_k(\Omega)$ admits a base over K consisting of the unique element $(t_0 \cdots t_r)^k / t_0 \cdots t_r$; the image in $H^r(\Omega)$ of that element will be denoted by ξ; ξ constitutes a base of $H^r(\Omega)$.

We will now define a scalar product $\langle h, \varphi\rangle$ between elements $h \in B^r(M)_{-n}$ and $\varphi \in \mathrm{Hom}_S(M, \Omega)_n$, with M being an arbitrary graded S-module. The element φ can be identified with an element of $\mathrm{Hom}_S(M(-n), \Omega)_0$, which is to say a homomorphism from $M(-n)$ to Ω; therefore it defines, by passage to cohomology groups, a homomorphism from $H^r(M(-n)) = B^r(M)_{-n}$ to $H^r(\Omega)$, that we also denote by φ. The image of h under this homomorphism is then a scalar multiple of ξ, and we define $\langle h, \varphi\rangle$ by the formula:

$$\varphi(h) = \langle h, \varphi\rangle \xi.$$

For all $\varphi \in \mathrm{Hom}_S(M, \Omega)_n$, the function $h \to \langle h, \varphi\rangle$ is a linear form on $B^r(M)_{-n}$, so it may be identified with an element $\nu(\varphi)$ of the dual of $B^r(M)_{-n}$, which is none other than $T^r(M)$. We have defined in this way a homogeneous function of degree 0

$$\mu : \mathrm{Hom}_S(M, \Omega) \to T^r(M),$$

and the formula $\langle P \cdot h, \varphi\rangle = \langle h, P \cdot \varphi\rangle$ shows that ν is an S-homomorphism.

PROPOSITION 2. *The homomorphism* $\nu : \mathrm{Hom}_S(M, \Omega) \to T^r(M)$ *is bijective.*

We first establish the proposition when M is a *free* module. If M is the direct sum of homogeneous submodules M^α, we have:

$$\mathrm{Hom}_S(M, \Omega)_n = \prod_\alpha \mathrm{Hom}_S(M^\alpha, \Omega)_n \quad \text{and} \quad T^r(M)_n = \prod_\alpha T^r(M^\alpha)_n.$$

Therefore, if the proposition is true for the M^α, then it is also for M, and this reduces the case of a free module to the particular case of a free module with a single generator, which is to say the case where $M = S(m)$. One can then identify $\mathrm{Hom}_S(M, \Omega)_n$ with $\mathrm{Hom}_S(S, S(n-m-r-1))_0$, which is the vector space of the homogeneous polynomials of degree $n-m-r-1$. Therefore $\mathrm{Hom}_S(M, \Omega)$ admits for a base the family of monomials $t_0^{\gamma_0} \cdots t_r^{\gamma_r}$, with $\gamma_i \geq 0$ and $\sum_{i=0}^r \gamma_i = n-m-r-1$. On the other hand, we saw in no. 62 that $\mathrm{Hom}_S^r(S(m-n))$ has a base (if k is sufficiently large) consisting of the family of monomials $(t_0 \cdots t_r)^k / t_0^{\beta_0} \cdots t_r^{\beta_r}$, with $\beta_i > 0$ and $\sum_{i=0}^r \beta_i = n-m$. By setting $\beta_i = \gamma_i' + 1$, one can write these monomials in the form $(t_0 \cdots t_r)^{k-1} / t_0^{\gamma_0'} \cdots t_r^{\gamma_r'}$, with $\gamma_i' \geq 0$ and $\sum_{i=0}^r \gamma_i' = n-m-r-1$. Recalling the definition of $\langle h, \varphi\rangle$, we observe that the scalar product:

$$\langle (t_0 \cdots t_r)^{k-1} / t_0^{\gamma_0'} \cdots t_r^{\gamma_r'}, t_0^{\gamma_0} \cdots t_r^{\gamma_r}\rangle$$

is always null, except when $\gamma_i = \gamma_i'$ for all i, in which case it is equal to 1. This means that ν transforms the base of the $t_0^{\gamma_0} \cdots t_r^{\gamma_r}$ into the dual base of the base of monomials $(t_0 \cdots t_r)^{k-1} / t_0^{\gamma_0} \cdots t_r^{\gamma_r}$, so it is bijective, which completes the proof of the Proposition in the case where M is free.

We now pass to the general case. Choose an exact sequence

$$L^1 \to L^0 \to M \to 0$$

where L^0 and L^1 are free. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_S(M,\Omega) & \to & \mathrm{Hom}_S(L^0,\Omega) & \to & \mathrm{Hom}_S(L^1,\Omega) \\
\nu\downarrow & & \nu\downarrow & & \nu\downarrow & & \nu\downarrow \\
0 & \longrightarrow & T^r(M) & \longrightarrow & T^r(L^0) & \longrightarrow & T^r(L^1).
\end{array}$$

The first line of this diagram is an exact sequence, from the general properties of the functor Hom_S; the second is also an exact sequence, because it is the dual of the sequence

$$B^r(L^1) \to B^r(L^0) \to B^r(M) \to 0,$$

which is exact, as a part of the exact sequence of cohomology of the B^q, and from the property $B^{r+1}(M) = 0$ of the graded S-module M. On the other hand the two vertical homomorphisms

$$\nu : \mathrm{Hom}_S(L^0,\Omega) \to T^r(L^0) \quad \text{and} \quad \mathrm{Hom}_S(L^1,\Omega) \to T^r(L^1)$$

are bijective, as we have already seen. It follows that

$$\mathrm{Hom}_S(M,\Omega) \to T^r(M)$$

is also bijective, which completes the proof.

Add a pair of zeros at the left end and apply the five lemma.

72. Determination of the $T^q(M)$. We will prove the following theorem, which generalizes Proposition 2.

THEOREM 1. *Let M be a graded S-module. For $q \neq r$, the graded S-modules $T^{r-q}(M)$ and $\mathrm{Ext}^q_S(M,\Omega)$ are isomorphic. Moreover, there is an exact sequence:*

$$0 \to \mathrm{Ext}^r_S(M,\Omega) \to T^0(M) \xrightarrow{\alpha^*} M^* \to \mathrm{Ext}^{r+1}_S(M,\Omega) \to 0.$$

We will use the axiomatic characterization of derived functors given in [6], Chap. III, §5. For that, we first define new functors as follows:

$$\begin{array}{ll}
\text{For } q \neq r, r+1, & E^q(M) = T^{r-q}(M) \\
\text{For } q = r, & E^r(M) = \mathrm{Ker}(\alpha^*) \\
\text{For } q = r+1, & E^{r+1}(M) = \mathrm{Coker}(\alpha^*).
\end{array}$$

The $E^q(M)$ are additive contravariant functors, which enjoy the following properties:

(i) $E^0(M)$ *is isomorphic to* $\mathrm{Hom}_S(M, \Omega)$.

This is what Proposition 2 asserts.

(ii) *If* L *is free,* $E^q(L) = 0$ *for all* $q > 0$.

It suffices to verify this for $L = S(n)$, and that case follows from no. 62.

(iii) *To every exact sequence* $0 \to M \to N \to P \to 0$ *is associated a sequence of coboundary operators* $d^q : E^q(M) \to E^{q+1}(P)$*, and the sequence:*

$$\cdots \to E^q(P) \to E^q(N) \to E^q(M) \xrightarrow{d^q} E^{q+1}(P) \to \cdots$$

is exact.

The definition of d^q is evident if $q \neq r-1$; it is the homomorphism from $T^{r-q}(M)$ to $T^{r-q-1}(P)$ defined in no. 70. For $q = r-1$ or r, we use the following commutative diagram:

$$\begin{array}{ccccccccc} T^1(M) & \longrightarrow & T^0(P) & \to & T^0(N) & \to & T^0(M) & \to & 0 \\ & & \downarrow{\scriptstyle\alpha^*} & & \downarrow{\scriptstyle\alpha^*} & & \downarrow{\scriptstyle\alpha^*} & & \downarrow{\scriptstyle\alpha^*} \\ 0 & \to & P^* & \to & N^* & \to & M^* & \to & 0. \end{array}$$

This diagram shows right away that the image of $T^1(M)$ is contained in the kernel of $\alpha^* : T^0(P) \to P^*$, which is none other than $E^r(P)$. Thus the definition of $d^{r-1} : E^{r-1}(M) \to E^r(P)$.

To define $d^r : \mathrm{Ker}(T^0(M) \to M^*) \to \mathrm{Coker}(T^0(P) \to P^*)$, we use the procedure of [6], Chap. III, Lemma 3.3: if $x \in \mathrm{Ker}(T^0(M) \to M^*)$, there is $y \in P^*$ and $z \in T^0(N)$ such that x is the image of z and y and z have the same image in N^*; we then set $d^r(x) = y$.

Chap. III, Lemma 3.3 of [6], and Lemma B1.3, are the snake lemma.

The exactness of the sequence

$$\cdots \to E^q(P) \to E^q(N) \to E^q(M) \xrightarrow{d^q} E^{q+1}(P) \to \cdots$$

follows from the exactness of the sequence

$$\cdots \to T^{r-q}(P) \to T^{r-q}(N) \to T^{r-q}(M) \to T^{r-q-1}(P) \to \cdots$$

and from [6], loc. cit.

(iv) *The isomorphism of* (i) *and the operators* d^q *of* (iii) *are "natural".*

This follows immediately from their definitions.

Since properties (i) to (iv) characterize the functors derived from the functor $\mathrm{Hom}_S(M, \Omega)$, we have $E^q(M) \approx \mathrm{Ext}^q_S(M, \Omega)$, which proves the Theorem.

The result in CE that Serre is appealing to is essentially Theorem C6.2, although the two axiom systems differs in some details. For this sort of result an author faces a tradeoff between making the axioms strong (useful properties are easily accessible) making them weak (strengthening the uniqueness assertion) or maintaining a clear distinction between the minimal collection of properties that the suffice for uniqueness and the others (correct in principle but often excessively fussy in practice). Suffice it to say that Serre would have no difficulty adjusting the details of his argument to the version of this result presented herein.

COROLLARY 1. *If M satisfies (TF), $H^q(M)$ is isomorphic to the dual vector space of $\mathrm{Ext}_S^{r-q}(M,\Omega)_0$ for all $q \geq 1$.*

In effect, we know that $H^q(M)$ is a finite dimensional vector space whose dual is isomorphic to $\mathrm{Ext}_S^{r-q}(M,\Omega)_0$.

COROLLARY 2. *If M satisfies (TF), the $T^q(M)$ are graded S-modules of finite type for $q \geq 1$, and $T^0(M)$ satisfies (TF).*

One can replace M with a module of finite type without changing the $B^q(M)$, hence also the $T^q(M)$. The $\mathrm{Ext}^{r-q}(M,\Omega)$ are consequently S-modules of finite type, and we have $M^* \in \mathcal{C}$, hence the Corollary.

§5. Applications to coherent algebraic sheaves

73. Relation between the functors Ext_S^q and $\mathrm{Ext}_{\mathcal{O}_x}^q$. Let M and N be two graded S-modules. If x is a point of $X = \mathbf{P}_r(K)$, we have defined in no. 57 the $\mathcal{O}_x$-modules M_x and N_x; we will display a relation between the $\mathrm{Ext}_{\mathcal{O}_x}^q(M_x,N_x)$ and the graded S-module $\mathrm{Ext}_S^q(M,N)$:

PROPOSITION 1. *Suppose that M is of finite type. Then:*

(a) The sheaf $\mathcal{A}(\mathrm{Hom}_S(M,N))$ is isomorphic to the sheaf $\mathrm{Hom}_{\mathcal{O}_x}(\mathcal{A}(M),\mathcal{A}(N))$.

(b) For all $x \in X$, the $\mathcal{O}_x$-module $\mathrm{Ext}_S^q(M,N)_x$ is isomorphic to the $\mathcal{O}_x$-module $\mathrm{Ext}_{\mathcal{O}_x}^q(M_x,N_x)$.

First we define a homomorphism $\iota_x : \mathrm{Hom}_S(M,N)_x \to \mathrm{Hom}_{\mathcal{O}_x}(M_x,N_x)$. An element of the first module is a fraction φ/P, with $\varphi \in \mathrm{Hom}_S(M,N)_n$, $P \in S(x)$, and P homogeneous of degree n; if m/P' is an element of M_x, $\varphi(m)/PP'$ is an element of N_x that depends only on φ/P and on m/P', and the function $m/P' \to \varphi(m)/PP'$ is a homomorphism $\iota_x(\varphi/P) : M_x \to N_x$; this defines ι_x. From Proposition 5 of no. 14, $\mathrm{Hom}_{\mathcal{O}_x}(M_x,N_x)$ can be identified with
$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{A}(M),\mathcal{A}(N))_x;$$
that identification transforms ι_x to
$$\iota_x : \mathcal{A}(\mathrm{Hom}_S(M,N))_x \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{A}(M),\mathcal{A}(N))_x,$$

and it is easy to verify that the collection of the ι_x is a homomorphism

$$\iota : \mathcal{A}(\mathrm{Hom}_S(M, N)) \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{A}(M), \mathcal{A}(N)).$$

When M is a free module of finite type, ι_x is bijective: in effect, it suffices to show this when $M = S(n)$, and this case is immediate.

If M is now a graded S-module of finite type (that is, M need no longer be free) choose a resolution of M:

$$\cdots \to L^{q+1} \to \cdots \to L^0 \to M \to 0,$$

where the L^q are free of finite type, and consider the complex C formed by the $\mathrm{Hom}_S(L^q, N)$. The cohomology groups of C are the $\mathrm{Ext}^q_S(M, N)$; in other words, if we denote by B^q and Z^q the submodules of C^q consisting respectively of the coboundaries and the cocycles, there are exact sequences:

$$0 \to Z^q \to C^q \to B^{q+1} \to 0,$$

and

$$0 \to B^q \to Z^q \to \mathrm{Ext}^q_S(M, N) \to 0.$$

Since the functor $\mathcal{A}(M)$ is exact, the sequences

$$0 \to Z^q_x \to C^q_x \to B^{q+1}_x \to 0$$

and

$$0 \to B^q_x \to Z^q_x \to \mathrm{Ext}^q_S(M, N)_x \to 0$$

are also exact.

But in view of what came before, C^q_x is isomorphic to $\mathrm{Hom}_{\mathcal{O}_x}(L^q_x, N_x)$; the $\mathrm{Ext}^q_S(M, N)_x$ are isomorphic to the cohomology groups of the complex formed by the $\mathrm{Hom}_{\mathcal{O}_x}(L^q_x, N_x)$, and, since the L^q_x are evidently $\mathcal{O}_x$-free, we recover the definition of the $\mathrm{Ext}^q_{\mathcal{O}_x}(M_x, N_x)$, which proves (b); for $q = 0$, the preceeding shows that ι_x is bijective, so it is an isomorphism, hence (a).

74. Vanishing of the cohomology groups $H^q(X, \mathcal{F}(-n))$ as $n \to \infty$.

Theorem 1. *Let $\mathcal{F}$ be a coherent algebraic sheaf on X, and let q be an integer ≥ 0. The following two conditions are equivalent:*

(a) $H^q(X, \mathcal{F}(-n)) = 0$ *for sufficiently large n.*

(b) $\mathrm{Ext}^{r-q}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{O}_x) = 0$ *for all $x \in X$.*

Part (a) has come to be known as Serre's vanishing theorem.

From Theorem 2 of no. 60, we may suppose that $\mathcal{F} = \mathcal{A}(M)$, where M is a graded S-module of finite type, and, from no. 64, $H^q(X, \mathcal{F}(-n))$ is isomorphic to $H^q(M(-n)) = B^q(M)_{-n}$; therefore condition (a) is equivalent to

$$T^q(M)_n = 0$$

for sufficiently large n, which is to say that $T^q(M) \in \mathcal{C}$. From Theorem 1 of no. 72 and the fact that $M^* \in \mathcal{C}$ when M is of finite type, this last condition is equivalent to $\mathrm{Ext}_S^{r-s}(M, \Omega) \in \mathcal{C}$; since $\mathrm{Ext}_S^{r-s}(M, \Omega)$ is an S-module of finite type,

$$\mathrm{Ext}_S^{r-s}(M, \Omega) \in \mathcal{C}$$

is equivalent to $\mathrm{Ext}_S^{r-s}(M, \Omega)_x = 0$ for all $x \in X$, from Proposition 5 of no. 58; at last, Proposition 1 shows that $\mathrm{Ext}_S^{r-s}(M, \Omega)_x = \mathrm{Ext}_{\mathcal{O}_x}^{r-s}(M_x, \Omega_x)$, and since M_x is isomorphic to $\mathcal{F}_x$, and Ω_x is isomorphic to $\mathcal{O}(-r-1)_x$, hence to $\mathcal{O}_x$, this implies the claim.

To state Theorem 2 we need the notion of the *dimension* of a $\mathcal{O}_x$-module. Recall ([6], Chap. VI, §2) that an $\mathcal{O}_x$-module of finite type P is said to be of dimension $\leq p$ if there is an exact sequence of $\mathcal{O}_x$-modules:

$$0 \to L_p \to L_{p-1} \to \cdots \to L_0 \to P \to 0,$$

where each L_p is free (this definition is equivalent to that of [6], loc. cit., from the fact that all projective $\mathcal{O}_x$-modules of finite type are free—cf. [6], Chap. VIII, Th. 6.1′). Each $\mathcal{O}_x$ module of finite type is of dimension $\leq r$, from the theorem of syzygies (cf. [6], Chap. VIII, Th. 6.2′).

Th. 6.1′ is Theorem D5.3, taking into account that a projective R-module is flat. (Proposition B7.4) Th. 6.2′ is part of Theorem I3.4.

LEMMA 1. *Let P be an $\mathcal{O}_x$-module of finite type, and let p be an integer ≥ 0. The following two conditions are equivalent:*

(i) *P is of dimension $\leq p$.*

(ii) $\mathrm{Ext}_{\mathcal{O}_x}^m(P, \mathcal{O}_x) = 0$ *for all $m > p$.*

It is clear that (i) implies (ii). We will show that (ii) implies (i) by descending recurrence on p; for $p \geq r$, the Lemma is trivial, since (i) always holds; passing now to $p+1$ and p, let N be any $\mathcal{O}_x$-module of finite type. We can find an exact sequence $0 \to R \to L \to N \to 0$, where L is free of finite type (since $\mathcal{O}_x$ is Noetherian). The exact sequence

$$\mathrm{Ext}_{\mathcal{O}_x}^{p+1}(P, L) \to \mathrm{Ext}_{\mathcal{O}_x}^{p+1}(P, N) \to \mathrm{Ext}_{\mathcal{O}_x}^{p+2}(P, R)$$

shows that $\mathrm{Ext}_{\mathcal{O}_x}^{p+1}(P, N) = 0$: in effect, we have $\mathrm{Ext}_{\mathcal{O}_x}^{p+1}(P, L) = 0$ from condition (ii), and $\mathrm{Ext}_{\mathcal{O}_x}^{p+1}(P, R) = 0$ since $\dim P \leq p+1$ from the hypothesis of recurrence. Since this property characterizes the modules of dimension $\leq p$, the Lemma is proved.

The argument up to this point shows that $\mathrm{Ext}^{p+1}_{\mathcal{O}_x}(P, N) = 0$ whenever N is of finite type, and the desired conclusion, that $\dim_{\mathcal{O}_x} P \leq p$, follows from Proposition D6.4.

On combining the Lemma and Theorem 1, we obtain:

THEOREM 2. *Let $\mathcal{F}$ be a coherent algebraic sheaf on X, and let p be an integer ≥ 0. The following two conditions are equivalent:*

(a) $H^q(X, \mathcal{F}(-n)) = 0$ for sufficiently large n and $0 \leq q < p$.

(b) For all $x \in X$, the $\mathcal{O}_x$-module $\mathcal{F}_x$ is of dimension $\leq r - p$.

75. Varieties without singularities. The following result plays an essential role in the extension to the abstract case of the "theorem of duality" of [15]:

THEOREM 3. *Let V be a subvariety without singularities of the projective space $P_r(K)$; suppose that all the irreducible components of V have the same dimension p. Let $\mathcal{F}$ be a coherent algebraic sheaf on V such that, for all $x \in V$, $\mathcal{F}_x$ is a free module on $\mathcal{O}_{x,V}$. We then have $H^q(V, \mathcal{F}(-n)) = 0$ for all sufficiently large n and $0 \leq q \leq p$.*

From Theorem 2, it all comes down to showing that $\mathcal{O}_{x,V}$, considered as an $\mathcal{O}_x$-module, is of dimension $\leq r - p$. Denote by $\mathcal{I}_x(V)$ the kernel of the canonical homomorphism $\varepsilon_x : \mathcal{O}_x \to \mathcal{O}_{x,V}$; since the point x is simple on V, we know (cf. [18], th. 1) that this ideal is generated by $r-p$ elements $f_1, \ldots, f_{r-p}$, and the theorem of Cohen-Macaulay (cf. [13], p. 53, prop. 2) shows that we have

$$(f_1, \ldots, f_{i-1}) : f_i = (f_1, \ldots, f_{i-1}) \qquad \text{for } 1 \leq i \leq r - p.$$

The ideas of Zariski in [18] had recently changed the way that nonsingularity (that is, the notion of a simple point) was defined. While extracting the consequences in question from this new definition can be reduced to a matter of algebra, in spirit it is closer to algebraic geometry per se than to commutative algebra, and thus not appropriately treated in the chapters. Thus we simply accept what Serre is asserting while urging the interested reader to consult texts on algebraic geometry, where the issue is addressed from diverse points of view.

Now we denote by L_q the free $\mathcal{O}_x$-module admitting for a base the elements $e\langle i_1 \cdots i_q \rangle$ corresponding to the sequence $(i_1, \ldots, i_q)$ such that

$$1 \leq i_i < i_2 < \cdots < i_q \leq r - p;$$

for $q = 0$, let $L_0 = \mathcal{O}_x$, and set:

$$d(e\langle i_1 \cdots i_q \rangle) = \sum_{j=1}^{q} (-1)^j f_{i_j} \cdot e\langle i_1 \cdots \hat{i}_j \cdots i_q \rangle$$

$$d(e\langle i\rangle) = f_i.$$

From [6], Chap. VIII, prop. 4.3, the sequence

$$0 \to L_{r-p} \xrightarrow{d} L_{r-p-1} \xrightarrow{d} \cdots \xrightarrow{d} L_0 \xrightarrow{\varepsilon_x} \mathcal{O}_{x,V} \to 0$$

is exact, which shows that $\dim_{\mathcal{O}_x}(\mathcal{O}_{x,V}) \leq r-p$, qed.

In Proposition G6.1 we set $M = \mathcal{O}_x$ and $n = r-p$, and the claim is the acyclicity of $X^{(r-p)}$.

Corollary. *We have* $H^q(V, \mathcal{O}_V(-n)) = 0$ *for sufficiently large* n *and* $0 \leq q \leq p$.

Remark. The proof above applies, more generally, whenever the ideal $\mathcal{I}_x(V)$ admits a system of $r-p$ generators, in other words when the variety V is *locally a complete intersection*, at each point.

76. Normal varieties. We will need the following Lemma:

Lemma 2. *Let M be an $\mathcal{O}_x$-module of finite type, and let f be a noninvertible element of $\mathcal{O}_x$, such that the relation $f \cdot m = 0$ implies that $m = 0$ if $m \in M$. The dimension of the $\mathcal{O}_x$-module M/fM is then equal to the dimension of M augmented by a unit.*

By hypothesis, there is an exact sequence $0 \to M \xrightarrow{\alpha} M \longrightarrow M/fM \to 0$, where α is multiplication by f. If N is an $\mathcal{O}_x$-module of finite type, we then have an exact sequence:

$$\cdots \to \operatorname{Ext}^q_{\mathcal{O}_x}(M,N) \xrightarrow{\alpha} \operatorname{Ext}^q_{\mathcal{O}_x}(M,N) \to \operatorname{Ext}^q_{\mathcal{O}_x}(M/fM,N) \to \operatorname{Ext}^{q+1}_{\mathcal{O}_x}(M,N) \to \cdots$$

Let p be the dimension of M. On setting $q = p+1$ in the preceeding exact sequence, we see that $\operatorname{Ext}^{p+2}_{\mathcal{O}_x}(M/fM,N) = 0$, which implies ([6], Chap. VI, §2) that $\dim\,(M/fM) \leq p+1$.

The claim is implied by Proposition D6.6.

On the other hand, since $\dim M = p$, we can choose an N such that $\operatorname{Ext}^p_{\mathcal{O}_x}(M,N) \neq 0$; after setting $q = p$ in the exact sequence above, we see that $\operatorname{Ext}^{p+1}_{\mathcal{O}_x}(M/fM,N)$ is identified with the kernel of

$$\operatorname{Ext}^p_{\mathcal{O}_x}(M,N) \xrightarrow{\alpha} \operatorname{Ext}^p_{\mathcal{O}_x}(M,N);$$

since this last homomorphism is none other than multiplication by f, and since f is not invertible in the local ring $\mathcal{O}_x$, it follows from [6], Chap. VIII, prop. $5.1'$ that the kernel is $\neq 0$, which shows that $\dim M/fM \geq p+1$, completing the proof.

The desired consequence of the cited result follows from Proposition A2.15.

We will now prove a result that has a close connection with the "lemma of d'Enriques-Severi", due to Zariski [19]:

THEOREM 4. *Let V be an irreducible subvariety, normal, of dimension ≥ 2, of the projective space $\mathbf{P}_r(K)$. Let $\mathcal{F}$ be a coherent algebraic sheaf on V such that, for all $x \in V$, $\mathcal{F}_x$ is a free module on $\mathcal{O}_{x,V}$. Then $H^1(V, \mathcal{F}(-n)) = 0$ for sufficiently large n.*

By definition V is *normal* if, for each $x \in V$, $\mathcal{O}_{x,V}$ is a normal ring, which is to say that it is an integral domain that is integrally closed in its field of fractions.

From Theorem 2, this reduces to showing that $\mathcal{O}_{x,V}$, considered as a $\mathcal{O}_x$-module, is of dimension $\leq r-2$. First we choose an element $f \in \mathcal{O}_x$, such that $f(x) = 0$ and the image of f in $\mathcal{O}_{x,V}$ is not null; this is possible because $\dim V > 0$. Since V is irreducible, $\mathcal{O}_{x,V}$ is an integral domain, and we can apply Lemma 2 to the pair $(\mathcal{O}_{x,V}, f)$; thus:

$$\dim \mathcal{O}_{x,V} = \dim \mathcal{O}_{x,V}/(f) - 1, \quad \text{with} \quad (f) = f \cdot \mathcal{O}_{x,V}.$$

Since $\mathcal{O}_{x,V}$ is an integrally closed ring, all the primary ideals $\mathfrak{p}^\alpha$ of the principal ideal (f) are minimal (cf. [12], p. 136, or [9], no. 37),

This is Theorem A11.2.

and none of them are equal to the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{x,V}$ (otherwise one would have $\dim V \leq 1$). Therefore one can find an element $g \in \mathfrak{m}$ that is not a zero divisor in the quotient ring $\mathcal{O}_{x,V}/(f)$; on letting $\overline{g}$ be a representative of g in $\mathcal{O}_x$, we see that we can apply Lemma 2 to the pair $(\mathcal{O}_{x,V}/(f), \overline{g})$; therefore:

$$\dim \mathcal{O}_{x,V}/(f) = \dim \mathcal{O}_{x,V}/(f,g) - 1.$$

But, from the theorem of syzygies already cited, we have $\dim \mathcal{O}_{x,V}/(f,g) \leq r$; whence $\mathcal{O}_{x,V}/(f) \leq r-1$ and $\dim \mathcal{O}_{x,V} \leq r-2$, qed.

Again, this is the Hilbert syzygy theorem.

COROLLARY. *We have $H^1(V, \mathcal{O}_V(-n)) = 0$ for sufficiently large n.*

REMARKS. (1) The reasoning above is classical in the theory of syzygies. Cf. for example W. Gröbner, *Moderne Algebraische Geometrie*, 152.6 and 153.1.

(2) Even when the dimension of V is > 2, one can have $\dim \mathcal{O}_{x,V} = r-2$. This is notably the case when V is a cone of which the hyperplane section W is a projective variety that is normal and irregular (i.e., $H^1(W, \mathcal{O}_W) \neq 0$).

77. Homological characterization of the varieties "k-fois de première espèce.

Let M be a graded S-module of finite type. Using reasoning identical to that used to prove Lemma 1, one can show that:

LEMMA 3. *In order for* $\dim M \leq k$, *it is necessary and sufficient that* $\mathrm{Ext}_S^q(M,S) = 0$ *for* $q > k$.

Since M is graded, we have $\mathrm{Ext}_S^q(M,\Omega) = \mathrm{Ext}_S^q(M,S)(-r-1)$, so the condition above is equivalent to $\mathrm{Ext}_S^q(M,\Omega) = 0$ for $q > k$. Taking Theorem 1 of no. 72 into account, we have

PROPOSITION 2. *(a) For* $\dim M \leq r$ *it is necessary and sufficient that* $\alpha : M_n \to H^0(M(n))$ *is injective for all* $n \in \mathbb{Z}$.

(b) If k *is an integer* ≥ 1, *for* $\dim M \leq r-k$ *it is necessary and sufficient that* $\alpha : M_n \to H^0(M(n))$ *is bijective for all* $n \in \mathbb{Z}$, *and that* $H^q(M(n)) = 0$ *for* $0 < q < k$ *and all* $n \in \mathbb{Z}$.

Let V be a closed subvariety of $\mathbf{P}_r(K)$, and let $I(V)$ be the ideal of homogeneous polynomials that vanish on V. Set $S(V) = S/I(V)$; this is a graded S-module whose associated sheaf is none other than $\mathcal{O}_V$. We say[8] that V is a subvariety "k-fois de première espèce" of $\mathbf{P}_r(K)$ if the dimension of the S-module $S(V)$ is $\leq r-k$.

The phrase "k-fois de première espèce" could be translated as "k times of the first type." I have chosen to leave it untranslated in order to avoid suggesting that it is currently part of standard terminology, which does not seem to be the case.

It is immediate that $\alpha : S(V)_n \to H^0(V,\mathcal{O}_V(n))$ is injective for all $n \in \mathbb{Z}$, so all varieties are 0-fois de première espèce. On applying the preceeding Proposition to $M = S(V)$, we obtain:

PROPOSITION 3. *Let* k *be an integer* ≥ 1. *For the subvariety* V *to be* k*-fois de première espèce, it is necessary and sufficient that the following conditions hold for all* $n \in \mathbb{Z}$:

(i) $\alpha : S(V)_n \to H^0(V,\mathcal{O}_V(n))$ *is bijective.*

(ii) $H^q(V,\mathcal{O}_V(n)) = 0$ *for* $0 < q < k$.

(The condition (i) can also be expressed by saying that the linear series cut on V by the forms of degree n is complete, which is well known.)

On comparing with Theorem 2 (or on reasoning directly) we obtain: COROLLARY. *If* V *is* k*-fois de première espèce, we have* $H^q(V,\mathcal{O}_V) = 0$ *for* $0 < q < k$, *and, for all* $x \in V$, *the dimension of the* $\mathcal{O}_x$*-module* $\mathcal{O}_{x,V}$ *is* $\leq r-k$.

[8] Cf. P. Dubreil, *Sur la dimension des idéaux de polynômes*, J. Math. Pures App., 15, 1936, p. 271-283. See also W. Gröbner, Moderne Algebraische Geometrie, §5.

If m is an integer ≥ 1, denote by φ_m the embedding of $\mathbf{P}_r(K)$ in a projective space of dimension given by the monomials of degree m (cf. [8], Chap. XVI, §6, or the proof of Lemma 2 in no. 52). The corollary above has the following "recipocal:"

Proposition 4. *Let k be an integer ≥ 1, and let V be a closed connected subvariety of $\mathbf{P}_r(K)$. Assume that $H^q(V, \mathcal{O}_V) = 0$ for $0 < q < k$, and that, for all $x \in V$, the dimension of the $\mathcal{O}_x$-module $\mathcal{O}_{x,V}$ is $\leq r - k$.*

Then, for all sufficiently large m, $\varphi_m(V)$ is a subvariety k-fois de première espèce.

From the fact that V is connected, we have $H^0(V, \mathcal{O}_x) = K$. In effect, if V is irreducible, it is evident (otherwise $H^0(V, \mathcal{O}_V)$ contains an algebra of polynomials, and is not of finite dimension over K); if V is reducible, each element $f \in H^0(V, \mathcal{O}_V)$ induces a constant on each of the irreducible components of V, and those constants are the same, because V is connected.

From the fact that $\dim \mathcal{O}_{x,V} \leq r - 1$, the algebraic dimension of each of the irreducible components of V is at least equal to 1. It follows that

$$H^0(V, \mathcal{O}_V(-n)) = 0$$

for $n > 0$ (because if $f \in H^0(V, \mathcal{O}_V(-n))$ and $f \neq 0$, the $f^h \cdot g$ with $g \in S(V)_{nk}$ form a linear subspace of $H^0(V.\mathcal{O}_V)$ of dimension > 1).

In order to be precise, denote by V_m the subvariety $\varphi_m(V)$; evidently

$$\mathcal{O}_{V_m}(n) = \mathcal{O}_V(nm).$$

For sufficiently large m the following hold:

(a) $\alpha : S(V)_{nm} \to H^0(V, \mathcal{O}_V(nm))$ is bijective for all $n \geq 1$.

This follows from Proposition 5 of no. 65.

(b) $H^q(V, \mathcal{O}_V(nm)) = 0$ for $0 < q < k$ and for all $n \geq 1$.

This follows from Proposition 7 of no. 65.

(c) $H^q(V, \mathcal{O}_V(nm)) = 0$ for $0 < q < k$ and for all $n \leq -1$.

This follows from Theorem 2 of no. 74, and from the hypothesis made on the $\mathcal{O}_{x,V}$.

On the other hand, we have $H^0(V, \mathcal{O}_V) = K$, $H^0(V, \mathcal{O}_V(nm)) = 0$ for all $n \leq 1$, and $H^q(V, \mathcal{O}_V) = 0$ for $0 < q < k$, by virtue of the hypotheses. It follows that V_m satisfies all the hypotheses of Proposition 3, qed.

Corollary. *Let k be an integer ≥ 1, and let V be a nonsingular projective variety, of dimension $\geq k$. In order for V to be biregularly isomorphic to a subvariety k-fois première espèce of a suitable projective space, it is necessary and sufficient that V is connected and that $H^q(V, \mathcal{O}_V) = 0$ for $0 < q < k$.*

The necessity is evident, from Proposition 3. To prove sufficiency if suffices to remark that $\mathcal{O}_{x,V}$ is of dimension $\leq r-k$ (cf. no. 75) and apply the preceeding Proposition.

78. Complete intersections. A subvariety V of dimension p of the projective space $\mathbf{P}_r(K)$ is a *complete intersection* if the ideal $I(V)$ has a system of $r-p$ generators $P_1,\ldots,P_{r-p}$; in this case, all the irreducible components of V have dimension p, from the theorem of Macaulay (cf. [9], no. 17).

The relevant result of Macaulay is now known as the unmixedness theorem. The end of Chapter H discusses why it implies the conclusion asserted by Serre.

It is well known that such a variety is p-fois de première espèce, which already implies that $H^q(V,\mathcal{O}_V(n))=0$ for $0<q<p$, as we will see. We will determine $H^p(V,\mathcal{O}_V(n))$ as a function of the degrees $m_1,\ldots,m_{r-p}$ of the homogeneous polynomials $P_1,\ldots,P_{r-p}$.

Let $S(V)=S/I(V)$ be the projective coordinate ring of V. From Theorem 1 of no. 72, everything reduces to the determination of the S-module $\operatorname{Ext}_S^{r-p}(S(V),\Omega)$. Now, there is a resolution analogous to that of no. 75; we take for L^q the free graded S-module admitting for a base the elements $e\langle i_1\cdots i_q\rangle$ corresponding to the sequences $(i_1,\ldots,i_q)$ such that $1\leq i_1<\cdots<i_q\leq r-p$, and of degree $\sum_{j=1}^q m_j$; for L^0, we take S. We set:

$$d(e\langle i_1\cdots i_q\rangle)=\sum_{j=1}^{q}(-1)^j P_{i_j}\cdot e\langle i_1\cdots \hat{i}_j\cdots i_q\rangle$$

$$d(e\langle i\rangle)=P_i.$$

The sequence $0\to L^{r-p}\xrightarrow{d}\cdots\xrightarrow{d}L^0\to S(V)\to 0$ is exact ([6], Chap. VIII, Prop. 4.3).

In Proposition G6.1 we set $M=S$ and $n=r-p$, and the claim is the acyclicity of $X^{(r-p)}$.

It follows that the $\operatorname{Ext}_S^q(S(V),\Omega)$ are the cohomology groups of the complex formed by the $\operatorname{Hom}_S(L^q,\Omega)$; but we can identify an element of $\operatorname{Hom}_S(L^q,\Omega)_n$ with a system $f\langle i_1\cdots i_q\rangle$, where the $f\langle i_1\cdots i_q\rangle$ are the homogeneous polynomials of degrees $m_{i_1}+\cdots+m_{i_q}+n-r-1$; once this identification is made, the coboundary operator is given by the usual formula:

$$(df)\langle i_1\cdots i_{q+1}\rangle=\sum_{j=1}^{q+1}(-1)^j P_{i_j}\cdot f\langle i_1\cdots\hat{i}_j\cdots i_{q+1}\rangle.$$

The theorem of Macaulay already cited shows that we have the conditions of [11], The groups $\mathrm{Ext}^q_S(S(V), \Omega)$ are the homology of

$$0 \to \mathrm{Hom}_S(L^0, \Omega) \to \cdots \to \mathrm{Hom}_S(L^q, \Omega) \to \cdots \to \mathrm{Hom}_S(L^{r-p}, \Omega) \to 0.$$

Under the identification described by Serre this becomes

$$0 \to \Omega \otimes_S L^0 \to \cdots \to \Omega \otimes_S L^q \to \cdots \to \Omega \otimes_S L^{r-p} \to 0.$$

We would now like to use Theorem G4.4 to arrive at the conclusions asserted below. For this we need is to be the case that $P_1, \ldots, P_{r-p}$ is a regular sequence on $\Omega = S(-r-1)$, which means that for each $i = 1, \ldots, r-p$, P_i is not a zero divisor of $\Omega/(P_1, \ldots, P_{i-1})\Omega$. Now P_i is a zero divisor if and only if it vanishes identically on one of the irreducible components of the variety of common zeros of $P_1, \ldots, P_{i-1}$, and if this happened the principle ideal theorem (Theorem F6.1) would imply that some ireducible component of V had dimension greater than p. As we saw above, Macaulay's unmixedness theorem implies that this cannot happen.

and from there we retrieve the fact that $\mathrm{Ext}^q_S(S(V), \Omega) = 0$ for $q \neq r-p$. On the other hand, $\mathrm{Ext}^{r-p}_S(S(V), \Omega)_n$ is isomorphic to the subspace of $S(V)$ formed of the homogeneous elements of degree $N + n$, with $N = \sum_{i=1}^{r-p} m_i - r - 1$. Taking into account Theorem 1 of no. 72, we obtain:

PROPOSITION 5. *Let V be a complete intersection, defined by the homogeneous polynomials $P_1, \ldots, P_{r-p}$, of degrees $m_1, \ldots, m_{r-p}$.*

(a) *The function $\alpha : S(V)_n \to H^0(V, \mathcal{O}_V(n))$ is bijective for all $n \in \mathbb{Z}$.*

(b) *$H^q(V, \mathcal{O}_V(n)) = 0$ for $0 < q < p$ and all $n \in \mathbb{Z}$.*

(c) *$H^p(V, \mathcal{O}_V(n))$ is isomorphic to the dual space of $H^0(V, \mathcal{O}_V(N-n))$, with $N = \sum_{i=1}^{r-p} m_i - r - 1$.*

We note in particular that $H^p(V, \mathcal{O}_V)$ does not vanish unless $N < 0$.

§6. Characteristic function and arithmetic genus

79. The Euler-Poincaré characteristic.

Let V be a projective variety, and let $\mathcal{F}$ be a coherent algebraic sheaf on V. Set:

$$h^q(V, \mathcal{F}) = \dim_K H^q(V, \mathcal{F}).$$

We have seen (no. 66, Theorem 1) that the $h^q(V, \mathcal{F})$ are *finite* for all integers q, and vanish for $q > \dim V$. We can therefore define an integer $\chi(V, \mathcal{F})$ on setting:

$$\chi(V, \mathcal{F}) = \sum_{q=0}^{\infty} (-1)^q h^q(V, \mathcal{F}).$$

This is the *Euler-Poincaré characteristic* of V, with values in $\mathcal{F}$.

LEMMA 1. *Let $0 \to L_1 \to \cdots \to L_p \to 0$ be an exact sequence, where the L_i are finite dimensional vector spaces over K, and the homomorphisms $L_i \to L_{i+1}$ are K-linear. Then:*

$$\sum_{q=1}^{p}(-1)^q \dim_K L_q = 0.$$

We argue by recursion on p; the lemma is evident if $p \leq 3$; if L'_{p-1} denotes the kernel of $L_{p-1} \to L_p$, we have the two exact sequences:

$$0 \to L_1 \to \cdots \to L'_{p-1} \to 0$$

$$0 \to L'_{p-1} \to L_{p-1} \to L_p \to 0.$$

On applying the hypothesis of recurrence to each of these sequences, we see that $\sum_{q=1}^{p-2}(-1)^q \dim L_q + (-1)^{p-1} \dim L'_{p-1} = 0$, and

$$\dim L'_{p-1} - \dim L_{p-1} + \dim L_p = 0,$$

from which the lemma follows immediately.

PROPOSITION 1. *Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of coherent algebraic sheaves on a projective variety V, with the homomorphisms $\mathcal{A} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{C}$ being K-linear. Then:*

$$\chi(V, \mathcal{B}) = \chi(V, \mathcal{A}) + \chi(V, \mathcal{C}).$$

On applying Lemma 1 to this exact sequence of vector spaces, we obtain the Proposition.

PROPOSITION 2. *Let $0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_p \to 0$ be an exact sequence of coherent algebraic sheaves on a projective variety V, with the homomorphisms $\mathcal{F}_i \to \mathcal{F}_{i+1}$ being algebraic. Then:*

$$\sum_{q=1}^{p}(-1)^q \chi(V, \mathcal{F}_q) = 0.$$

We argue by recurrence on p, beginning with the Proposition being a particular case of Proposition 1 if $p \leq 3$. If we denote by $\mathcal{F}'_{p-1}$ the kernel of $\mathcal{F}_{p-1} \to \mathcal{F}_p$, the sheaf $\mathcal{F}'_{p-1}$ is coherent algebraic since $\mathcal{F}_{p-1} \to \mathcal{F}_p$ is an algebraic homomorphism. We can then apply the hypothesis of recurrence to the two exact sequences

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}'_{p-1} \to 0$$

$$0 \to \mathcal{F}'_{p-1} \to \mathcal{F}_{p-1} \to \mathcal{F}_p \to 0,$$

and the Proposition follows straight away.

80. Relation with the characteristic function of a graded S-module .

Let $\mathcal{F}$ be a coherent algebraic sheaf on the space $\mathbf{P}_r(K)$; we will write $\chi(\mathcal{F})$ in place of $\chi(\mathbf{P}_r(K), \mathcal{F})$. We have:

PROPOSITION 3. *$\chi(\mathcal{F}(n))$ is a polynomial in n of degree $\leq r$.*

From Theorem 2 of no. 60, there exists a graded S-module M, of finite type, such that $\mathcal{A}(M)$ is isomorphic to $\mathcal{F}$. On applying to M the theorem of syzygies of Hilbert,

The free dimension of M agrees with its projective dimension, by Proposition I3.1. The projective dimension is not greater than the global dimension of S, which is $r+1$. (Theorem I3.4.)

we obtain an exact sequence of graded S-modules:

$$0 \to L^{r+1} \to \cdots \to L^0 \to M \to 0,$$

where the L^q are free of finite type. On applying the functor $\mathcal{A}$ to this sequence, we obtain an exact sequence of sheaves:

$$0 \to \mathcal{L}^{r+1} \to \cdots \to \mathcal{L}^0 \to \mathcal{F} \to 0,$$

where each of the $\mathcal{L}^q$ is isomorphic to a finite direct sum of sheaves $\mathcal{O}(n_i)$. Proposition 2 shows that $\chi(\mathcal{F}(n))$ is equal to the alternating sum of the $\chi(\mathcal{L}^q(n))$, which we reduced to the case of the sheaf $\mathcal{O}(n_i)$. Now it follows from no. 62 that we have $\chi(\mathcal{O}(n)) = \binom{n+r}{n}$, which is indeed a polynomial in n of degree $\leq r$, as we wished to show.

PROPOSITION 4. *Let M be an graded S-module satisfying the condition (TF), and let $\mathcal{F} = \mathcal{A}(M)$. For all sufficiently large n we have $\chi(\mathcal{F}(n)) = \dim_K M_n$.*

In effect, we know (no. 65) that, for all sufficiently large n, the homomorphism $\alpha : M_n \to H^0(X, \mathcal{F}(n))$ is bijective, and $H^q(X, \mathcal{F}(n)) = 0$ for all $q > 0$; therefore $\chi(\mathcal{F}(n)) = h^0(X, \mathcal{F}(n)) = \dim_K M_n$.

We now recall the well known fact that $\dim_K M_n$ is a polynomial in n for sufficiently large n; this polynomial, which we denote by P_M, is called the *characteristic function* of M; for all $n \in \mathbb{Z}$, we have $P_M(n) = \chi(\mathcal{F}(n))$, and, in particular, for $n = 0$, we see that *the constant term of P_M is equal to* $\chi(\mathcal{F})$.

This passage might easily confuse a reader, since it is standard to define the *Poincaré-Hilbert series* to be $\sum_{n=0}^{\infty} (\dim_K M_n) t^n$, and to denote this series by $P_M(t)$. The fact that there is a $g \in \mathbb{Q}[t]$ such that $\dim_K M_n = g(n)$ for sufficiently large n is Proposition F2.4.

We apply this to $M = S/I(V)$, where $I(V)$ is the homogeneous ideal of S consisting of the polynomials that vanish on a closed subvariety V of $\mathbf{P}_r(K)$. The constant term of P_M is called, in this case, the *arithmetic genus* of V (cf. [19]); on the other hand we have $\mathcal{A}(M) = \mathcal{O}_V$, so we obtain:

PROPOSITION 5. *The arithmetic genus of a projective variety V is equal to*

$$\chi(V, \mathcal{O}_V) = \sum_{i=0}^{\infty} (-1)^q \dim_K H^q(V, \mathcal{O}_V).$$

REMARKS. (1) The preceeding proposition makes evident the fact that the arithmetic genus is independent of the embedding of V in a projective space, since all embeddings have the same $H^q(V, \mathcal{O}_V)$.

(2) The *virtual* arithmetic genus (defined by Zariski in [19]) can equally be reduced to an Euler-Poincaré characteristic. Ultimately we will return to this question, which is closely related to the Riemann-Roch theorem.

(3) For reasons of convenience, we have adopted a definition of the arithmetic genus that is slightly different from the classical definition (cf. [19]). If all the irreducible components of V have the same dimension p, the two definition are related by the following formula: $\chi(V, \mathcal{O}_V) = 1 + (-1)^p p_a(V)$.

81. Degree of the characteristic function .

If $\mathcal{F}$ is a coherent algebraic sheaf on an algebraic variety V, the *support* of $\mathcal{F}$, denoted by $\mathrm{Supp}(\mathcal{F})$, is the set of the points $x \in V$ such that $\mathcal{F}_x \neq 0$. From the fact that $\mathcal{F}$ is a sheaf of finite type, this set is *closed*: in effect, if we have $\mathcal{F}_x = 0$, the null section generates $\mathcal{F}_x$ and consequently also $\mathcal{F}_y$ for y sufficiently close to x (no. 12, Proposition 1), which means that the complement of $\mathrm{Supp}(\mathcal{F})$ is open.

Let M be a graded S-module of finite type, and let $\mathcal{F} = \mathcal{A}(M)$ be the sheaf defined by M on $\mathbf{P}_r(K) = X$. One can derive $\mathrm{Supp}(\mathcal{F})$ from M in the following manner:

Let $0 = \bigcap_\alpha M^\alpha$ be a decomposition of 0 as an intersection of primary homogeneous submodules M^α of M, where the M^α correspond to the homogeneous primary ideals $\mathfrak{p}^\alpha$ (cf. [12], chap. IV); we may assume that this decomposition is "as short as possible", i.e. none of the M^α are contained in the intersection of the others.

The existence of what is now called a primary decomposition of the submodule 0 is assertion of Theorem A12.3. Such a decomposition is finite, so redundant elements can be discarded until it is minimal.

For all $x \in X$, each $\mathfrak{p}^\alpha$ defines a primary ideal of the local ring $\mathcal{O}_x$, and we have $\mathfrak{p}^\alpha_x = \mathcal{O}_x$ if and only if x is not contained in the variety V^α defined by the ideal $\mathfrak{p}^\alpha$. In the same way $0 = \bigcap_\alpha M^\alpha_x$ in M_x, and it is easily verified that from this we obtain a primary decomposition of 0 in M_x, with the M^α_x

corresponding to the primary ideals $\mathfrak{p}_x^\alpha$; if $x \notin V^\alpha$, we have $M_x^\alpha = M_x$, and, if we only consider the M_x^α such that $x \in V^\alpha$, we obtain a decomposition that is "as short as possible" (cf. [12], chap. IV, th. 4, where analogous results are established). It follows immediately that $M_x \neq 0$ if and only if x is contained in one of the varieties V^α, which is to say that $\mathrm{Supp}(\mathcal{F}) = \bigcup_\alpha V^\alpha$.

PROPOSITION 6. *If $\mathcal{F}$ is a coherent algebraic sheaf on $\mathbf{P}_r(K)$, the degree of the polynomial $\chi(\mathcal{F}(n))$ is equal to the dimension of* $\mathrm{Supp}(\mathcal{F})$.

We reason by recurrence on r, with the case $r = 0$ being trivial. We may suppose that $\mathcal{F} = \mathcal{A}(M)$, where M is a graded S-module of finite type; using the notations introduced above, we will show that $\chi(\mathcal{F}(n))$ is a polynomial of degree $q = \mathrm{Sup}\dim V^\alpha$.

Let t be a homogeneous linear form not contained in any of the primary ideals $\mathfrak{p}^\alpha$, except the "improper" primary ideal $\mathfrak{p}^0 = (t_0, \ldots, t_r)$; such a form exists due to the fact that the field K is infinite. Let E be the hyperplane of X defined by the equation $t = 0$. Consider the exact sequence:

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_E \to 0,$$

where $\mathcal{O} \to \mathcal{O}_E$ is the restriction homomorphism, while $\mathcal{O}(-1) \to \mathcal{O}$ is the homomorphism $f \to t \cdot f$. By tensoring with $\mathcal{F}$, we obtain the exact sequence:

$$\mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_E \to 0, \quad \text{with} \quad \mathcal{F}_E = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}_E.$$

Above U_i, we can identify $\mathcal{F}(-1)$ with $\mathcal{F}$, and this identification transforms the homomorphism $\mathcal{F}(-1) \to \mathcal{F}$ defined above to the one defined by multiplication by t/t_i; since t has been chosen outside of the $\mathfrak{p}^\alpha$, t/t_i is not contained in each of the prime ideals of $M_x = \mathcal{F}_x$ if $x \in U_i$, and the preceeding homomorphism is injective (cf. [12], p. 122, th. 7, b‴).

Because t is not an element of any of the primary ideals, it is not a zero divisor of M, by Corollary A10.3.

Therefore we have the exact sequence:

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_E \to 0,$$

from which we obtain, for each n, the exact sequence:

$$0 \to \mathcal{F}(n-1) \to \mathcal{F}(n) \to \mathcal{F}_E(n) \to 0.$$

On applying Proposition 1, we see that

$$\chi(\mathcal{F}(n)) = \chi(\mathcal{F}(n-1)) = \chi(\mathcal{F}_E(n)).$$

But the sheaf $\mathcal{F}_E$ is a coherent sheaf of $\mathcal{O}_E$-modules, which is to say that it is a coherent algebraic sheaf on E, which is a projective space of dimension $r-1$. Moreover, $\mathcal{F}_{x,E} = 0$ means that the endomorphism of $\mathcal{F}_x$ defined by multiplication by t/t_i is surjective, which implies that $\mathcal{F}_x = 0$ (cf. [6], chap. VIII, prop. 5.1′).

The desired consequence of the cited result follows from Proposition A2.15, which is a consequence of Nakayama's lemma.

It follows that $\mathrm{Supp}(\mathcal{F}_E) = E \cap \mathrm{Supp}(\mathcal{F})$, and, since E does not contain any of the varieties V^α, if follows by a known result that the dimension of $\mathrm{Supp}(\mathcal{F}_E)$ is equal to $q-1$. The hypothesis of recurrence then shows that $\chi(\mathcal{F}_E(n))$ is a polynomial of degree $q-1$; since this is the first difference of the function $\chi(\mathcal{F}(n))$, the latter function is polynomial of degree q.

Remarks. (1) Proposition 6 is well known when $\mathcal{F} = \mathcal{O}/\mathcal{I}$, with $\mathcal{I}$ being a coherent sheaf of ideals. Cf. [9], no. 24, for example.

(2) The preceeding demonstration did not use Proposition 3, and gives a new proof of that result.

Paris

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Index